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General iterative methods for generalized equilibrium problems and fixed point problems of k-strict pseudo-contractions

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Abstract

In this paper, we modify the general iterative method to approximate a common element of the set of solutions of generalized equilibrium problems and the set of common fixed points of a finite family of k-strictly pseudo-contractive nonself mappings. Strong convergence theorems are established under some suitable conditions in a real Hilbert space, which also solves some variation inequality problems. Results presented in this paper may be viewed as a refinement and important generalizations of the previously known results announced by many other authors.

MSC: 47H05; 47H09; 47H10

Keywords: generalized equilibrium problem; k-strict pseudo-contractions; general iterative method; α-inverse strongly monotone; common fixed point; strong convergence

1 Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, respectively. Let K be a nonempty closed convex subset of H. Let $A : K \to H$ be a nonlinear mapping and $F : K \times K \to \mathbb{R}$ be a bi-function, where $\mathbb{R}$ denotes the set of real numbers. We consider the following generalized equilibrium problem: Find $x \in K$ such that

$$F(x, y) + \langle Ax, y - x \rangle \geq 0, \quad \forall y \in K. \tag{1.1}$$

We use $EP(F, A)$ to denote the solution set of the problem (1.1). If $A \equiv 0$, the zero mapping, then the problem (1.1) is reduced to the normal equilibrium problem: Find $x \in K$ such that

$$F(x, y) \geq 0, \quad \forall y \in K. \tag{1.2}$$

We use $EP(F)$ to denote the solution set of the problem (1.2). If $F \equiv 0$, then the problem (1.1) is reduced to the classical variational inequality problem: Find $x \in K$ such that

$$\langle Ax, y - x \rangle \geq 0, \quad \forall y \in K.$$

The generalized equilibrium problem (1.1) is very general in the sense that it includes, as special cases, optimization problems, variational inequalities, mini-max problems, the Nash equilibrium problem in noncooperative games and others (see, *e.g.*, [1–3]).

Recall that a nonself mapping $T : K \to H$ is called a k-strict pseudo-contraction if there exists a constant $k \in [0,1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\big\|(I - T)x - (I - T)y\big\|^2, \quad \forall x, y \in K. \tag{1.3}$$

We use $F(T)$ to denote the fixed point set of T, *i.e.*, $F(T) := \{x \in K : Tx = x\}$. As $k = 0$, T is said to be nonexpansive, *i.e.*,

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in K.$$

T is said to be pseudo-contractive if $k = 1$, and is also said to be strongly pseudo-contractive if there exists a positive constant $\lambda \in (0,1)$ such that $T + \lambda I$ is pseudo-contractive. Clearly, the class of k-strict pseudo-contractions falls into the one between classes of nonexpansive mappings and pseudo-contractions. We remark also that the class of strongly pseudo-contractive mappings is independent of the class of k-strict pseudo-contractions (see, *e.g.*, [4, 5]).

Iterative methods for equilibrium problems and fixed point problems of nonexpansive mappings have been extensively investigated. However, iterative schemes for strict pseudo-contractions are far less developed than those for nonexpansive mappings though Browder and Petryshyn [5] initiated their work in 1967; the reason is probably that the second term appearing in the right-hand side of (1.3) impedes the convergence analysis for iterative algorithms used to find a fixed point of the strict pseudo-contraction. On the other hand, strict pseudo-contractions have more powerful applications than nonexpansive mappings do in solving inverse problems; see, *e.g.*, [6–18, 20–27] and the references therein. Therefore it is interesting to develop the effective iterative methods for equilibrium problems and fixed point problems of strict pseudo-contractions.

In 2006, Marino and Xu [8] introduced a general iterative method and proved that for a given $x_0 \in H$, the sequence $\{x_n\}$ generated by

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n B)Tx_n, \quad \forall n \in N,$$

where T is a self-nonexpansive mapping on H, f is a contraction of H into itself and $\{\alpha_n\} \subseteq (0,1)$ satisfies certain conditions, B is a strongly positive bounded linear operator on H, converges strongly to $x^* \in F(T)$, which is the unique solution of the following variational inequality:

$$\langle (B - \gamma f)x^*, x^* - x\rangle \leq 0, \quad \forall x \in F(T),$$

and is also the optimality condition for some minimization problem.

Recently, Takahashi and Takahashi [12] considered the equilibrium problem and nonexpansive mapping by viscosity approximation methods. To be more precise, they proved the following theorem.

Theorem of TT *Let K be a nonempty closed convex subset of H. Let F be a bi-function from $K \times K$ to $\mathbb{R}$ satisfying (A1)-(A4) and let $T : K \to H$ be a nonexpansive mapping such that $F(T) \cap EP(F) \neq \phi$. Let $f : H \to H$ be a contraction and let $\{x_n\}$ and $\{y_n\}$ be sequences generated by $x_1 \in H$ and*

$$\begin{cases} F(y_n, z) + \frac{1}{r_n}\langle z - y_n, y_n - x_n\rangle \geq 0, & \forall z \in K, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Ty_n, & n \geq 1, \end{cases}$$

where $\{\alpha_n\} \subset [0,1]$ and $\{r_n\}$ satisfy

$$\lim_{n\to\infty} \alpha_n = 0, \qquad \sum_{n=0}^{\infty} \alpha_n = \infty, \qquad \sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty,$$
$$\liminf_{n\to\infty} r_n > 0, \qquad \sum_{n=0}^{\infty} |r_{n+1} - r_n| < \infty.$$

Then $\{x_n\}$ and $\{y_n\}$ converge strongly to $q \in F(T) \cap EP(F)$, where $q = P_{F(T)\cap EP(F)} f(q)$.

