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Convergence results for the zero-finding problem and fixed points of nonexpansive semigroups and strict pseudocontractions

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Abstract
In this work, we establish strong convergence theorems for solving the fixed point problem of nonexpansive semigroups and strict pseudocontractions, and the zero-finding problem of maximal monotone operators in a Hilbert space. We further apply our result to the convex minimization problem and commutative semigroups.
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1 Introduction

Let H be a real Hilbert space and K a nonempty, closed, and convex subset of H. Let $T : K \to K$ be a nonlinear mapping. Then T is said to be *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in K$. The fixed points set of T is denoted by $F(T)$.

In 1953, Mann [21] introduced the following classical iteration for a nonexpansive mapping $T : K \to K$ in a real Hilbert space: $x_1 \in K$ and

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad n \geq 1, \tag{1.1}$$

where $\{\alpha_n\} \subset (0, 1)$.

In 1967, Halpern [13] introduced another classical iteration for a nonexpansive mapping $T : K \to K$ in a real Hilbert space: $x_1 \in K$ and

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)Tx_n, \quad n \geq 1,$$

where $\{\alpha_n\} \subset (0, 1)$ and $u \in K$ is fixed.

Let $f : K \to K$ be a contraction (*i.e.*, $\|f(x) - f(y)\| \leq \alpha\|x - y\|$ for all $x, y \in K$ and $\alpha \in [0, 1)$). In 2000, Moudafi [25] introduced the viscosity approximation method for a nonexpansive mapping T as follows: $x_1 \in K$ and

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Tx_n, \quad n \geq 1, \tag{1.2}$$

where $\{\alpha_n\} \subset (0, 1)$. It was proved, in a Hilbert space that the sequence $\{x_n\}$ generated by (1.2) strongly converges to a fixed point of T under suitable conditions.

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Let A be a strongly positive bounded linear operator on H: that is, there is a constant $\bar{\gamma}$ with property

$$\langle Ax, x\rangle \geq \bar{\gamma}\|x\|^2 \quad \forall x \in H.$$

A typical problem is to minimize a quadratic function over the set of the fixed points of a nonexpansive mapping on a real Hilbert space H:

$$\min_{x\in K}\langle Ax, x\rangle - \langle x, b\rangle,$$

where K is the fixed point set of a nonexpansive mapping T on H and b is a given point in H.

Recently, Marino-Xu [22] introduced the following general iterative method for a nonexpansive mapping T in a Hilbert space: $x_1 \in H$ and

$$x_{n+1} = \alpha_n\gamma f(x_n) + (I - \alpha_n A)Tx_n, \quad n \geq 1,$$

where $\{\alpha_n\} \subset (0,1)$, f is a contraction and A is a strongly positive bounded linear operator.

Since then, there have been a number of modified viscosity approximation methods for nonexpansive mappings or nonexpansive semigroups (see, for example, [6, 7, 9, 26, 32, 35, 38, 42, 43]).

Recall that $T : K \to K$ is called a *κ-strict pseudocontraction* if there exists a constant $0 \leq \kappa < 1$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \kappa\left\|(I - T)x - (I - T)y\right\|^2 \tag{1.3}$$

for all $x, y \in K$. It is known that (1.3) is equivalent to the following:

$$\langle Tx - Ty, x - y\rangle \leq \|x - y\|^2 - \frac{1-\kappa}{2}\left\|(I - T)x - (I - T)y\right\|^2$$

for all $x, y \in K$.

The class of strict pseudocontractions was introduced, in 1967, by Browder-Petryshyn [3]. The existence and weak convergence theorems were proved in a real Hilbert space by using Mann iterative algorithm (1.1) with a constant sequence $\alpha_n = \alpha$ for all $n \geq 1$. Recently, Marino-Xu [23] and Zhou [44] extended the results of Browder-Petryshyn [3] to Mann's iteration process (1.1). Since 1967, the study of fixed points for strict pseudocontractions has been investigated by many authors (see, *e.g.*, [1, 28]).

A set-valued mapping $M : H \to 2^H$ is called *monotone* if for all $x, y \in H$, $f \in M(x)$, and $g \in M(y)$ imply $\langle x - y, f - g\rangle \geq 0$. A monotone mapping M is *maximal* if its graph $G(M) := \{(f, x) \in H \times H : f \in M(x)\}$ of M is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping M is *maximal* if and only if for $(x, f) \in H \times H$, $\langle x - y, f - g\rangle \geq 0$ for all $(y, g) \in G(M)$ imply $f \in M(x)$. Let $J^M_\lambda = (I + \lambda M)^{-1}$, $\lambda > 0$ be *the resolvent* of M. It is well known that J^M_λ is single-valued and $D(J^M_\lambda) = H$ for any $\lambda > 0$. For each $\lambda > 0$, the *Yosida approximation* of M is defined by $A_\lambda = \frac{I-J^M_\lambda}{\lambda}$. We know that $(J^M_\lambda x, A_\lambda x) \in G(M)$ for all $\lambda > 0$ and $x \in H$.

A fundamental problem of monotone operators is that of finding an element x such that $0 \in Mx$. Such a problem is called *the zero-finding problem* (denoted by $M^{-1}(0)$ the set of solutions) and also includes many concrete examples, such as convex programming and monotone variational inequalities. It is known that if $g : H \to (-\infty, \infty]$ is a proper lower semicontinuous convex function, then ∂g is maximal monotone and the equation $0 \in \partial g(x)$ is reduced to $g(x) = \min\{g(y) : y \in H\}$ (see [29, 30]).

Initiated by Martinet [24], Rockafellar [30] introduced the following iterative scheme: $x_1 \in H$ and

$$x_{n+1} = J^M_{\lambda_n} x_n, \quad n \geq 1, \tag{1.4}$$

where $\{\lambda_n\} \subset (0, \infty)$ and M is a maximal monotone operator on H. Such an algorithm is called the *proximal point algorithm*. It was proved that the sequence $\{x_n\}$ generated by (1.4) converges weakly to an element in $M^{-1}(0)$ if $\liminf_{n\to\infty} \lambda_n > 0$.

