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Fixed point theorems on ordered gauge spaces with applications to nonlinear integral equations

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Abstract
We establish coincidence and fixed point theorems for mappings satisfying generalized weakly contractive conditions on the setting of ordered gauge spaces. Presented theorems extend and generalize many existing studies in the literature. We apply our obtained results to the study of existence and uniqueness of solutions to some classes of nonlinear integral equations.

Keywords: Gauge spaceordered setcoincidence pointfixed pointaltering distance function

1 Introduction

Fixed point theory is considered as one of the most important tools of nonlinear analysis that widely applied to optimization, computational algorithms, physics, variational inequalities, ordinary differential equations, integral equations, matrix equations and so on (see, for example, [1-6]). The Banach contraction principle [7] is a fundamental result in fixed point theory. It consists of the following theorem.

Theorem 1.1 (Banach [7]) *Let (X, d) be a complete metric space and let $T : X \to X$ be a contraction, i.e., there exists $k \in [0, 1)$ such that $d(Tx, Ty) \le kd(x, y)$ for all $x, y \in X$. Then T has a unique fixed point, that is, there exists a unique $x^* \in X$ such that $Tx^* = x^*$. Moreover, for any $x \in X$, the sequence $\{T^n x\}$ converges to x^*.*

Generalization of the above principle has been a heavily investigated branch of research (see, for example, [8-10]). In particular, there has been a number of studies involving altering distance functions. There are control functions which alter the distance between two points in a metric space. Such functions were introduced by Khan et al. [11], where they present some fixed point theorems with the help of such functions.

Definition 1.1 *An altering distance function is a function $\psi : [0, \infty) \to [0, \infty)$ which satisfies*

(a) ψ is continuous and nondecreasing;

(b) $\psi(t) = 0$ if and only if $t = 0$.

In [11], Khan et al. proved the following result.

Theorem 1.2 (Khan et al. [11]) *Let (X, d) be a complete metric space, ψ be an altering distance function, $c \in [0, 1)$ and $T : X \to X$ satisfying*

$$\psi(d(Tx, Ty)) \le c\psi(d(x, y)),$$

for all $x, y \in X$. Then T has an unique fixed point.

Altering distance has been used in metric fixed point theory in many studies (see, for example, [2,3,12-19]). On the other hand, Alber and Guerre-Delabriere in [12] introduced a new class of contractive mappings on closed convex sets of Hilbert spaces, called weakly contractive maps.

Definition 1.2 (Alber and Guerre-Delabriere [12]) *Let $(E, \|\cdot\|)$ be a Banach space and $C \subseteq E$ a closed convex set. A map $T : C \to C$ is called weakly contractive if there exists an altering distance function $\psi : [0, \infty) \to [0, \infty)$ with $\lim_{t\to\infty} \psi(t) = \infty$ such that*

$$\|Tx - Ty\| \le \|x - y\| - \psi(\|x - y\|),$$

for all $x, y \in X$.

In [12], Alber and Guerre-Delabriere proved the following result.

Theorem 1.3 (Alber and Guerre-Delabriere [12]) *Let H be a Hilbert space and $C \subseteq H$ a closed convex set. If $T : C \to C$ is a weakly contractive map, then it has a unique fixed point $x^* \in C$.*

Rhoades [18] proved that the previous result is also valid in complete metric spaces without the condition $\lim_{t\to\infty} \psi(t) = \infty$.

Theorem 1.4 (Rhoades [18]) *Let (X, d) be a complete metric space, ψ be an altering distance function and $T : X \to X$ satisfying*

$$d(Tx, Ty) \le d(x, y) - \psi(d(x, y))$$

for all $x, y \in X$. Then T has a unique fixed point.

Dutta and Choudhury [20] present a generalization of Theorems 1.2 and 1.4 proving the following result.

Theorem 1.5 (Dutta and Choudhury [20]) *Let (X, d) be a complete metric space and $T : X \to X$ be a mapping satisfying*

$$\psi(d(Tx, Ty)) \le \psi(d(x, y)) - \varphi(d(x, y)),$$

for all $x, y \in X$, where ψ and ϕ are altering distance functions. Then T has an unique fixed point.

An extension of Theorem 1.5 was considered by Dorić [13].