In 2009, Ceng *et al.* [15] further studied the equilibrium problem and fixed point problems of strict pseudo-contraction mappings T by an iterative scheme for finding an element of $EP(F) \cap F(T)$. Very recently, by using the general iterative method Liu [16] proposed the implicit and explicit iterative processes for finding an element of $EP(F) \cap F(T)$ and then obtained some strong convergence theorems, respectively. On the other hand, Takahashi and Takahashi [18] considered the generalized equilibrium problem and nonexpansive mapping in a Hilbert space. Moreover, they constructed an iterative scheme for finding an element of $EP(F,A) \cap F(T)$ and then proved a strong convergence of the iterative sequence under some suitable conditions.

In this paper, inspired and motivated by research going on in this area, we introduce a general iterative method for generalized equilibrium problems and strict pseudo-contractive nonself mappings, which is defined in the following way:

$$\begin{cases} F(u_n, y) + \langle Ax_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, & \forall y \in K, \\ y_n = \beta_n u_n + (1 - \beta_n)\sum_{i=1}^{N} \eta_i^{(n)} T_i u_n, \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n B)y_n, & n \geq 1, \end{cases} \tag{1.4}$$

where constant $\gamma > 0$, f is a contraction and A, B are two operators, $\{T_i\}_{i=i}^{N} : K \to H$ is a finite family of k_i-strict pseudo-contractions, $\{\eta_i^{(n)}\}_{i=1}^{N}$ is a finite sequence of positive numbers, $\{\alpha_n\}$, $\{\beta_n\}$ and $\{r_n\}$ are some sequences with certain conditions.

Our purpose is not only to modify the general iterative method to the case of a finite family of k_i-strictly pseudo-contractive nonself mappings, but also to establish strong convergence theorems for a generalized equilibrium problem and k_i-strict pseudo-contractions in a real Hilbert space, which also solves some variation inequality problems. Our theorems presented in this paper improve and extend the corresponding results of [12, 15, 16, 18, 20, 21, 25].

2 Preliminaries

Let K be a nonempty closed convex subset of a real Hilbert H space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, respectively. Recall that a mapping $f: K \to K$ is a contraction, if there exists a constant $\rho \in (0,1)$ such that

$$\|f(x) - f(y)\| \leq \rho\|x - y\|, \quad \forall x, y \in K.$$

We use Π_K to denote the collection of all contractions on K. The operator $A: K \to H$ is said to be monotone if

$$\langle Ax - Ay, x - y\rangle \geq 0, \quad \forall x, y \in K.$$

$A: K \to H$ is said to be r-strongly monotone if there exists a constant $r > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq r\|x - y\|^2, \quad \forall x, y \in K.$$

$A: K \to H$ is said to be α-inverse strongly monotone if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha\|Ax - Ay\|^2, \quad \forall x, y \in K.$$

Recall that an operator B is strongly positive if there exists a constant $\overline{\gamma} > 0$ with the property

$$\langle Bx, x\rangle \geq \overline{\gamma}\|x\|^2, \quad \forall x \in H.$$

To study the generalized equilibrium problem (1.1), we may assume that the bi-function $F: K \times K \to \mathbb{R}$ satisfies the following conditions:

(A1) $F(x,x) = 0$ for all $x \in K$;
(A2) F is monotone, *i.e.*, $F(x,y) + F(y,x) \leq 0$ for all $x, y \in K$;
(A3) for each $x, y, z \in K$, $\lim_{t\to 0} F(tz + (1-t)x, y) \leq F(x,y)$;
(A4) for each $x \in K$, $y \mapsto F(x,y)$ is convex and lower semi-continuous.

In order to prove our main results, we need the following lemmas and propositions.

Lemma 2.1 [1, 3] *Let $F: K \times K \to \mathbb{R}$ be a bi-function satisfying (A1)-(A4). Then, for any $r > 0$ and $x \in H$, there exists $z \in K$ such that*

$$F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in K.$$

Further, if $T_r x = \{z \in K : F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in K\}$, then the following hold:

(1) T_r is single-valued;
(2) T_r is firmly nonexpansive, i.e, $\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle$ for all $x, y \in H$;
(3) $F(T_r) = EP(F)$;
(4) $EP(F)$ is closed and convex.

Lemma 2.2 [8] *In the Hilbert space H, there hold the following identities:*

(i) $\|x+y\|^2 = \|x\|^2 + 2\langle x,y\rangle + \|y\|^2 \le \|x\|^2 + 2\langle y,(x+y)\rangle$, $\forall x,y \in H$;

(ii) $\|tx+(1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|^2$, $\forall t \in [0,1]$, $\forall x,y \in H$.

Lemma 2.3 [8] *Assume that B is a strongly positive linear bounded operator on the Hilbert space H with a coefficient $\overline{\gamma} > 0$ and $0 < \varrho < \|B\|^{-1}$. Then $\|I - \varrho B\| \le 1 - \varrho\overline{\gamma}$.*

Lemma 2.4 [10] *If $T : K \to H$ is a k-strict pseudo-contraction, then the fixed point set $F(T)$ is closed convex so that the projection $P_{F(T)}$ is well defined.*

Lemma 2.5 [2, 10] *Let $T : K \to H$ be a k-strict pseudo-contraction. For $\lambda \in [k,1)$, define $S : K \to H$ by $Sx = \lambda x + (1-\lambda)Tx$ for each $x \in K$. Then S is a nonexpansive mapping such that $F(S) = F(T)$.*

Lemma 2.6 [19] *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \le (1-\gamma_n)a_n + \gamma_n\delta_n, \quad n \ge 0,$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a real sequence such that

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(ii) $\limsup_{n\to\infty} \delta_n \le 0$ *or* $\sum_{n=1}^{\infty} |\gamma_n\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Proposition 2.1 (See, *e.g.*, Acedo and Xu [20]) *Let K be a nonempty closed convex subset of the Hilbert space H. Given an integer $N \ge 1$, assume that $\{T_i\}_{i=1}^N : K \to H$ is a finite family of k_i-strict pseudo-contractions. Suppose that $\{\lambda_i\}_{i=1}^N$ is a positive sequence such that $\sum_{i=1}^N \lambda_i = 1$. Then $\sum_{i=1}^N \lambda_i T_i$ is a k-strict pseudo-contraction with $k = \max\{k_i : 1 \le i \le N\}$.*