The convergence of the zero-finding problem of monotone operators has been studied by many authors in several setting (see, for example, [8, 10, 14, 15, 27, 34]).

In this work, motivated by Lau *et al.* [16–20], Marino-Xu [22], and Saeidi [32], we introduce a new general iterative scheme for solving the fixed- point problem of a nonexpansive semigroup involving a strict pseudocontraction and the zero-finding problem of a maximal monotone operator in the framework of a Hilbert space. Some applications concerning the convex minimization problem and commutative semigroups are also presented.

2 Preliminaries and lemmas

In this section, we state some preliminaries and lemmas which will be used in the sequel.

Let S be a semigroup. We denote by $\ell^\infty(S)$ the Banach space of all bounded real-valued functionals on S with supremum norm. For each $s \in S$, we define the left and right translation operators $l(s)$ and $r(s)$ on $\ell^\infty(S)$ by

$$\big(l(s)f\big)(t) = f(st) \quad \text{and} \quad \big(r(s)f\big)(t) = f(ts)$$

for each $t \in S$ and $f \in \ell^\infty(S)$, respectively. Let X be a subspace of $\ell^\infty(S)$ containing 1. An element μ in the dual space X^* of X is said to be a *mean* on X if $\|\mu\| = \mu(1) = 1$. It is well known that μ is a mean on X if and only if

$$\inf_{s\in S} f(s) \leq \mu(f) \leq \sup_{s\in S} f(s)$$

for each $f \in X$. We often write $\mu_t(f(t))$ instead of $\mu(f)$ for $\mu \in X^*$ and $f \in X$.

Let X be a translation invariant subspace of $\ell^\infty(S)$ (*i.e.*, $l(s)X \subset X$ and $r(s)X \subset X$ for each $s \in S$) containing 1. Then a mean μ on X is said to be *left invariant* (resp. *right invariant*) if $\mu(l(s)f) = \mu(f)$ (resp. $\mu(r(s)f) = \mu(f)$) for each $s \in S$ and $f \in X$. A mean μ on X is said to be *invariant* if μ is both left and right invariant [16–18]. S is said to be *left* (resp. *right*) *amenable* if X has a left (resp. right) invariant mean. S is a amenable if S is left and right amenable. In this case, $\ell^\infty(S)$ also has an invariant mean. It is known that $\ell^\infty(S)$ is amenable when S is commutative semigroup or solvable group. However, the free group or semigroup of two generators is not left or right amenable (see [11, 20]). A net $\{\mu_\alpha\}$ of means on X is said to be *left regular* [11] if

$$\lim_\alpha \big\| l_s^* \mu_\alpha - \mu_\alpha \big\| = 0$$

for each $s \in S$, where l_s^* is the adjoint operator of l_s.

Let K be a nonempty, closed, and convex subset of H. A family $\mathcal{S} = \{T(s) : s \in S\}$ is called a nonexpansive semigroup on K if for each $s \in S$, the mapping $T(s) : K \to K$ is nonexpansive and $T(st) = T(s)T(t)$ for each $s, t \in S$. We denote by $F(\mathcal{S})$ the set of common fixed points of $\mathcal{S}$, *i.e.*,

$$F(\mathcal{S}) = \bigcap_{s\in S} F\big(T(s)\big) = \bigcap_{s\in S} \big\{x \in K : T(s)x = x\big\}.$$

Throughout this article, we denote the open ball of radius r centered at 0 by B_r and also denote the closed and convex hull of $A \subset H$ by $\overline{co}A$. For $\varepsilon > 0$ and a mapping $T : D \to H$, the set of ε-approximate fixed points of T will be denoted by $F_\varepsilon(T, D)$, *i.e.* $F_\varepsilon(T, D) = \{x \in D : \|x - Tx\| \le \varepsilon\}$.

The following lemmas are important in order to prove our main theorem.

Lemma 2.1 [20, 31, 39] *Let f be a function of a semigroup S into a Banach space E such that the weak closure of $\{f(t) : t \in S\}$ is weakly compact and let X be a subspace of $\ell^\infty(S)$ containing all the functions $t \mapsto \langle f(t), x^*\rangle$ with $x^* \in E^*$. Then, for any $\mu \in X^*$, there exists a unique element f_μ in E such that*

$$\langle f_\mu, x^*\rangle = \mu_t\langle f(t), x^*\rangle$$

for all $x^* \in E^*$. Moreover, if μ is a mean on X then

$$\int f(t)\, d\mu(t) \in \overline{co}\big\{f(t) : t \in S\big\}.$$

We can write f_μ by $\int f(t)\, d\mu(t)$.

Lemma 2.2 [20, 31, 39] *Let K be a closed and convex subset of a Hilbert space H, $\mathcal{S} = \{T(s) : s \in S\}$ be a nonexpansive semigroup from K into K such that $F(\mathcal{S}) \neq \emptyset$ and X be a subspace of $\ell^\infty(S)$ containing* 1 *and the mapping $t \mapsto \langle T(t)x, y\rangle$ be an element of X for each $x \in K$ and $y \in H$, and μ be a mean on X.*

If we write $T(\mu)x$ instead of $\int T_t x\, d\mu(t)$, then the following hold:

(i) $T(\mu)$ is a nonexpansive mapping from K into K;
(ii) $T(\mu)x = x$ for each $x \in F(\mathcal{S})$;
(iii) $T(\mu)x \in \overline{co}\{T_t x : t \in S\}$ for each $x \in K$;
(iv) if μ is left invariant, then $T(\mu)$ is a nonexpansive retraction from K onto $F(\mathcal{S})$.

Let K be a nonempty, closed, and convex subset of a real Hilbert space H. Then, for any $x \in H$, there exists a unique nearest point in K, denoted by $P_K x$, such that

$$\|x - P_K x\| \le \|x - y\|$$

for all $y \in K$. Such a projection P_K is called the *metric projection* of H onto K. We also know that for $x \in H$ and $z \in K$, $z = P_K x$ if and only if

$$\langle x - z, y - z\rangle \le 0, \quad \forall y \in K.$$

We know the following subdifferential inequality.