Theorem 1.6 (Dorić [13]) *Let (X, d) be a complete metric space and $T : X \to X$ be a mapping satisfying*

$$\psi(d(Tx, Ty)) \le \psi(M(x, y)) - \varphi(M(x, y)),$$

for all $x, y \in X$, where

$$M(x, y) = \max\left\{d(x, y), d(Tx, x), d(Ty, y), \frac{1}{2}[d(y, Tx) + d(x, Ty)]\right\},$$

ψ is an altering distance function and ϕ is a lower semi-continuous function with $\phi(t) = 0$ if and only if $t = 0$. Then T has a unique fixed point.

Very recently, Eslamian and Abkar [14] (see also, Choudhury and Kundu [2]) introduced the concept of (ψ, α, β)-weak contraction and established the following result.

Theorem 1.7 (Eslamian and Abkar [14]) *Let (X, d) be a complete metric space and $T : X \to X$ be a mapping satisfying*

$$\psi(d(Tx, Ty)) \leq \alpha(d(x, y)) - \beta(d(x, y)), \tag{1}$$

for all $x, y \in X$, where $\psi, \alpha, \beta : [0, \infty) \to [0, \infty)$ are such that ψ is an altering distance function, α is continuous, β is lower semi-continuous,

$$\alpha(0) = \beta(0) = 0 \quad \text{and} \quad \psi(t) - \alpha(t) + \beta(t) > 0 \quad \text{for all} \quad t > 0.$$

Then T has a unique fixed point.

Note that Theorem 1.7 seems to be new and original. Unfortunately, it is not the case. Indeed, the contractive condition (1) can be written as follows:

$$\psi(d(Tx, Ty)) \leq \psi(d(x, y)) - \varphi(d(x, y)),$$

where $\phi : [0, \infty) \to [0, \infty)$ is given by

$$\varphi(t) = \psi(t) - \alpha(t) + \beta(t), \quad t \geq 0.$$

Clearly, from the hypotheses of Theorem 1.7, the function ϕ is lower semi-continuous with $\phi(t) = 0$ if and only if $t = 0$. So Theorem 1.7 is similar to Theorem 1.6 of Dorić [13].

On the other hand, Ran and Reurings [6] proved the following Banach-Caccioppoli type principle in ordered metric spaces.

Theorem 1.8 (Ran and Reurings [6]**)** *Let $(X, \preccurlyeq)$ be a partially ordered set such that every pair $x, y \in X$ has a lower and an upper bound. Let d be a metric on X such that the metric space (X, d) is complete. Let $f : X \to X$ be a continuous and monotone (i.e., either decreasing or increasing with respect to $\preccurlyeq$) operator. Suppose that the following two assertions hold:*

1. there exists $k \in [0, 1)$ such that $d(fx, fy) \leq kd(x, y)$ for each $x, y \in X$ with $x \preccurlyeq y$;

2. there exists $x_0 \in X$ such that $x_0 \preccurlyeq f x_0$ or $x_0 \succcurlyeq f x_0$.

Then f has an unique fixed point $x^* \in X$.

Nieto and Rodŕiguez-López [4] extended the result of Ran and Reurings for non-continuous mappings.

Theorem 1.9 (Nieto and Rodŕiguez-López [4]**)** *Let $(X, \preccurlyeq)$ be a partially ordered set and suppose that there exists a metric d in X such that the metric space (X, d) is complete. Let $T : X \to X$ be a nondecreasing mapping. Suppose that the following assertions hold:*

1. there exists $k \in [0, 1)$ such that $d(Tx, Ty) \leq kd(x, y)$ for all $x, y \in X$ with $x \preccurlyeq y$;

2. there exists $x_0 \in X$ such that $x_0 \preccurlyeq Tx_0$;

3. if $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \to x \in X$ as $n \to \infty$, then $x_n \preccurlyeq x$ for all n.

Then T has a fixed point.

Since then, several authors considered the problem of existence (and uniqueness) of a fixed point for contraction type operators on partially ordered metric spaces (see, for example, [2,3,5,15-17,19,21-38]).

In [3], Harjani and Sadarangani extended Theorem 1.5 of Dutta and Choudhury [20] to the setting of ordered metric spaces.