Proposition 2.2 (See, *e.g.*, Acedo and Xu [20]) *Let $\{T_i\}_{i=1}^N$ and $\{\lambda_i\}_{i=1}^N$ be given as in Proposition 2.1 above. Then $F(\sum_{i=1}^N \lambda_i T_i) = \bigcap_{i=1}^N F(T_i)$.*

3 Main results

Theorem 3.1 *Let K be a nonempty closed convex subset of the Hilbert space H and $F : K \times K \to \mathbb{R}$ be a bi-function satisfying (A1)-(A4). Let A be an α-inverse strongly monotone mapping and B be a strongly positive bounded linear operator on H with $\overline{\gamma} > 0$. Assume that $\{T_i\}_{i=1}^N : K \to H$ be a finite family of k_i-strict pseudo-contractions such that $\mathcal{F} = \bigcap_{i=1}^N F(T_i) \cap EP(F,A) \neq \phi$. Suppose $f \in \Pi_K$ with a coefficient $\rho \in (0,1)$ and $\{\eta_i^{(n)}\}_{i=1}^N$ are finite sequences of positive numbers such that $\sum_{i=1}^N \eta_i^{(n)} = 1$ for all $n \ge 0$, for a given point $x_0 \in K$, $\alpha_n, \beta_n \in (0,1)$, $r_n \in (0, 2\alpha)$ and $0 < \gamma < \frac{\overline{\gamma}}{\rho}$, the following control conditions are satisfied:*

(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ *and* $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$;

(ii) $k_i \le \beta_n \le \lambda < 1$, $\lim_{n\to\infty} \beta_n = \lambda$ *and* $\sum_{n=1}^{\infty} |\beta_n - \beta_{n-1}| < \infty$;

(iii) $\sum_{n=1}^{\infty} \sum_{i=1}^N |\eta_i^{(n)} - \eta_i^{(n-1)}| < \infty$;

(iv) $\liminf_{n\to\infty} r_n > 0$ *and* $\sum_{n=1}^{\infty} |r_n - r_{n-1}| < \infty$.

Then the sequence $\{x_n\}$ generated by (1.4) converges strongly to $q \in \mathcal{F}$, which solves the variational inequality

$$\big\langle (B - \gamma f)q, q - p \big\rangle \le 0, \quad \forall p \in \mathcal{F}.$$

Proof Putting $W_n = \sum_{i=1}^N \eta_i^{(n)} T_i$, we have $W_n : K \to H$ is a k-strict pseudo-contraction and $F(W_n) = \bigcap_{i=1}^N F(T_i)$ by Proposition 2.1 and 2.2, where $k = \max\{k_i : 1 \le i \le N\}$.

First, we show that the mapping $I - r_n A$ is nonexpansive. Indeed, for each $x, y \in K$, we have

$$\begin{aligned}\|(I - r_n A)x - (I - r_n A)y\|^2 &= \|x - y\|^2 - 2r_n\langle x - y, Ax - Ay\rangle + r_n^2\|Ax - Ay\|^2 \\ &\le \|x - y\|^2 - 2\alpha r_n\|Ax - Ay\|^2 + r_n^2\|Ax - Ay\|^2 \\ &= \|x - y\|^2 - r_n(2\alpha - r_n)\|Ax - Ay\|^2.\end{aligned}$$

It follows from the condition $r_n \in (0, 2\alpha)$ that the mapping $I - r_n A$ is nonexpansive. From Lemma 2.1, we see that $EP(F, A) = F(T_{r_n}(I - r_n A))$. Note that u_n can be rewritten as $u_n = T_{r_n}(I - r_n A)x_n$ and $p = T_{r_n}(I - r_n A)p$ for each $n \ge 1$ as $p \in \mathcal{F}$.

From (1.4), condition (ii) and Lemma 2.2, we have

$$\begin{aligned}\|y_n - p\|^2 &= \|\beta_n(u_n - p) + (1 - \beta_n)(W_n u_n - p)\|^2 \\ &= \beta_n\|u_n - p\|^2 + (1 - \beta_n)\|W_n u_n - p\|^2 - \beta_n(1 - \beta_n)\|u_n - W_n u_n\|^2 \\ &\le \beta_n\|u_n - p\|^2 + (1 - \beta_n)\big[\|u_n - p\|^2 + k\|u_n - W_n u_n\|^2\big] \\ &\quad - \beta_n(1 - \beta_n)\|u_n - W_n u_n\|^2 \\ &= \|u_n - p\|^2 - (1 - \beta_n)(\beta_n - k)\|u_n - W_n u_n\|^2 \\ &\le \|u_n - p\|^2. \end{aligned} \tag{3.1}$$

By $u_n = T_{r_n}(I - r_n A)x_n$, we obtain

$$\|u_n - p\| = \|T_{r_n}(I - r_n A)x_n - p\| \le \|x_n - p\|.$$

This together with (3.1), we see that

$$\|y_n - p\| \le \|u_n - p\| \le \|x_n - p\|. \tag{3.2}$$

Furthermore, by Lemma 2.3, we have

$$\begin{aligned}\|x_{n+1} - p\| &= \|\alpha_n[\gamma f(x_n) - Bp] + (I - \alpha_n B)(y_n - p)\| \\ &\le (1 - \alpha_n\overline{\gamma})\|y_n - p\| + \alpha_n\|\gamma f(x_n) - Bp\| \\ &\le (1 - \alpha_n\overline{\gamma})\|y_n - p\| + \alpha_n\big[\|\gamma f(x_n) - \gamma f(p)\| + \|\gamma f(p) - Bp\|\big] \\ &\le \big[1 - (\overline{\gamma} - \gamma\rho)\alpha_n\big]\|x_n - p\| + \alpha_n\|\gamma f(p) - Bp\|.\end{aligned}$$

It follows from induction that

$$\|x_n - p\| \le \max\left\{\|x_0 - p\|, \frac{1}{\overline{\gamma} - \gamma\rho}\|\gamma f(p) - Bp\|\right\}, \quad n \ge 1, \tag{3.3}$$

which gives that sequence $\{x_n\}$ is bounded, and so are $\{u_n\}$ and $\{y_n\}$.