Lemma 2.3 *For all $x, y \in H$, there holds the inequality*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle.$$

Lemma 2.4 [22] *Let A be a strongly positive bounded linear operator on a Hilbert space H with coefficient $\bar{\gamma}$ and $0 < \rho \leq \|A\|^{-1}$. Then $\|I - \rho A\| \leq 1 - \rho\bar{\gamma}$.*

In the sequel, we need the following crucial lemmas.

Lemma 2.5 [41] *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\rho_n)a_n + \rho_n\delta_n, \quad n \geq 1,$$

where $\{\rho_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that
(a) $\sum_{n=1}^{\infty} \rho_n = \infty$;
(b) $\limsup_{n\to\infty} \delta_n \leq 0$ or $\sum_{n=1}^{\infty} |\rho_n\delta_n| < \infty$.
Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.6 [36] *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space E such that*

$$x_{n+1} = (1-\beta_n)y_n + \beta_n x_n, \quad \forall n \geq 1,$$

where $\{\beta_n\}$ is a real sequence in $(0,1)$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. If $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$, then $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

The following crucial results can be found in [1].

Lemma 2.7 [1] *Let K be a nonempty, closed, and convex subset of a real Hilbert space H and let $T : K \to K$ be a κ-strict pseudocontraction such that $F(T) \neq \emptyset$, then $I - T$ is demiclosed at zero, that is, for all sequence $\{x_n\} \subset K$ with $x_n \rightharpoonup y$ and $\|x_n - Tx_n\| \to 0$ it follows that $y = Ty$.*

Lemma 2.8 [1] *Let K be a nonempty, closed, and convex subset of a real Hilbert space H and let $T_i : K \to K$ $(i = 1, 2, \ldots, N)$ be a family of κ_i-strict pseudocontractions for some $0 \leq \kappa_i < 1$. Assume $\{\eta_i\}_{i=1}^N$ is a positive sequence such that $\sum_{i=1}^N \eta_i = 1$. Then $\sum_{i=1}^N \eta_i T_i$ is a κ-strict pseudocontraction with $\kappa = \max\{\kappa_i : 1 \leq i \leq N\}$. Moreover, if $\{T_i\}_{i=1}^N$ has a common fixed point, then $F(\sum_{i=1}^N \eta_i T_i) = \bigcap_{i=1}^N F(T_i)$.*

Lemma 2.9 [40] *Let the resolvent J_λ^M be defined by $J_\lambda^M = (I + \lambda M)^{-1}$, $\lambda > 0$. Then the following holds:*

$$\left\|J_s^M x - J_t^M x\right\| \leq \left|\frac{s-t}{t}\right| \left\|x - J_t^M x\right\|$$

for all $s, t > 0$ and $x \in H$.

3 Main result

In this section, we are now ready to prove our main theorem.

Theorem 3.1 *Let H be a real Hilbert space and $\mathcal{S} = \{T(t) : t \in S\}$ a nonexpansive semigroup on H. Let $M : H \to 2^H$ be a maximal monotone operator and $T : H \to H$ a κ-strict pseudocontraction such that $F := F(\mathcal{S}) \cap M^{-1}(0) \cap F(T) \neq \emptyset$. Let X be a left invariant subspace of $\ell^\infty(S)$ such that $1 \in X$, and the function $t \mapsto \langle T(t)x, y\rangle$ is an element of X for each $x, y \in H$. Let $\{\mu_n\}$ be a left regular sequence of means on X such that $\lim_{n\to\infty} \|\mu_{n+1} - \mu_n\| = 0$. Let f be an α-contraction on H and A a strongly positive bounded linear operator with coefficient $\bar{\gamma}$. Let β and γ be real numbers such that $0 < \beta < 1$ and $0 < \gamma < \bar{\gamma}/\alpha$. Let $\{x_n\}$ be generated by $x_1 \in H$ and*

$$\begin{cases} y_n = J_{\lambda_n}^M(\delta_n x_n + (1-\delta_n)Tx_n), \\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta x_n + ((1-\beta)I - \alpha_n A)T(\mu_n)y_n, \quad n \geq 1, \end{cases}$$

where $\{\alpha_n\} \subset (0,1)$, $\{\delta_n\} \subset (\kappa, 1)$ and $\{\lambda_n\} \subset (0,\infty)$ satisfying the conditions:

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^\infty \alpha_n = \infty$;

(C2) $\lim_{n\to\infty} |\delta_{n+1} - \delta_n| = 0$;

(C3) $\kappa < \liminf_{n\to\infty} \delta_n \leq \limsup_{n\to\infty} \delta_n < 1$;

(C4) $\liminf_{n\to\infty} \lambda_n > 0$ *and* $\lim_{n\to\infty} |\lambda_{n+1} - \lambda_n| = 0$.

Then $\{x_n\}$ converges strongly to $p \in F$ which also solves the following variational inequality:

$$\big\langle (\gamma f - A)p, q - p \big\rangle \leq 0, \quad \forall q \in F. \tag{3.1}$$

Proof Since $\alpha_n \to 0$, we shall assume that $\alpha_n \leq (1-\beta)\|A\|^{-1}$ and $1 - \alpha_n(\bar{\gamma} - \alpha\gamma) > 0$. So by Lemma 2.4, we have $\|(1-\beta)I - \alpha_n A\| \leq 1 - \beta - \alpha_n\bar{\gamma}$.