Theorem 1.10 (Harjani and Sadarangani [3]**)** *Let $(X, \preccurlyeq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space. Let $T : X \to X$ be a nondecreasing mapping such that*

$$\psi(d(Tx, Ty)) \leq \psi(d(x, y)) - \varphi(d(x, y)),$$

for all $x, y \in X$ *with* $x \preccurlyeq y$, *where* ψ *and* ϕ *are altering distance functions. Also suppose either*

(I) T *is continuous or*

(II) If $\{x_n\} \subset X$ *is a nondecreasing sequence with* $x_n \to x \in X$, *then* $x_n \preccurlyeq x$ *for all* n.

If there exists $x_0 \in X$ *with* $x_0 \preccurlyeq Tx_0$, *then* T *has a fixed point.*

In [16], Jachymski established a nice geometric lemma and proved that Theorem 1.10 of Harjani and Sadarangani can be deuced from an earlier result of O'Regan and Petruşel [33].

In this article, we present new coincidence and fixed point theorems in the setting of ordered gauge spaces for mappings satisfying generalized weak contractions involving two families of functions. Presented theorems extend and generalize many existing results in the literature, in particular Harjani and Sadarangani [3, Theorem 1.10], Nieto and Rodŕiguez-López [4, Theorem 1.9], Ran and Reurings [6, Theorem 1.8], and Dorić [13, Theorem 1.6]. As an application, existence results for some integral equations on the positive real axis are given.

Now, we shall recall some preliminaries on ordered gauge spaces and introduce some definitions.

2 Preliminaries

Definition 2.1 *Let* X *be a nonempty set. A map* $d : X \times X \to [0, \infty)$ *is called a pseudo-metric in* X *whenever*

(i) $d(x, x) = 0$ *for all* $x \in X$;

(ii) $d(x, y) = d(y, x)$ *for all* $x, y \in X$;

(iii) $d(x, y) \leq d(x, z) + d(z, y)$ *for all* $x, y, z \in X$.

Definition 2.2 *Let* X *be a nonempty set endowed with a pseudo-metric* d. *The* d*-ball of radius* $\varepsilon > 0$ *centered at* $x \in X$ *is the set*

$$B(x; d, \varepsilon) = \{y \in X | d(x, y) < \varepsilon\}.$$

Definition 2.3 *A family* $\mathcal{F} = \{d_\lambda | \lambda \in \mathcal{A}\}$ *of pseudo-metrics is called separating if for each pair* $x \neq y$, *there is a* $d_\lambda \in \mathcal{F}$ *such that* $d_\lambda(x, y) \neq 0$.

Definition 2.4 *Let* X *be a nonempty set and* $\mathcal{F} = \{d_\lambda | \lambda \in \mathcal{A}\}$ *be a separating family of pseudo-metrics on X. The topology* $\mathcal{T}(\mathcal{F})$ *having for a subbasis the family*

$$\mathcal{B}(\mathcal{F}) = \{B(x; d_\lambda, \varepsilon) | x \in X, d_\lambda \in \mathcal{F}, \varepsilon > 0\}$$

of balls is called the topology in X induced by the family $\mathcal{F}$. *The pair* $(X, \mathcal{T}(\mathcal{F}))$ *is called a gauge space. Note that* $(X, \mathcal{T}(\mathcal{F}))$ *is Hausdorff because we require* $\mathcal{F}$ *to be separating.*

Definition 2.5 *Let* $(X, \mathcal{T}(\mathcal{F}))$ *be a gauge space with respect to the family* $\mathcal{F} = \{d_\lambda | \lambda \in \mathcal{A}\}$ *of pseudo-metrics on X. Let* $\{x_n\}$ *be a sequence in X and* $x \in X$.

(a) The sequence $\{x_n\}$ *converges to x if and only if*

$$\forall \lambda \in \mathcal{A}, \ \ \forall \varepsilon > 0, \ \ \exists N \in N | d_\lambda(x_n, x) < \varepsilon, \ \ \forall n \geq N.$$

In this case, we denote $x_n \xrightarrow{\mathcal{F}} x$.