Define a mapping $S_nx := \beta_n x + (1-\beta_n)W_nx$ for each $x \in K$. Then $S_n : K \to H$ is nonexpansive. Indeed, by using (1.3), Lemma 2.2 and condition (ii), we have for all $x, y \in K$ that

$$\begin{aligned}
\|S_nx - S_ny\|^2 &= \bigl\|\beta_n(x-y) + (1-\beta_n)(W_nx - W_ny)\bigr\|^2 \\
&= \beta_n\|x-y\|^2 + (1-\beta_n)\|W_nx - W_ny\|^2 \\
&\quad - \beta_n(1-\beta_n)\bigl\|x - W_nx - (y - W_ny)\bigr\|^2 \\
&\le \beta_n\|x-y\|^2 + (1-\beta_n)\bigl[\|x-y\|^2 + k\bigl\|x - W_nx - (y-W_ny)\bigr\|^2\bigr] \\
&\quad - \beta_n(1-\beta_n)\bigl\|x - W_nx - (y - W_ny)\bigr\|^2 \\
&= \|x-y\|^2 - (1-\beta_n)(\beta_n - k)\bigl\|x - W_nx - (y - W_ny)\bigr\|^2 \\
&\le \|x-y\|^2,
\end{aligned}$$

which shows that $S_n : K \to H$ is nonexpansive.

Next, we show that $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$. From (1.4) and Lemma 2.3, we have

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \bigl\|\alpha_n\gamma f(x_n) + (I - \alpha_nB)y_n - \bigl[\alpha_{n-1}\gamma f(x_{n-1}) + (I - \alpha_{n-1}B)y_{n-1}\bigr]\bigr\| \\
&\le \alpha_n\gamma\bigl\|f(x_n) - f(x_{n-1})\bigr\| + |\alpha_n - \alpha_{n-1}|\bigl[\gamma\bigl\|f(x_{n-1})\bigr\| + \|By_{n-1}\|\bigr] \\
&\quad + \bigl\|(I - \alpha_nB)(y_n - y_{n-1})\bigr\| \\
&\le \alpha_n\gamma\rho\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|M_1 + (1 - \alpha_n\overline{\gamma})\|y_n - y_{n-1}\|,
\end{aligned} \tag{3.4}$$

where $M_1 = \sup_{n\ge1}\{\gamma\|f(x_n)\| + \|By_n\|\} < \infty$. Moreover, we note that $y_n = S_nu_n$ and

$$\begin{aligned}
\|y_n - y_{n-1}\| &\le \|S_nu_n - S_nu_{n-1}\| + \|S_nu_{n-1} - S_{n-1}u_{n-1}\| \\
&\le \|u_n - u_{n-1}\| + \bigl\|\beta_nu_{n-1} + (1-\beta_n)W_nu_{n-1} \\
&\quad - \bigl[\beta_{n-1}u_{n-1} + (1-\beta_{n-1})W_{n-1}u_{n-1}\bigr]\bigr\| \\
&\le \|u_n - u_{n-1}\| + |\beta_n - \beta_{n-1}|\|u_{n-1} - W_{n-1}u_{n-1}\| \\
&\quad + (1-\beta_n)\|W_nu_{n-1} - W_{n-1}u_{n-1}\| \\
&\le \|u_n - u_{n-1}\| + |\beta_n - \beta_{n-1}|M_2 + (1-\beta_n)\sum_{i=1}^{N}\bigl|\eta_i^{(n)} - \eta_i^{(n-1)}\bigr|\|T_iu_{n-1}\|,
\end{aligned} \tag{3.5}$$

where $M_2 = \sup_{n\ge1}\{\|u_{n-1} - W_{n-1}u_{n-1}\|\}$. On the other hand, we note that

$$\begin{cases}
F(u_n, y) + \langle Ax_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0, \\
F(u_{n-1}, y) + \langle Ax_{n-1}, y - u_{n-1}\rangle + \frac{1}{r_{n-1}}\langle y - u_{n-1}, u_{n-1} - x_{n-1}\rangle \ge 0.
\end{cases} \tag{3.6}$$

Putting $y = u_{n-1}$ and $y = u_n$ in (3.6) respectively, we have

$$\begin{cases}
F(u_n, u_{n-1}) + \langle Ax_n, u_{n-1} - u_n\rangle + \frac{1}{r_n}\langle u_{n-1} - u_n, u_n - x_n\rangle \ge 0, \\
F(u_{n-1}, u_n) + \langle Ax_{n-1}, u_n - u_{n-1}\rangle + \frac{1}{r_{n-1}}\langle u_n - u_{n-1}, u_{n-1} - x_{n-1}\rangle \ge 0.
\end{cases} \tag{3.7}$$

It follows from (A2) that

$$\left\langle u_n - u_{n-1}, \frac{u_{n-1} - (I - r_{n-1}A)x_{n-1}}{r_{n-1}} - \frac{u_n - (I - r_nA)x_n}{r_n} \right\rangle \geq 0,$$

and hence

$$\left\langle u_n - u_{n-1}, u_{n-1} - u_n + u_n - (I - r_{n-1}A)x_{n-1} - \frac{r_{n-1}}{r_n}\big[u_n - (I - r_nA)x_n\big] \right\rangle \geq 0.$$