First, we show that $\{x_n\}$ is bounded. Let $w \in F$. Put $z_n = \delta_n x_n + (1-\delta_n)Tx_n$ for all $n \in \mathbb{N}$. Then

$$\begin{aligned} \|z_n - w\|^2 &= \big\|\delta_n x_n + (1-\delta_n)Tx_n - w\big\|^2 \\ &= \big\|\delta_n(x_n - w) + (1-\delta_n)(Tx_n - w)\big\|^2 \\ &= \delta_n\|x_n - w\|^2 + (1-\delta_n)\|Tx_n - w\|^2 - \delta_n(1-\delta_n)\|x_n - Tx_n\|^2 \\ &\leq \delta_n\|x_n - w\|^2 + (1-\delta_n)\|x_n - w\|^2 + (1-\delta_n)\kappa\|x_n - Tx_n\|^2 \\ &\quad - \delta_n(1-\delta_n)\|x_n - Tx_n\|^2 \\ &= \|x_n - w\|^2 + (1-\delta_n)(\kappa - \delta_n)\|x_n - Tx_n\|^2 \\ &\leq \|x_n - w\|^2, \end{aligned} \tag{3.2}$$

which yields

$$\|z_n - w\| \leq \|x_n - w\|.$$

Moreover, since $J_{\lambda_n}^M$ is firmly nonexpansive,

$$\|y_n - w\| = \big\|J_{\lambda_n}^M z_n - w\big\| \leq \|z_n - w\| \leq \|x_n - w\|. \tag{3.3}$$

From (3.3), we have

$$\begin{aligned}\|x_{n+1}-w\| &\le \big\|\big[(1-\beta)I-\alpha_n A\big]\big[T(\mu_n)y_n-w\big]\big\| + \big\|\alpha_n\gamma\big[f(x_n)-f(w)\big]\big\| \\ &\quad + \big\|\alpha_n\big[\gamma f(w)-Aw\big]\big\| + \big\|\beta(x_n-w)\big\| \\ &\le \big[1-\alpha_n(\bar{\gamma}-\alpha\gamma)\big]\|x_n-w\| + \alpha_n\big\|\gamma f(w)-Aw\big\| \\ &\le \max\left\{\|x_n-w\|, \frac{\|\gamma f(w)-Aw\|}{(\bar{\gamma}-\gamma\alpha)}\right\}.\end{aligned}$$

By an induction, we can show that

$$\|x_n-w\| \le \max\left\{\|x_1-w\|, \frac{\|\gamma f(w)-Aw\|}{(\bar{\gamma}-\gamma\alpha)}\right\}, \quad \forall n \ge 1.$$

Therefore, $\{x_n\}$ is bounded. So are $\{f(x_n)\}$, $\{y_n\}$, $\{z_n\}$, and $\{T(\mu_n)y_n\}$.

We next show that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0.$$

Observe that

$$\lim_{n\to\infty}\big\|T(\mu_{n+1})y_n - T(\mu_n)y_n\big\| = 0. \tag{3.4}$$

Indeed,

$$\begin{aligned}\big\|T(\mu_{n+1})y_n - T(\mu_n)y_n\big\| &= \sup_{\|z\|=1}\big|\big\langle T(\mu_{n+1})y_n - T(\mu_n)y_n, z\big\rangle\big| \\ &= \sup_{\|z\|=1}\big|(\mu_{n+1})_s\big\langle T(s)y_n, z\big\rangle - (\mu_n)_s\big\langle T(s)y_n, z\big\rangle\big| \\ &\le \|\mu_{n+1}-\mu_n\|\sup_{s\in S}\big\|T(s)y_n\big\|.\end{aligned}$$

Since $\{y_n\}$ is bounded and $\lim_{n\to\infty}\|\mu_{n+1}-\mu_n\| = 0$, (3.4) holds.

For each $n \in \mathbb{N}$, define $T_n x = \delta_n x + (1-\delta_n)Tx$. Then T_n is nonexpansive, and hence

$$\begin{aligned}\|z_{n+1}-z_n\| &= \|T_{n+1}x_{n+1} - T_n x_n\| \\ &\le \|T_{n+1}x_{n+1} - T_{n+1}x_n\| + \|T_{n+1}x_n - T_n x_n\| \\ &\le \|x_{n+1}-x_n\| + |\delta_{n+1}-\delta_n| M_1 \end{aligned}\tag{3.5}$$

for some big enough constant $M_1 > 0$.

On the other hand, since $y_n = J_{\lambda_n}^M z_n$ and $y_{n+1} = J_{\lambda_{n+1}}^M z_{n+1}$,

$$\begin{aligned}\|y_{n+1}-y_n\| &= \big\|J_{\lambda_{n+1}}^M z_{n+1} - J_{\lambda_n}^M z_n\big\| \\ &\le \big\|J_{\lambda_{n+1}}^M z_{n+1} - J_{\lambda_n}^M z_{n+1}\big\| + \big\|J_{\lambda_n}^M z_{n+1} - J_{\lambda_n}^M z_n\big\| \\ &\le \big\|J_{\lambda_{n+1}}^M z_{n+1} - J_{\lambda_n}^M z_{n+1}\big\| + \|z_{n+1}-z_n\|. \end{aligned}\tag{3.6}$$

Put $w_n = \frac{x_{n+1}-\beta x_n}{1-\beta}$. Then

$$\begin{aligned} w_{n+1} - w_n &= \frac{1}{1-\beta}\big[(x_{n+2} - \beta x_{n+1}) - (x_{n+1} - \beta x_n)\big] \\ &= \frac{1}{1-\beta}\big[\alpha_{n+1}\big(\gamma f(x_{n+1}) - AT(\mu_{n+1})y_{n+1}\big) + (1-\beta)T(\mu_{n+1})y_{n+1}\big] \\ &\quad - \frac{1}{1-\beta}\big[\alpha_n\big(\gamma f(x_n) - AT(\mu_n)y_n\big) + (1-\beta)T(\mu_n)y_n\big] \\ &= \frac{\alpha_{n+1}}{1-\beta}\big(\gamma f(x_{n+1}) - AT(\mu_{n+1})y_{n+1}\big) + T(\mu_{n+1})y_{n+1} \\ &\quad - \frac{\alpha_n}{1-\beta}\big(\gamma f(x_n) - AT(\mu_n)y_n\big) - T(\mu_n)y_n \\ &= \frac{\alpha_{n+1}}{1-\beta}\big(\gamma f(x_{n+1}) - AT(\mu_{n+1})y_{n+1}\big) + \big(T(\mu_{n+1})y_{n+1} - T(\mu_{n+1})y_n\big) \\ &\quad - \frac{\alpha_n}{1-\beta}\big(\gamma f(x_n) - AT(\mu_n)y_n\big) - \big(T(\mu_n)y_n - T(\mu_{n+1})y_n\big) \end{aligned}$$

which implies

$$\begin{aligned} \|w_{n+1} - w_n\| &\le \frac{\alpha_{n+1}}{1-\beta}\big\|\gamma f(x_{n+1}) - AT(\mu_{n+1})y_{n+1}\big\| + \|y_{n+1} - y_n\| \\ &\quad + \frac{\alpha_n}{1-\beta}\big\|\gamma f(x_n) - AT(\mu_n)y_n\big\| + \big\|T(\mu_n)y_n - T(\mu_{n+1})y_n\big\|. \end{aligned} \tag{3.7}$$