(b) The sequence $\{x_n\}$ *is Cauchy if and only if*

$$\forall\lambda \in \mathcal{A},\ \ \forall\varepsilon > 0,\ \ \exists N \in \mathbb{N} | d_\lambda(x_{n+p}, x_n) < \varepsilon,\ \ \forall n \geq N, p \in \mathbb{N}.$$

(c) $(X, \mathcal{T}(\mathcal{F}))$ *is complete if and only if any Cauchy sequence in* $(X, \mathcal{T}(\mathcal{F}))$ *is convergent to an element of X.*

(d) A subset of X is said to be closed if it contains the limit of any convergent sequence of its elements.

Definition 2.6 *Let* $\mathcal{F} = \{d_\lambda | \lambda \in \mathcal{A}\}$ *be a family of pseudo-metrics on X.* $(X, \mathcal{F}, \preccurlyeq)$ *is called an ordered gauge space if* $(X, \mathcal{T}(\mathcal{F}))$ *is a gauge space and* $(X, \preccurlyeq)$ *is a partially ordered set.*

For more details on gauge spaces, we refer the reader to [39].

Now, we introduce the concept of compatibility of a pair of self mappings on a gauge space.

Definition 2.7 *Let* $(X, \mathcal{T}(\mathcal{F}))$ *be a gauge space and f, g : X → X are giving mappings. We say that the pair {f, g} is compatible if for all* $\lambda \in \mathcal{A}$, $d_\lambda(fgx_n, gfx_n) \to 0$ *as n* $\to \infty$ *whenever* $\{x_n\}$ *is a sequence in X such that* $fx_n \xrightarrow{\mathcal{F}} t$ *and* $gx_n \xrightarrow{\mathcal{F}} t$ *for some* $t \in X$.

Definition 2.8 (Ćirić et al. [29]**)** *Let* $(X, \preccurlyeq)$ *be a partially ordered set and f, g : X → X are two giving mappings. The mapping f is said to be g-nondecreasing if for all x, y* ∈ *X, we have*

$$gx \preccurlyeq gy \Rightarrow fx \preccurlyeq fy.$$

Definition 2.9 *Let* $(X, \preccurlyeq)$ *be a partially ordered set. We say that* $(X, \preccurlyeq)$ *is directed if every pair of elements has an upper bound, that is, for every a, b* ∈ *X, there exists c* ∈ X *such that* $a \preccurlyeq c$ *and* $b \preccurlyeq c$.

3 Main results

Let $(X, \mathcal{T}(\mathcal{F}))$ be a gauge space.

We consider the class of functions $\{\psi_\lambda\}_{\lambda\in\mathcal{A}}$ and $\{\varphi_\lambda\}_{\lambda\in\mathcal{A}}$ such that for all $\lambda \in \mathcal{A}$, ψ_λ, ϕ_λ,: $[0, \infty) \to [0, \infty)$ satisfy the following conditions:

(C1) ψ_λ is an altering distance function.

(C2) ϕ_λ is a lower semi-continuous function with $\phi_\lambda(t) = 0$ if and only if $t = 0$.

Our first result is the following.

Theorem 3.1 *Let* $(X, \mathcal{F}, \preccurlyeq)$ *be an ordered complete gauge space and let f, g : X → X be two continuous mappings such that f is g-nondecreasing,* $f(X) \subseteq g(X)$ *and the pair {f, g} is compatible. Suppose that*

$$\psi_\lambda(d_\lambda(fx, fy)) \leq \psi_\lambda(d_\lambda(gx, gy)) - \varphi_\lambda(d_\lambda(gx, gy)) \tag{2}$$

for all $\lambda \in \mathcal{A}$, *for all x, y* ∈ *X for which* $gx \preccurlyeq gy$. *If there exists* x_0 *such that* $gx_0 \preccurlyeq fx_0$, *then f and g have a coincidence point, that is, there exists a z* ∈ *X such that fz = gz.*

Proof. Let $x_0 \in X$ such that $gx_0 \preccurlyeq fx_0$ (such a point exists by hypothesis). Since $f(X) \subseteq g(X)$, we can choose $x_1 \in X$ such that $fx_0 = gx_1$. Then $gx_0 \preccurlyeq fx_0 = gx_1$. As f is g-nondecreasing, we get $fx_0 \preccurlyeq fx_1$. Continuing this process, we can construct a sequence $\{x_n\}$ in X such that

$$gx_{n+1} = fx_n, \quad n = 0, 1, \ldots$$

for which

$$gx_0 \preccurlyeq fx_0 = gx_1 \preccurlyeq fx_1 = gx_2 \preccurlyeq \cdots \preccurlyeq fx_{n-1} = gx_n \preccurlyeq \cdots$$