Since $\lim_{n\to\infty} r_n > 0$, we assume that there exists a real number μ such that $r_n > \mu > 0$ for all $n \in N$. Consequently, we have

$$\begin{aligned} \|u_n - u_{n-1}\|^2 &\leq \Big\langle u_n - u_{n-1}, (I - r_nA)x_n - (I - r_{n-1}A)x_{n-1} \\ &\quad + \left(1 - \frac{r_{n-1}}{r_n}\right)\big[u_n - (I - r_nA)x_n\big] \Big\rangle \\ &\leq \|u_n - u_{n-1}\| \bigg[\|x_n - x_{n-1}\| + |r_n - r_{n-1}| \|Ax_{n-1}\| \\ &\quad + \frac{r_n - r_{n-1}}{r_n} \big\| u_n - (I - r_nA)x_n \big\| \bigg], \end{aligned}$$

and hence

$$\begin{aligned} \|u_n - u_{n-1}\| &\leq \|x_n - x_{n-1}\| + |r_n - r_{n-1}| \|Ax_{n-1}\| + \frac{r_n - r_{n-1}}{r_n} \big\| u_n - (I - r_nA)x_n \big\| \\ &\leq \|x_n - x_{n-1}\| + |r_n - r_{n-1}| \left[\|Ax_{n-1}\| + \frac{1}{\mu} \big\| u_n - (I - r_nA)x_n \big\| \right] \\ &\leq \|x_n - x_{n-1}\| + |r_n - r_{n-1}| M_3, \end{aligned} \tag{3.8}$$

where $M_3 = \sup\{\|Ax_{n-1}\| + \frac{1}{\mu}\|u_n - (I - r_nA)x_n\|, n \in N\}$. Combining (3.4), (3.5) and (3.8), we have

$$\begin{aligned} \|x_{n+1} - x_n\| &\leq \alpha_n \gamma \rho \|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}| M_1 + (1 - \alpha_n \overline{\gamma}) \bigg[\|x_n - x_{n-1}\| \\ &\quad + |\beta_n - \beta_{n-1}| M_2 + |r_n - r_{n-1}| M_3 + (1 - \beta_n) \sum_{i=1}^{N} \big|\eta_i^{(n)} - \eta_i^{(n-1)}\big| \|T_i u_{n-1}\| \bigg] \\ &\leq \big[1 - (\overline{\gamma} - \gamma\rho)\alpha_n\big] \|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}| M_1 + |\beta_n - \beta_{n-1}| M_2 \\ &\quad + |r_n - r_{n-1}| M_3 + \sum_{i=1}^{N} \big|\eta_i^{(n)} - \eta_i^{(n-1)}\big| \|T_i u_{n-1}\|. \end{aligned}$$

It follows from $0 < \gamma < \frac{\overline{\gamma}}{\rho}$ and Lemma 2.6 that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.9}$$

Moreover, we observe that

$$\|x_n - y_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| \leq \|x_n - x_{n+1}\| + \alpha_n \big\| \gamma f(x_n) - By_n \big\|.$$

It follows from $\lim_{n\to\infty}\alpha_n = 0$ and (3.9) that

$$\lim_{n\to\infty}\|x_n - y_n\| = 0. \tag{3.10}$$

For $p \in F(S_n)\cap EP(F,A)$, we note that $u_n = T_{r_n}(I - r_nA)x_n$ and

$$\begin{aligned}\|u_n - p\|^2 &= \bigl\|T_{r_n}(I - r_nA)x_n - T_{r_n}(I - r_nA)p\bigr\|^2 \le \langle x_n - p, u_n - p\rangle \\ &= \frac{1}{2}\bigl(\|x_n - p\|^2 + \|u_n - p\|^2 - \|x_n - u_n\|^2\bigr),\end{aligned}$$

which implies that

$$\|u_n - p\|^2 \le \|x_n - p\|^2 - \|x_n - u_n\|^2. \tag{3.11}$$

From (1.4), (3.2) and (3.11), we have

$$\begin{aligned}\|x_{n+1} - p\|^2 &= \bigl\|\alpha_n\bigl[\gamma f(x_n) - Bp\bigr] + (I - \alpha_nB)(y_n - p)\bigr\|^2 \\ &\le (1 - \alpha_n\overline{\gamma})^2\|y_n - p\|^2 + \alpha_n^2\bigl\|\gamma f(x_n) - Bp\bigr\|^2 \\ &\quad + 2\alpha_n(1 - \alpha_n\overline{\gamma})\bigl\|\gamma f(x_n) - Bp\bigr\|\|y_n - p\| \\ &\le \|u_n - p\|^2 + \alpha_n^2\bigl\|\gamma f(x_n) - Bp\bigr\|^2 + 2\alpha_n\bigl\|\gamma f(x_n) - Bp\bigr\|\|y_n - p\| \\ &\le \|x_n - p\|^2 - \|x_n - u_n\|^2 + \alpha_n^2\bigl\|\gamma f(x_n) - Bp\bigr\|^2 \\ &\quad + 2\alpha_n\bigl\|\gamma f(x_n) - Bp\bigr\|\|y_n - p\|.\end{aligned}$$

Using $\lim_{n\to\infty}\alpha_n = 0$ and (3.9) again, we obtain

$$\lim_{n\to\infty}\|x_n - u_n\| = \lim_{n\to\infty}\bigl\|x_n - T_{r_n}(I - r_nA)x_n\bigr\| = 0. \tag{3.12}$$

By the nonexpansion of S_n, we have

$$\begin{aligned}\|x_n - S_nx_n\| &\le \|x_n - x_{n+1}\| + \|x_{n+1} - S_nx_n\| \\ &\le \|x_n - x_{n+1}\| + \bigl\|\alpha_n\gamma f(x_n) + (I - \alpha_nB)y_n - S_nx_n\bigr\| \\ &\le \|x_n - x_{n+1}\| + \alpha_n\bigl[\bigl\|\gamma f(x_n)\bigr\| + \|By_n\|\bigr] + \|S_nu_n - S_nx_n\| \\ &\le \|x_n - x_{n+1}\| + \alpha_n\bigl[\bigl\|\gamma f(x_n)\bigr\| + \|By_n\|\bigr] + \|u_n - x_n\|.\end{aligned}$$