Substituting (3.5) and (3.6) into (3.7), we obtain

$$\begin{aligned} \|w_{n+1} - w_n\| &\le \frac{\alpha_{n+1}}{1-\beta}\big\|\gamma f(x_{n+1}) - AT(\mu_{n+1})y_{n+1}\big\| + \big\|J^M_{\lambda_{n+1}} z_{n+1} - J^M_{\lambda_n} z_{n+1}\big\| \\ &\quad + \|x_{n+1} - x_n\| + |\delta_{n+1} - \delta_n| M_1 + \frac{\alpha_n}{1-\beta}\big\|\gamma f(x_n) - AT(\mu_n)y_n\big\| \\ &\quad + \big\|T(\mu_n)y_n - T(\mu_{n+1})y_n\big\|. \end{aligned}$$

Using Lemma 2.9, (3.4), (C1), (C2), and (C4), we have

$$\limsup_{n\to\infty}\big(\|w_{n+1} - w_n\| - \|x_{n+1} - x_n\|\big) \le 0.$$

From Lemma 2.6, we derive

$$\lim_{n\to\infty} \|w_n - x_n\| = 0.$$

It also follows that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.8}$$

We next show that

$$\lim_{n\to\infty} \big\|x_n - T(t)x_n\big\| = 0, \quad \forall t \in S.$$

Put

$$K = \max\left\{\|x_1 - w\|, \frac{\|\gamma f(w) - Aw\|}{(\bar{\gamma} - \gamma\alpha)}\right\}.$$

Set $D = \{y \in H : \|y - w\| \leq K\}$. Then D is a nonempty bounded closed convex set. Moreover, $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ are in D. To complete our proof, we follow the proof line as in [2] (see also [19, 20, 33]). Let $\varepsilon > 0$. From [5], there exists $\delta > 0$ such that

$$\overline{co}F_\delta\big(T(t);D\big) + B_\delta \subseteq F_\varepsilon\big(T(t);D\big), \quad \forall t \in S. \tag{3.9}$$

From Corollary 1.1 in [5], there exists a natural number N such that

$$\left\|\frac{1}{N+1}\sum_{i=0}^{N} T\big(t^i s\big)y - T(t)\left(\frac{1}{N+1}\sum_{i=0}^{N} T\big(t^i s\big)y\right)\right\| \leq \delta, \tag{3.10}$$

for all $t, s \in S$ and $y \in D$. Let $t \in S$. Since $\{\mu_n\}$ is left regular, there exists $n_0 \in \mathbb{N}$ such that

$$\|\mu_n - l^*_{t^i}\mu_n\| \leq \frac{\delta}{3(K + \|w\|)}$$

for all $n \geq n_0$ and $i = 1, 2, \ldots, N$. So we have for all $n \geq n_0$

$$\begin{aligned}
&\sup_{y\in D}\left\|T(\mu_n)y - \int \frac{1}{N+1}\sum_{i=0}^{N} T\big(t^i s\big)y\,d\mu_n(s)\right\| \\
&\quad = \sup_{y\in D}\sup_{\|z\|=1}\left|(\mu_n)_s\langle T(s)y, z\rangle - (\mu_n)_s\left\langle \frac{1}{N+1}\sum_{i=0}^{N} T\big(t^i s\big)y, z\right\rangle\right| \\
&\quad \leq \frac{1}{N+1}\sum_{i=0}^{N}\sup_{y\in D}\sup_{\|z\|=1}\left|(\mu_n)_s\langle T(s)y, z\rangle - \big(l^*_{t^i}\mu_n\big)_s\langle T(s)y, z\rangle\right| \\
&\quad \leq \max_{i=1,2,\ldots,N}\|\mu_n - l^*_{t^i}\mu_n\|\big(K + \|w\|\big) \leq \frac{\delta}{3}.
\end{aligned} \tag{3.11}$$

Observe, by Lemma 2.2

$$\int \frac{1}{N+1}\sum_{i=0}^{N} T\big(t^i s\big)y\,d\mu_n(s) \in \overline{co}\left\{\frac{1}{N+1}\sum_{i=0}^{N} T(t)^i\big(T(s)y\big) : s \in S\right\}. \tag{3.12}$$

Combining (3.10)-(3.12), we derive

$$\begin{aligned}
T(\mu_n)y &= \int \frac{1}{N+1}\sum_{i=0}^{N} T\big(t^i s\big)y\,d\mu_n(s) + \left(T(\mu_n)y - \int \frac{1}{N+1}\sum_{i=0}^{N} T\big(t^i s\big)y\,d\mu_n(s)\right) \\
&\in \overline{co}\left\{\frac{1}{N+1}\sum_{i=0}^{N} T(t)^i\big(T(s)y\big) : s \in S\right\} + B_{\delta/3} \\
&\subseteq \overline{co}F_\delta\big(T(t);D\big) + B_{\delta/3},
\end{aligned} \tag{3.13}$$

for all $y \in D$ and $n \geq n_0$. Let $t \in S$ and $\varepsilon > 0$. Then there exists $\delta > 0$ which satisfies (3.9). Observe

$$x_{n+1} = T(\mu_n)y_n + \frac{\beta}{1-\beta}(x_{n+1} - x_n) + \frac{\alpha_n}{1-\beta}\big(\gamma f(x_n) - AT(\mu_n)y_n\big).$$