Then from (2), for all p, $q \in \mathbb{N}$, for all $\lambda \in \mathcal{A}$, we have

$$\psi_\lambda(d_\lambda(fx_p, fx_q)) \leq \psi_\lambda(d_\lambda(gx_p, gx_q)) - \varphi_\lambda(d_\lambda(gx_p, gx_q)). \quad (3)$$

We complete the proof in the following three steps.

Step 1. We will prove that

$$d_\lambda(fx_n, fx_{n+1}) \to 0 \quad \text{as } n \to +\infty, \quad \text{for all } \lambda \in \mathcal{A}. \quad (4)$$

Let $\lambda \in \mathcal{A}$. We distinguish two cases.

• *First case*: We suppose that there exists $m \in \mathbb{N}$ such that $d_\lambda(fx_m, fx_{m+1}) = 0$. Applying (3), we get that

$$\begin{aligned}\psi_\lambda(d_\lambda(fx_{m+1}, fx_{m+2})) &\leq \psi_\lambda(d_\lambda(gx_{m+1}, gx_{m+2})) - \varphi_\lambda(d_\lambda(gx_{m+1}, gx_{m+2})) \\ &= \psi_\lambda(d_\lambda(fx_m, fx_{m+1})) - \varphi_\lambda(d_\lambda(fx_m, fx_{m+1})) \\ &= \psi_\lambda(0) - \varphi_\lambda(0) \\ (\text{from (C1), (C2)}) &= 0.\end{aligned}$$

Then it follows from (C1) that $d_\lambda(fx_{m+1}, fx_{m+2}) = 0$. Continuing this process, one can show that $d_\lambda(fx_n, fx_{n+1}) = 0$ for all $n \geq m$. Then our claim (4) holds.

• *Second case*: We suppose that

$$d_\lambda(fx_n, fx_{n+1}) > 0, \quad \text{for all } n \in \mathbb{N}. \quad (5)$$

Let, if possible, for some $n_0 \in \mathbb{N}$,

$$d_\lambda(fx_{n_0-1}, fx_{n_0}) < d_\lambda(fx_{n_0}, fx_{n_0+1}).$$

By the monotone property of ψ_λ, and using (3), we get

$$\begin{aligned}\psi_\lambda(d_\lambda(fx_{n_0-1}, fx_{n_0})) \leq \psi_\lambda(d_\lambda(fx_{n_0}, fx_{n_0+1})) &\leq \psi_\lambda(d_\lambda(gx_{n_0}, gx_{n_0+1})) - \varphi_\lambda(d_\lambda(gx_{n_0}, gx_{n_0+1})) \\ &= \psi_\lambda(d_\lambda(fx_{n_0-1}, fx_{n_0})) - \varphi_\lambda(d_\lambda(fx_{n_0-1}, fx_{n_0})).\end{aligned}$$

Then, by (C2), we have that $d_\lambda(fx_{n_0-1}, fx_{n_0}) = 0$, which contradicts (5). Therefore, we deduce that

$$d_\lambda(fx_n, fx_{n+1}) \leq d_\lambda(fx_{n-1}, fx_n), \quad \text{for all } n \geq 1.$$

So, it follows that $\{d_\lambda(fx_n, fx_{n+1})\}$ is a decreasing sequence of non-negative real numbers. Hence, there is $r \geq 0$ such that

$$d_\lambda(fx_n, fx_{n+1}) \to r \quad \text{as } n \to +\infty. \quad (6)$$

On the other hand, from (3), we have

$$\begin{aligned}\psi_\lambda(d_\lambda(fx_n, fx_{n+1})) &\leq \psi_\lambda(d_\lambda(gx_n, gx_{n+1})) - \varphi_\lambda(d_\lambda(gx_n, gx_{n+1})) \\ &= \psi_\lambda(d_\lambda(fx_{n-1}, fx_n)) - \varphi_\lambda(d_\lambda(fx_{n-1}, fx_n)).\end{aligned}$$