This together with (3.9) and (3.12), we obtain

$$\lim_{n\to\infty}\|x_n - S_nx_n\| = 0. \tag{3.13}$$

Furthermore, we note that

$$\|x_n - S_nx_n\| = (1 - \beta_n)\|x_n - W_nx_n\|.$$

It follows from condition (ii) that

$$\lim_{n\to\infty}\|x_n - W_nx_n\| = 0. \tag{3.14}$$

On the other hand, by condition (iii), we may assume that $\eta_i^{(n)} \to \eta_i$ as $n \to \infty$ for every $1 \le i \le N$. It is easily seen that each $\eta_i > 0$ and $\sum_{i=1}^N \eta_i = 1$. Define $W = \sum_{i=1}^N \eta_i T_i$, then $W : K \to H$ is a k-strict pseudo-contraction such that $F(W) = F(W_n) = \bigcap_{i=1}^N F(T_i)$ by Proposition 2.1 and 2.2. Consequently,

$$\begin{aligned}\|x_n - Wx_n\| &\le \|x_n - W_n x_n\| + \|W_n x_n - Wx_n\| \\ &\le \|x_n - W_n x_n\| + \sum_{i=1}^N \bigl|\eta_i^{(n)} - \eta_i\bigr| \|T_i x_n\|,\end{aligned}$$

which implies that

$$\lim_{n\to\infty} \|x_n - Wx_n\| = 0. \tag{3.15}$$

Combining (3.14) and (3.15), we obtain

$$\lim_{n\to\infty} \|W_n x_n - Wx_n\| = 0. \tag{3.16}$$

Define $S : K \to H$ by $Sx = \lambda x + (1 - \lambda)Wx$. By condition (ii) again, we have $\lim_{n\to\infty} \beta_n = \lambda \in [k, 1)$. Then, S is nonexpansive with $F(S) = F(W)$ by Lemma 2.5. Notice that

$$\begin{aligned}\|x_n - Sx_n\| &\le \|x_n - S_n x_n\| + \|S_n x_n - Sx_n\| \\ &= \|x_n - S_n x_n\| + \bigl\|\beta_n x_n + (1 - \beta_n)W_n x_n - \lambda x_n - (1 - \lambda)Wx_n\bigr\| \\ &\le \|x_n - S_n x_n\| + |\beta_n - \lambda|\|x_n - Wx_n\| + (1 - \beta_n)\|W_n x_n - Wx_n\|.\end{aligned}$$

It follows from (3.13), (3.15) and (3.16) that

$$\lim_{n\to\infty} \|x_n - Sx_n\| = 0. \tag{3.17}$$

Now we claim that $\limsup_{n\to\infty} \langle (B - \gamma f)q, q - x_n\rangle \le 0$, where $q = \lim_{t\to 0} x_t$ with x_t being the fixed point of the contraction Ψ_n on H defined by

$$\Psi_n x = t\gamma f(x) + (I - tB)S_n T_{r_n}(I - r_n A)x, \quad \forall x \in H, n \in N,$$

where $t \in (0, 1)$. Indeed, by Lemma 2.1 and 2.3, we have

$$\begin{aligned}\|\Psi_n x - \Psi_n y\| &\le t\gamma \bigl\|f(x) - f(y)\bigr\| + (1 - t\overline{\gamma})\bigl\|S_n T_{r_n}(I - r_n A)x - S_n T_{r_n}(I - r_n A)y\bigr\| \\ &\le t\gamma\rho\|x - y\| + (1 - t\overline{\gamma})\bigl\|T_{r_n}(I - r_n A)x - T_{r_n}(I - r_n A)y\bigr\| \\ &\le t\gamma\rho\|x - y\| + (1 - t\overline{\gamma})\|x - y\| \\ &= \bigl[1 - (\overline{\gamma} - \gamma\rho)t\bigr]\|x - y\|,\end{aligned}$$

for all $x, y \in H$. Since $0 < 1 - (\overline{\gamma} - \gamma\rho)t < 1$, it follows that Ψ_n is a contraction. Therefore, by the Banach contraction principle, Ψ_n has a unique fixed point $x_t \in H$ such that

$$x_t = t\gamma f(x_t) + (I - tB)S_n T_{r_n}(I - r_n A)x_t.$$

By Lemma 2.2 and (3.10), we have

$$\begin{aligned}\|x_t - x_n\|^2 &= \big\|(I - tB)\big[S_n T_{r_n}(I - r_n A)x_t - x_n\big] + t\big[\gamma f(x_t) - Bx_n\big]\big\|^2 \\ &\le (1 - \overline{\gamma} t)^2 \big\|S_n T_{r_n}(I - r_n A)x_t - x_n\big\|^2 + 2t\big\langle \gamma f(x_t) - Bx_n, x_t - x_n\big\rangle \\ &= (1 - \overline{\gamma} t)^2 \big\|S_n T_{r_n}(I - r_n A)x_t - S_n T_{r_n}(I - r_n A)x_n + S_n T_{r_n}(I - r_n A)x_n - x_n\big\|^2 \\ &\quad + 2t\big\langle \gamma f(x_t) - Bx_n, x_t - x_n\big\rangle \\ &\le (1 - \overline{\gamma} t)^2 \big[\|x_t - x_n\| + \|y_n - x_n\|\big]^2 + 2t\big\langle \gamma f(x_t) - Bx_n, x_t - x_n\big\rangle \\ &\le (1 - \overline{\gamma} t)^2 \|x_t - x_n\|^2 + \psi_n(t) + 2t\big\langle \gamma f(x_t) - Bx_t, x_t - x_n\big\rangle \\ &\quad + 2t\langle Bx_t - Bx_n, x_t - x_n\rangle, \end{aligned} \tag{3.18}$$

where $\psi_n(t) = (1 - \overline{\gamma} t)^2 (2\|x_t - x_n\| + \|y_n - x_n\|)\|y_n - x_n\| \to 0$ as $n \to \infty$. Observe B is strongly positive, we obtain

$$\langle Bx_t - Bx_n, x_t - x_n\rangle = \big\langle B(x_t - x_n), x_t - x_n\big\rangle \ge \overline{\gamma}\|x_t - x_n\|^2. \tag{3.19}$$