Since $\|x_{n+1} - x_n\| \to 0$ and $\alpha_n \to 0$, there exists $k \in \mathbb{N}$ such that

$$\begin{aligned} x_{n+1} &= T(\mu_n)y_n + \frac{\beta}{1-\beta}(x_n - x_{n+1}) + \frac{\alpha_n}{1-\beta}\big(\gamma f(x_n) - AT(\mu_n)y_n\big) \\ &\in \overline{co}F_\delta\big(T(t);D\big) + B_{\delta/3} + B_{\delta/3} + B_{\delta/3} \\ &\subseteq \overline{co}F_\delta\big(T(t);D\big) + B_\delta \subseteq F_\varepsilon\big(T(t);D\big), \end{aligned}$$

for all $n > k$. Hence, $\limsup_{n\to\infty} \|x_n - T(t)x_n\| \leq \varepsilon$. Since $\varepsilon > 0$ is arbitrary,

$$\lim_{n\to\infty} \big\|x_n - T(t)x_n\big\| = 0. \tag{3.14}$$

We next show that

$$\lim_{n\to\infty} \|y_n - z_n\| = 0. \tag{3.15}$$

Since $J^M_{\lambda_n}$ is firmly nonexpansive and $y_n = J^M_{\lambda_n} z_n$,

$$\begin{aligned} \|y_n - w\|^2 &= \big\|J^M_{\lambda_n} z_n - J^M_{\lambda_n} w\big\|^2 \\ &\leq \big\langle J^M_{\lambda_n} z_n - J^M_{\lambda_n} w, z_n - w\big\rangle \\ &= \langle y_n - w, z_n - w\rangle \\ &= \frac{1}{2}\big(\|y_n - w\|^2 + \|z_n - w\|^2 - \|z_n - y_n\|^2\big), \end{aligned}$$

which implies

$$\|y_n - w\|^2 \leq \|z_n - w\|^2 - \|z_n - y_n\|^2.$$

Therefore,

$$\begin{aligned} \|x_{n+1} - w\|^2 &= \big\|\big[(1-\beta)\big(T(\mu_n)y_n - w\big) + \beta(x_n - w)\big] + \alpha_n\big[\gamma f(x_n) - AT(\mu_n)y_n\big]\big\|^2 \\ &\leq \big\|(1-\beta)\big(T(\mu_n)y_n - w\big) + \beta(x_n - w)\big\|^2 \\ &\quad + 2\alpha_n\big\langle \gamma f(x_n) - AT(\mu_n)y_n, x_{n+1} - w\big\rangle \\ &\leq (1-\beta)\|y_n - w\|^2 + \beta\|x_n - w\|^2 \\ &\quad + 2\alpha_n\big\langle \gamma f(x_n) - AT(\mu_n)y_n, x_{n+1} - w\big\rangle \\ &\leq (1-\beta)\big(\|z_n - w\|^2 - \|z_n - y_n\|^2\big) + \beta\|x_n - w\|^2 \\ &\quad + 2\alpha_n\big\langle \gamma f(x_n) - AT(\mu_n)y_n, x_{n+1} - w\big\rangle \\ &\leq \|x_n - w\|^2 - (1-\beta)\|z_n - y_n\|^2 \\ &\quad + 2\alpha_n\big\langle \gamma f(x_n) - AT(\mu_n)y_n, x_{n+1} - w\big\rangle, \end{aligned}$$

which yields

$$(1-\beta)\|z_n - y_n\|^2 \le \alpha_n M_2 + \left(\|x_n - w\|^2 - \|x_{n+1} - w\|^2\right)$$

for some $M_2 > 0$. Thus, (3.15) holds by (3.8) and $\alpha_n \to 0$.

We next show that

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0. \tag{3.16}$$

From (3.2), we have

$$\|y_n - w\|^2 \le \|z_n - w\|^2 \le \|x_n - w\|^2 + (1-\delta_n)(\kappa - \delta_n)\|x_n - Tx_n\|^2.$$

So, we obtain

$$\begin{aligned}
\|x_{n+1} - w\|^2 &\le (1-\beta)\|y_n - w\|^2 + \beta\|x_n - w\|^2 \\
&\quad + 2\alpha_n\langle \gamma f(x_n) - AT(\mu_n)y_n, x_{n+1} - w\rangle \\
&\le (1-\beta)\left(\|x_n - w\|^2 + (1-\delta_n)(\kappa - \delta_n)\|x_n - Tx_n\|^2\right) + \beta\|x_n - w\|^2 \\
&\quad + 2\alpha_n\langle \gamma f(x_n) - AT(\mu_n)y_n, x_{n+1} - w\rangle \\
&\le \|x_n - w\|^2 + (1-\beta)(1-\delta_n)(\kappa - \delta_n)\|x_n - Tx_n\|^2 + \alpha_n M_2.
\end{aligned}$$

It follows that

$$(1-\beta)(1-\delta_n)(\delta_n - \kappa)\|x_n - Tx_n\|^2 \le \alpha_n M_2 + \|x_n - w\|^2 - \|x_{n+1} - w\|^2.$$

From (C1) and (C3), we conclude that (3.16) holds. Moreover, we get that

$$\lim_{n\to\infty} \|x_n - z_n\| = 0. \tag{3.17}$$

It is easy to see that $P_F(\gamma f + (I - A))$ is a contraction. So, by Banach's contraction principle, there exists a unique point p which satisfies the following variational inequality:

$$\langle (\gamma f - A)p, q - p\rangle \le 0, \quad \forall q \in F.$$

We next show that

$$\limsup_{n\to\infty} \langle (\gamma f - A)p, x_n - p\rangle \le 0.$$

To this end, we choose a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty} \langle (\gamma f - A)p, x_n - p\rangle = \lim_{k\to\infty} \langle (\gamma f - A)p, x_{n_k} - p\rangle.$$

Since $\{x_n\}$ is bounded and H is reflexive, there exists a point $z \in H$ such that $x_{n_k} \rightharpoonup z$. From (3.15) and (3.17), there exists a corresponding subsequence $\{y_{n_k}\}$ of $\{y_n\}$ (resp. $\{z_{n_k}\}$ of $\{z_n\}$) such that $y_{n_k} \rightharpoonup z$ (resp. $z_{n_k} \rightharpoonup z$).