This implies that

$$\limsup_{n\to\infty} \psi_\lambda(d_\lambda(fx_n, fx_{n+1})) \leq \limsup_{n\to\infty} \psi_\lambda(d_\lambda(fx_{n-1}, fx_n)) - \liminf_{n\to\infty} \varphi_\lambda(d_\lambda(fx_{n-1}, fx_n)).$$

Then, using (6), the continuity hypothesis of ψ_λ and the lower semi-continuity of ϕ_λ, we get that

$$\psi_\lambda(r) \leq \psi_\lambda(r) - \varphi_\lambda(r),$$

which, by condition (C2) implies that $r = 0$. Thus, we proved (4).

Step 2. We will prove that $\{fx_n\}$ is a Cauchy sequence in the gauge space $(X, \mathcal{T}(\mathcal{F}))$. Suppose that $\{fx_n\}$ is not a Cauchy sequence. Then there exists $(\lambda, \varepsilon) \in \mathcal{A} \times (0, \infty)$ for which we can find two sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ such that for all positive integers k,

$$n(k) > m(k) > k, \quad d_\lambda(fx_{m(k)}, fx_{n(k)}) \geq \varepsilon, \quad d_\lambda(fx_{m(k)}, fx_{n(k)-1}) < \varepsilon. \tag{7}$$

Using (7) and the triangular inequality, we get that

$$\begin{aligned} \varepsilon &\leq d_\lambda(fx_{n(k)}, fx_{m(k)}) \\ &\leq d_\lambda(fx_{m(k)}, fx_{n(k)-1}) + d_\lambda(fx_{n(k)-1}, fx_{n(k)}) \\ &< \varepsilon + d_\lambda(fx_{n(k)}, fx_{n(k)-1}). \end{aligned}$$

Thus we have

$$\varepsilon \leq d_\lambda(fx_{n(k)}, fx_{m(k)}) < \varepsilon + d_\lambda(fx_{n(k)}, fx_{n(k)-1}).$$

Letting $k \to +\infty$ in the above inequality and using (4), we obtain

$$d_\lambda(fx_{n(k)}, fx_{m(k)}) \to \varepsilon \quad \text{as } k \to +\infty. \tag{8}$$

On the other hand, we have

$$d_\lambda(fx_{n(k)}, fx_{m(k)}) \leq d_\lambda(fx_{n(k)}, fx_{n(k)-1}) + d_\lambda(fx_{n(k)-1}, fx_{m(k)-1}) + d_\lambda(fx_{m(k)-1}, fx_{m(k)})$$

and

$$d_\lambda(fx_{n(k)-1}, fx_{m(k)-1}) \leq d_\lambda(fx_{n(k)-1}, fx_{n(k)}) + d_\lambda(fx_{n(k)}, fx_{m(k)}) + d_\lambda(fx_{m(k)}, fx_{m(k)-1}).$$

Thus we have

$$\begin{cases} d_\lambda(fx_{n(k)-1}, fx_{m(k)-1}) \geq d_\lambda(fx_{n(k)}, fx_{m(k)}) - d_\lambda(fx_{n(k)}, fx_{n(k)-1}) - d_\lambda(fx_{m(k)-1}, fx_{m(k)}) \\ d_\lambda(fx_{n(k)-1}, fx_{m(k)-1}) \leq d_\lambda(fx_{n(k)-1}, fx_{n(k)}) + d_\lambda(fx_{n(k)}, fx_{m(k)}) + d_\lambda(fx_{m(k)-1}) \end{cases}$$

which implies that

$$\left| d_\lambda(fx_{n(k)-1}, fx_{m(k)-1}) - d_\lambda(fx_{n(k)}, fx_{m(k)}) \right| \leq d_\lambda(fx_{n(k)-1}, fx_{n(k)}) + d_\lambda(fx_{m(k)}, fx_{m(k)-1}).$$

Letting $k \to \infty$ in the above inequality, using (4) and (8), we get that

$$d_\lambda(fx_{n(k)-1}, fx_{m(k)-1}) \to \varepsilon \quad \text{as } k \to +\infty. \tag{9}$$