Combining (3.18) and (3.19), we have

$$\begin{aligned} 2t\big\langle Bx_t - \gamma f(x_t), x_t - x_n\big\rangle &\le \big(\overline{\gamma}^2 t^2 - 2\overline{\gamma} t\big)\|x_t - x_n\|^2 + \psi_n(t) + 2t\langle Bx_t - Bx_n, x_t - x_n\rangle \\ &\le \big(\overline{\gamma} t^2 - 2t\big)\big\langle B(x_t - x_n), x_t - x_n\big\rangle + \psi_n(t) \\ &\quad + 2t\langle Bx_t - Bx_n, x_t - x_n\rangle \\ &= \overline{\gamma} t^2 \langle Bx_t - Bx_n, x_t - x_n\rangle + \psi_n(t). \end{aligned}$$

It follows that

$$\big\langle Bx_t - \gamma f(x_t), x_t - x_n\big\rangle \le \frac{\overline{\gamma} t}{2}\langle Bx_t - Bx_n, x_t - x_n\rangle + \frac{1}{2t}\psi_n(t). \tag{3.20}$$

Let $n \to \infty$ in (3.20) and note that $\psi_n(t) \to 0$ as $n \to \infty$ yields

$$\limsup_{n\to\infty}\big\langle Bx_t - \gamma f(x_t), x_t - x_n\big\rangle \le \frac{t}{2}M_4, \tag{3.21}$$

where M_4 is an appropriate positive constant such that $M_4 \ge \overline{\gamma}\langle Bx_t - Bx_n, x_t - x_n\rangle$ for all $t \in (0,1)$ and $n \ge 1$. Taking $t \to 0$ from (3.21), we have

$$\limsup_{t\to 0}\limsup_{n\to\infty}\big\langle Bx_t - \gamma f(x_t), x_t - x_n\big\rangle \le 0. \tag{3.22}$$

On the other hand, we have

$$\begin{aligned} \big\langle \gamma f(q) - Bq, x_n - q\big\rangle &= \big\langle \gamma f(q) - Bq, x_n - q\big\rangle - \big\langle \gamma f(q) - Bq, x_n - x_t\big\rangle + \big\langle \gamma f(q) - Bq, x_n - x_t\big\rangle \\ &\quad - \big\langle \gamma f(q) - Bx_t, x_n - x_t\big\rangle + \big\langle \gamma f(q) - Bx_t, x_n - x_t\big\rangle \\ &\quad - \big\langle \gamma f(x_t) - Bx_t, x_n - x_t\big\rangle + \big\langle \gamma f(x_t) - Bx_t, x_n - x_t\big\rangle. \end{aligned}$$

It follows that

$$\begin{aligned}
&\limsup_{n\to\infty}\langle \gamma f(q) - Bq, x_n - q\rangle \\
&\quad\le \big\|\gamma f(q) - Bq\big\| \|x_t - q\| + \|B\| \|x_t - q\| \lim_{n\to\infty} \|x_n - x_t\| \\
&\qquad + \gamma\rho \|x_t - q\| \lim_{n\to\infty} \|x_n - x_t\| + \limsup_{n\to\infty}\langle \gamma f(x_t) - Bx_t, x_n - x_t\rangle.
\end{aligned}$$

Therefore, from (3.22) and $\lim_{t\to 0} x_t = q$, we have

$$\begin{aligned}
\limsup_{n\to\infty}\langle \gamma f(q) - Bq, x_n - q\rangle &= \limsup_{t\to 0}\limsup_{n\to\infty}\langle \gamma f(q) - Bq, x_n - q\rangle \\
&\le \limsup_{t\to 0}\big\|\gamma f(q) - Bq\big\| \|x_t - q\| \\
&\quad + \limsup_{t\to 0} \|B\| \|x_t - q\| \lim_{n\to\infty} \|x_n - x_t\| \\
&\quad + \limsup_{t\to 0} \gamma\rho \|x_t - q\| \lim_{n\to\infty} \|x_n - x_t\| \\
&\quad + \limsup_{t\to 0}\limsup_{n\to\infty}\langle \gamma f(x_t) - Bx_t, x_n - x_t\rangle \\
&\le 0.
\end{aligned} \tag{3.23}$$

Finally, we prove that $x_n \to q$ as $n \to \infty$. From (1.4) and (3.2) again, we have

$$\begin{aligned}
\|x_{n+1} - q\|^2 &= \big\langle \alpha_n\gamma f(x_n) + (I - \alpha_n B)y_n - q, x_{n+1} - q\big\rangle \\
&= \alpha_n\big\langle \gamma f(x_n) - Bq, x_{n+1} - q\big\rangle + \big\langle (I - \alpha_n B)(y_n - q), x_{n+1} - q\big\rangle \\
&\le \alpha_n\gamma\big\langle f(x_n) - f(q), x_{n+1} - q\big\rangle + \alpha_n\big\langle \gamma f(q) - Bq, x_{n+1} - q\big\rangle \\
&\quad + (1 - \alpha_n\overline{\gamma})\|y_n - q\| \|x_{n+1} - q\| \\
&\le \alpha_n\gamma\rho\|x_n - q\| \|x_{n+1} - q\| + \alpha_n\big\langle \gamma f(q) - Bq, x_{n+1} - q\big\rangle \\
&\quad + (1 - \alpha_n\overline{\gamma})\|x_n - q\| \|x_{n+1} - q\| \\
&= \big[1 - (\overline{\gamma} - \gamma\rho)\alpha_n\big]\|x_n - q\| \|x_{n+1} - q\| + \alpha_n\big\langle \gamma f(q) - Bq, x_{n+1} - q\big\rangle \\
&\le \frac{1 - (\overline{\gamma} - \gamma\rho)\alpha_n}{2}\big(\|x_n - q\|^2 + \|x_{n+1} - q\|^2\big) + \alpha_n\big\langle \gamma f(q) - Bq, x_{n+1} - q\big\rangle \\
&\le \frac{1 - (\overline{\gamma} - \gamma\rho)\alpha_n}{2}\|x_n - q\|^2 + \frac{1}{2}\|x_{n+1} - q\|^2 + \alpha_n\big\langle \gamma f(q) - Bq, x_{n+1} - q\big\rangle.
\end{aligned}$$