We next show that $z \in M^{-1}(0)$. Since $y_n = J_{\lambda_n}^M z_n$,

$$\|A_{\lambda_n} z_n\| = \frac{1}{\lambda_n}\|y_n - z_n\|.$$

From (3.15) and $\liminf_{n\to\infty} \lambda_n > 0$, we have

$$\lim_{n\to\infty} \|A_{\lambda_n} z_n\| = 0. \tag{3.18}$$

Noting that $(z_n, A_{\lambda_n} z_n) \in G(M)$, by the monotonicity of M, we have

$$\langle s - z_n, s^* - A_{\lambda_n} z_n\rangle \geq 0$$

for all $(s, s^*) \in G(M)$. So we obtain

$$\langle s - z, s^*\rangle \geq 0$$

for all $(s, s^*) \in G(M)$. Hence, $z \in M^{-1}(0)$ by the maximality of M.

On the other hand, from (3.14), we get that $z \in F(\mathcal{S})$ by the demiclosedness of a nonexpansive mapping [4, 12]. Applying Lemma 2.7 to (3.16), we also get that $z \in F(T)$. This shows that $z \in F$, and hence

$$\limsup_{n\to\infty}\langle(\gamma f - A)p, x_n - p\rangle = \langle(\gamma f - A)p, z - p\rangle \leq 0. \tag{3.19}$$

We finally show that $x_n \to p$ as $n \to \infty$. From Lemmas 2.3 and 2.4, we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \big\|\big[\big((1-\beta)I - \alpha_n A\big)\big(T(\mu_n)y_n - p\big) + \beta(x_n - p)\big] + \alpha_n\big(\gamma f(x_n) - Ap\big)\big\|^2 \\
&\leq \big\|\big((1-\beta)I - \alpha_n A\big)\big(T(\mu_n)y_n - p\big) + \beta(x_n - p)\big\|^2 \\
&\quad + 2\alpha_n\langle\gamma f(x_n) - Ap, x_{n+1} - p\rangle \\
&= \left\|(1-\beta)\frac{(1-\beta)I - \alpha_n A}{(1-\beta)}\big(T(\mu_n)y_n - p\big) + \beta(x_n - p)\right\|^2 \\
&\quad + 2\alpha_n\gamma\langle f(x_n) - f(p), x_{n+1} - p\rangle + 2\alpha_n\langle\gamma f(p) - Ap, x_{n+1} - p\rangle \\
&\leq (1-\beta)\left\|\frac{(1-\beta)I - \alpha_n A}{(1-\beta)}\big(T(\mu_n)y_n - p\big)\right\|^2 + \beta\|x_n - p\|^2 \\
&\quad + 2\alpha_n\gamma\alpha\|x_n - p\|\|x_{n+1} - p\| + 2\alpha_n\langle\gamma f(p) - Ap, x_{n+1} - p\rangle \\
&\leq \frac{\|(1-\beta)I - \alpha_n A\|^2}{1-\beta}\big\|T(\mu_n)y_n - p\big\|^2 + \beta\|x_n - p\|^2 \\
&\quad + \alpha_n\gamma\alpha\big(\|x_n - p\|^2 + \|x_{n+1} - p\|^2\big) + 2\alpha_n\langle\gamma f(p) - Ap, x_{n+1} - p\rangle \\
&\leq \left(\frac{((1-\beta) - \bar{\gamma}\alpha_n)^2}{1-\beta} + \beta + \alpha_n\gamma\alpha\right)\|x_n - p\|^2 + \alpha_n\gamma\alpha\|x_{n+1} - p\|^2 \\
&\quad + 2\alpha_n\langle\gamma f(p) - Ap, x_{n+1} - p\rangle \\
&= \left(1 - (2\bar{\gamma} - \alpha\gamma)\alpha_n + \frac{\bar{\gamma}^2\alpha_n^2}{1-\beta}\right)\|x_n - p\|^2 + \alpha_n\gamma\alpha\|x_{n+1} - p\|^2 \\
&\quad + 2\alpha_n\langle\gamma f(p) - Ap, x_{n+1} - p\rangle.
\end{aligned}$$

It follows that

$$\begin{aligned}\|x_{n+1}-p\|^2 &\le \left(1-\frac{2\alpha_n(\bar{\gamma}-\alpha\gamma)}{1-\alpha\gamma\alpha_n}\right)\|x_n-p\|^2 \\ &\quad + \frac{2\alpha_n(\bar{\gamma}-\alpha\gamma)}{1-\alpha\gamma\alpha_n}\left(\frac{1}{\bar{\gamma}-\alpha\gamma}\langle\gamma f(p)-Ap, x_{n+1}-p\rangle\right. \\ &\quad \left.+ \frac{\bar{\gamma}^2\alpha_n}{2(1-\beta)(\bar{\gamma}-\alpha\gamma)}\|x_n-p\|^2\right).\end{aligned}$$

From (3.19) and (C1), we can apply Lemma 2.5 to conclude that $x_n \to p$ as $n \to \infty$. This completes the proof. □

From Rockafellar's theorem [29, 30], we next apply our result to the convex minimization problem in a Hilbert space.

Corollary 3.2 *Let H be a real Hilbert space and $\mathcal{S} = \{T(t) : t \in S\}$ a nonexpansive semigroup on H. Let $g : H \to (-\infty, \infty]$ be a proper lower semi-continuous convex function and $T : H \to H$ a κ-strict pseudocontraction such that $F := F(\mathcal{S}) \cap \partial g^{-1}(0) \cap F(T) \neq \emptyset$. Let X be a left invariant subspace of $\ell^\infty(S)$ such that $1 \in X$, and the function $t \mapsto \langle T(t)x, y\rangle$ is an element of X for each $x, y \in H$. Let $\{\mu_n\}$ be a left regular sequence of means on X such that $\lim_{n\to\infty} \|\mu_{n+1} - \mu_n\| = 0$. Let f be an α-contraction on H and A a strongly positive bounded linear operator with coefficient $\bar{\gamma}$. Let $\{\alpha_n\}$, β, γ, $\{\delta_n\}$ and $\{\lambda_n\}$ be as in Theorem 3.1. Then the sequence $\{x_n\}$ generated by $x_1 \in H$ and*

$$\begin{cases} z_n = \delta_n x_n + (1-\delta_n)Tx_n, \\ y_n = \operatorname{argmin}_{y\in H}\{g(y) + \frac{1}{2\lambda_n}\|z_n - y\|^2\}, \\ x_{n+1} = \alpha_n\gamma f(x_n) + \beta x_n + ((1-\beta)I - \alpha_n A)T(\mu_n)y_n, \quad n \ge 1, \end{cases}$$

converges strongly to $p \in F$ which also solves the variational inequality (3.1).