Applying inequality (3) with $p = n(k)$ and $q = m(k)$, we get that

$$\psi_\lambda(d_\lambda(fx_{n(k)}, fx_{m(k)})) \leq \psi_\lambda(d_\lambda(gx_{n(k)}, gx_{m(k)})) - \varphi_\lambda(d_\lambda(gx_{n(k)}, gx_{m(k)})),$$

that is,

$$\psi_\lambda(d_\lambda(fx_{n(k)}, fx_{m(k)})) \leq \psi_\lambda(d_\lambda(fx_{n(k)-1}, fx_{m(k)-1})) - \varphi_\lambda(d_\lambda(fx_{n(k)-1}, fx_{m(k)-1})).$$

Letting $k \to +\infty$ in the above inequality, using (8), (9), the continuity hypothesis of ψ_λ and the lower semi-continuity of ϕ_λ, we obtain

$$\psi_\lambda(\varepsilon) \le \psi_\lambda(\varepsilon) - \varphi_\lambda(\varepsilon),$$

which implies from (C2) that $\varepsilon = 0$, which is a contradiction with $\varepsilon > 0$. Finally, we deduce that $\{fx_n\}$ is a Cauchy sequence.

Step 3. Existence of a coincidence point.

Since $\{fx_n\}$ is a Cauchy sequence in the complete gauge space $(X, \mathcal{T}(\mathcal{F}))$, then there exists a $z \in X$ such that $fx_n \xrightarrow{\mathcal{F}} z$. Since f and g are continuous, we get that $ffx_n \xrightarrow{\mathcal{F}} fz$ and $gfx_n \xrightarrow{\mathcal{F}} gz$. On the other hand, from $gx_{n+1} = fx_n$, we have also $gx_n \xrightarrow{\mathcal{F}} z$. Thus, we

$$fx_n \xrightarrow{\mathcal{F}} z, \quad ffx_n \xrightarrow{\mathcal{F}} fz, \quad gfx_n \xrightarrow{\mathcal{F}} gz, \quad gx_n \xrightarrow{\mathcal{F}} z. \tag{10}$$

From the compatibility hypothesis of the pair $\{f, g\}$, we get that for all $\lambda \in \mathcal{A}$,

$$d_\lambda(fgx_n, gfx_n) \to 0 \quad \text{as } n \to \infty. \tag{11}$$

Now, using the triangular inequality, for all $\lambda \in \mathcal{A}$, we have

$$d_\lambda(fz, gz) \le d_\lambda(fz, ffx_n) + d_\lambda(fgx_{n+1}, gfx_{n+1}) + d_\lambda(gfx_{n+1}, gz).$$

Letting $n \to \infty$ in the above inequality, and using (10) and (11), we get that $d_\lambda(fz, gz) = 0$ for all $\lambda \in \mathcal{A}$. In the virtue of the separating structure of $\mathcal{F}$, this implies that $fz = gz$, that
is, z is a coincidence point of f and g.

Let $(X, \mathcal{F}, \preccurlyeq)$ be an ordered gauge space. We consider the following assumption: (H): If $\{u_n\} \subset X$ is a nondecreasing sequence with $u_n \xrightarrow{\mathcal{F}} u \in X$, then $u_n \preccurlyeq u$ for all n.

Theorem 3.2 Let $(X, \mathcal{F}, \preccurlyeq)$ be *an ordered complete gauge space satisfying the assumption (H). Let f, g* : $X \to X$ *be two mappings such that f is g-nondecreasing,* $f(X) \subseteq g(X)$ *and* $g(X)$ *is closed. Suppose that*

$$\psi_\lambda(d_\lambda(fx, fy)) \le \psi_\lambda(d_\lambda(gx, gy)) - \varphi_\lambda(d_\lambda(gx, gy)) \tag{12}$$

for all $\lambda \in \mathcal{A}$ *, for all* $x, y \in X$ *for which* $gx \preccurlyeq gy$*. If there exists* x_0 *such that* $gx_0 \preccurlyeq fx_0$*, then f and g have a coincidence point.*

Proof. Following the proof of Theorem 3.1, we know that $\{gx_n\}$ is a Cauchy sequence in the ordered complete gauge space $(X, \mathcal{F}, \preccurlyeq)$. Since $g(X)$ is closed, there exists $z \in X$ such that $gx_n \xrightarrow{\mathcal{F}} gz$. Then we have

$$fx_n \xrightarrow{\mathcal{F}} gz \quad \text{and} \quad gx_n \xrightarrow{\mathcal{F}} gz. \tag{13}$$