It follows that

$$\|x_{n+1} - q\|^2 \le \big[1 - (\overline{\gamma} - \gamma\rho)\alpha_n\big]\|x_n - q\|^2 + 2\alpha_n\big\langle \gamma f(q) - Bq, x_{n+1} - q\big\rangle. \tag{3.24}$$

From $0 < \gamma < \frac{\overline{\gamma}}{\rho}$, condition (i) and (3.23), we can arrive at the desired conclusion $\lim_{n\to\infty} \|x_n - q\| = 0$ by Lemma 2.6. This completes the proof. □

Theorem 3.2 *Let K be a nonempty closed convex subset of the Hilbert space H and $F : K \times K \to \mathbb{R}$ be a bi-function satisfying (A1)-(A4). Let A be an α-inverse strongly monotone mapping, $f \in \Pi_K$ with a coefficient $\rho \in (0,1)$ and B be a strongly positive bounded linear operator on H with $\overline{\gamma} > 0$ and $0 < \gamma < \frac{\overline{\gamma}}{\rho}$. Let $T : K \to H$ be a k-strict pseudo-contraction* *such that $\mathcal{F} = F(T) \cap EP(F,A) \neq \phi$. Let $\{x_n\}$ be a sequence generated by $x_0 \in K$ in the following manner:*

$$\begin{cases} F(u_n,y) + \langle Ax_n, y-u_n\rangle + \frac{1}{r}\langle y-u_n, u_n-x_n\rangle \geq 0, \quad \forall y \in K, \\ y_n = \beta_n u_n + (1-\beta_n)Tu_n, \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I-\alpha_n B)y_n, \quad n \geq 1, \end{cases}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $(0,1)$, constant $r \in (0,2\alpha)$. If the following control conditions are satisfied:

(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ *and* $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$;

(ii) $k \leq \beta_n \leq \lambda < 1$, $\lim_{n\to\infty} \beta_n = \lambda$ *and* $\sum_{n=1}^{\infty} |\beta_n - \beta_{n-1}| < \infty$.

Then the sequence $\{x_n\}$ converges strongly to $q \in \mathcal{F}$, which solves the variational inequality

$$\big\langle (B-\gamma f)q, q-p\big\rangle \leq 0, \quad \forall p \in \mathcal{F}.$$

Proof Putting $r_n = r$ and $N = 1$, *i.e.*, $W_n = T$, the desired conclusion follows immediately from Theorem 3.1. This completes the proof. □

Theorem 3.3 *Let K be a nonempty closed convex subset of the Hilbert space H and $F: K \times K \to \mathbb{R}$ be a bi-function satisfying (A1)-(A4). Let $f \in \Pi_K$ with a coefficient $\rho \in (0,1)$ and B be a strongly positive bounded linear operator on H with $\overline{\gamma} > 0$ and $0 < \gamma < \frac{\overline{\gamma}}{\rho}$. Let $T: K \to H$ be a k-strict pseudo-contraction such that $\mathcal{F} = F(T) \cap EP(F) \neq \phi$. Let $\{x_n\}$ be a sequence generated by $x_0 \in K$ in the following manner:*

$$\begin{cases} F(u_n,y) + \frac{1}{r_n}\langle y-u_n, u_n-x_n\rangle \geq 0, \quad \forall y \in K, \\ y_n = \beta_n u_n + (1-\beta_n)Tu_n, \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I-\alpha_n B)y_n, \quad n \geq 1, \end{cases}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $(0,1)$, sequence $\{r_n\} \subset (0,2\alpha)$. If the following control conditions are satisfied:

(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ *and* $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$;

(ii) $k \leq \beta_n \leq \lambda < 1$, $\lim_{n\to\infty} \beta_n = \lambda$ *and* $\sum_{n=1}^{\infty} |\beta_n - \beta_{n-1}| < \infty$;

(iii) $\liminf_{n\to\infty} r_n > 0$ *and* $\sum_{n=1}^{\infty} |r_n - r_{n-1}| < \infty$.

Then the sequence $\{x_n\}$ converges strongly to $q \in \mathcal{F}$, which solves the variational inequality

$$\big\langle (B-\gamma f)q, q-p\big\rangle \leq 0, \quad \forall p \in \mathcal{F}.$$

Proof Putting $N = 1$ and $A = 0$, *i.e.*, the generalized equilibrium problem (1.1) reduces to the normal equilibrium problem (1.2), the desired conclusion follows immediately from Theorem 3.1. This completes the proof. □

Remark 3.1 Theorem 3.1 and 3.2 improve and extend the main results of Takahashi and Takahashi [18] and Qin *et al.* [21] in different directions.

Remark 3.2 Theorem 3.3 is mainly due to Liu [16], which improves and extends the main results of Takahashi and Takahashi [12].

Remark 3.3 If $F = A = 0$ and $u_n = x_n$, then the algorithm (1.4) reduces to approximate the fixed point of k-strict pseudo-contractions, which includes the general iterative method of Marino and Xu [8] and the parallel algorithm of Acedo and Xu [20] as special cases.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Wen, DJ carried out the primary studies for the generalized equilibrium problems and fixed point problems of k-strict pseudo-contractions, participated in the design of iterative methods and drafted the manuscript. Chen YA participated in the convergence analysis and coordination. All authors read and approved the final manuscript.

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