Using Lemma 2.8, we next apply our result to a finite family of strict pseudocontractions in a Hilbert space.

Corollary 3.3 *Let H be a real Hilbert space and $\mathcal{S} = \{T(t) : t \in S\}$ a nonexpansive semigroup on H. Let $M : H \to 2^H$ be a maximal monotone operator and $\{T_i\}_{i=1}^N : H \to H$ a family of κ_i-strict pseudocontractions such that $F := F(\mathcal{S}) \cap M^{-1}(0) \cap F(T_1) \cap \cdots \cap F(T_N) \neq \emptyset$. Let $\kappa = \max\{\kappa_i : 1 \le i \le N\}$. Let X be a left invariant subspace of $\ell^\infty(S)$ such that $1 \in X$, and the function $t \mapsto \langle T(t)x, y\rangle$ is an element of X for each $x, y \in H$. Let $\{\mu_n\}$ be a left regular sequence of means on X such that $\lim_{n\to\infty} \|\mu_{n+1} - \mu_n\| = 0$. Let f be an α-contraction on H and A a strongly positive bounded linear operator with coefficient $\bar{\gamma}$. Let $\{\alpha_n\}$, β, γ, $\{\delta_n\}$ and $\{\lambda_n\}$ be as in Theorem 3.1 and $\eta_i \in (0,1)$ with $\sum_{i=1}^N \eta_i = 1$. Then the sequence $\{x_n\}$ generated by $x_1 \in H$ and*

$$\begin{cases} y_n = J_{\lambda_n}^M(\delta_n x_n + (1-\delta_n)\sum_{i=1}^N \eta_i T_i x_n), \\ x_{n+1} = \alpha_n\gamma f(x_n) + \beta x_n + ((1-\beta)I - \alpha_n A)T(\mu_n)y_n, \quad n \ge 1, \end{cases}$$

converges strongly to $p \in F$ which also solves the variational inequality (3.1).

Using the results proved in [37] (see also [19]), we obtain the following corollaries.

Corollary 3.4 *Let H be a real Hilbert space. Let S_1 and S_2 be nonexpansive mappings on H with $S_1S_2 = S_2S_1$. Let $M : H \to 2^H$ be a maximal monotone operator and let $T : H \to H$ be a κ-strict pseudocontraction such that $F := F(S_1) \cap F(S_2) \cap M^{-1}(0) \cap F(T) \neq \emptyset$. Let f be an α-contraction on H and A a strongly positive bounded linear operator with coefficient $\bar{\gamma}$. Let $\{\alpha_n\}$, β, γ, $\{\delta_n\}$, and $\{\lambda_n\}$ be as in Theorem 3.1. Then the sequence $\{x_n\}$ generated by $x_1 \in H$ and*

$$\begin{cases} y_n = J^M_{\lambda_n}(\delta_n x_n + (1-\delta_n)Tx_n), \\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta x_n + ((1-\beta)I - \alpha_n A)(\frac{1}{n^2}\sum_{i=0}^{n-1}\sum_{j=0}^{n-1} S_1^i S_2^j y_n), \quad n \geq 1, \end{cases}$$

converges strongly to $p \in F$ which also solves the variational inequality (3.1).

Corollary 3.5 *Let H be a real Hilbert space. Let $\mathcal{S} = \{T(t) : t \in \mathbb{R}_+\}$ be a strongly continuous nonexpansive semigroup on H. Let $M : H \to 2^H$ be a maximal monotone operator and $T : H \to H$ a κ-strict pseudocontraction such that $F := F(\mathcal{S}) \cap M^{-1}(0) \cap F(T) \neq \emptyset$. Let f be an α-contraction on H and A a strongly positive bounded linear operator with coefficient $\bar{\gamma}$. Let $\{\alpha_n\}$, β, γ, $\{\delta_n\}$, and $\{\lambda_n\}$ be as in Theorem 3.1. Then the sequence $\{x_n\}$ generated by $x_1 \in H$ and*

$$\begin{cases} y_n = J^M_{\lambda_n}(\delta_n x_n + (1-\delta_n)Tx_n), \\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta x_n + ((1-\beta)I - \alpha_n A)(\frac{1}{t_n}\int_0^{t_n} T(s)y_n\,d(s)), \quad n \geq 1, \end{cases}$$

where $\{t_n\}$ is an increasing sequence in $(0,\infty)$ such that $\lim_{n\to\infty} t_n = \infty$ and $\lim_{n\to\infty} t_n/t_{n+1} = 1$, converges strongly to $p \in F$ which also solves the variational inequality (3.1).

Corollary 3.6 *Let H be a real Hilbert space. Let $\mathcal{S} = \{T(t) : t \in \mathbb{R}_+\}$ be a strongly continuous nonexpansive semigroup on H. Let $M : H \to 2^H$ be a maximal monotone operator and $T : H \to H$ a κ-strict pseudocontraction such that $F := F(\mathcal{S}) \cap M^{-1}(0) \cap F(T) \neq \emptyset$. Let f be an α-contraction on H and A a strongly positive bounded linear operator with coefficient $\bar{\gamma}$. Let $\{\alpha_n\}$, β, γ, $\{\delta_n\}$ and $\{\lambda_n\}$ be as in Theorem 3.1. Then the sequence $\{x_n\}$ generated by $x_1 \in H$ and*

$$\begin{cases} y_n = J^M_{\lambda_n}(\delta_n x_n + (1-\delta_n)Tx_n), \\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta x_n + ((1-\beta)I - \alpha_n A)(a_n\int_0^{\infty} \exp(-a_n s)T(s)y_n\,d(s)), \quad n \geq 1, \end{cases}$$

where $\{a_n\}$ is a decreasing sequence in $(0,\infty)$ such that $\lim_{n\to\infty} a_n = 0$, converges strongly to $p \in F$ which also solves the variational inequality (3.1).

Competing interests

The authors declare that they have no competing interests.

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