Since $\{gx_n\}$ is a nondecreasing sequence, from (H), we have $gx_n \preccurlyeq gz$ for all $n \ge 1$. Then we can apply (12) with $x = x_n$ and $y = z$, we obtain

$$\psi_\lambda(d_\lambda(fx_n, fz)) \le \psi_\lambda(d_\lambda(gx_n, gz)) - \varphi_\lambda(d_\lambda(gx_n, gz))$$

for all $\lambda \in \mathcal{A}$ and $n \ge 1$. Let $\lambda \in \mathcal{A}$ be fixed. Letting $n \to \infty$ in the above inequality, using (C1), (C2) and (13), we obtain that $\psi_\lambda(d_\lambda(gz, fz)) = 0$, which implies from (C1) that $d_\lambda(gz, fz) = 0$. Thus, we proved that $d_\lambda(gz, fz) = 0$ for all $\lambda \in \mathcal{A}$. Then $gz = fz$ and z is a coincidence point of g and f.

Theorem 3.3 *Let* $(X, \mathcal{F}, \preccurlyeq)$ *be an ordered complete gauge space and* $f : X \to X$ be a *nondecreasing mapping. Suppose that*

$$\psi_\lambda(d_\lambda(fx, fy)) \leq \psi_\lambda(d_\lambda(x, y)) - \varphi_\lambda(d_\lambda(x, y)) \tag{14}$$

for all $(X, \mathcal{F}, \preccurlyeq)$ *, for all* $x, y \in X$ *with* $x \preccurlyeq y$*. Also suppose either*

(I) *f is continuous or*

(II) *If* $\{x_n\} \subset X$ *is a nondecreasing sequence with* $x_n \xrightarrow{\mathcal{F}} z \in X$ *, then* $x_n \preccurlyeq z$ *for all n.*

If there exists x_0 *such that* $x_0 \preccurlyeq fx_0$*, then f has a fixed point, that is, there exists* $z \in X$ *such that* $z = fz$*. Moreover, if*$(X, \preccurlyeq)$ *is directed, we obtain the uniqueness of the fixed point of f.*

Proof. The existence of a fixed point of *f* follows immediately from Theorems 3.1 and 3.2 by taking $g = I_X$ (the identity mapping on *X*). Now, suppose that $z' \in X$ is another fixed point of *f*, that is, $z' = fz'$. Since $(X, \preccurlyeq)$ is a directed set, there exists $w \in X$ such that $z \preccurlyeq w$ and $z' \preccurlyeq w$. Monotonicity of *f* implies that $f^n(z) \preccurlyeq f^n(w)$ and $f^n(z') \preccurlyeq f^n(w)$. Then we have

$$\begin{aligned} \psi_\lambda(d_\lambda(z, f^n(w))) &\leq \psi_\lambda(d_\lambda(f^{n-1}(z), f^{n-1}(w))) - \varphi_\lambda(d_\lambda(f^{n-1}(z), f^{n-1}(w))) \\ &\leq \psi_\lambda(d_\lambda(f^{n-1}(z), f^{n-1}(w))) \\ &= \psi_\lambda(d_\lambda(z, f^{n-1}(w))). \end{aligned} \tag{15}$$

Since ψ_λ is a nondecreasing function, we get that

$$d_\lambda(z, f^n(w)) \leq d_\lambda(z, f^{n-1}(w)), \quad \text{for all } n \geq 1,\ \lambda \in \mathcal{A}.$$

Then there exists $r_\lambda \geq 0$ such that $d_\lambda(z, f^n(w)) \to r_\lambda$ as $n \to \infty$. Letting $n \to \infty$ in (15), we get that

$$\psi_\lambda(r_\lambda) \leq \psi_\lambda(r_\lambda) - \varphi_\lambda(r_\lambda),$$

which implies that $r_\lambda = 0$. Then we have $f^n(w) \xrightarrow{\mathcal{F}} z$. Similarly, one can show that $f^n(w) \xrightarrow{\mathcal{F}} z'$. Since $(X, \mathcal{T}(\mathcal{F}))$ is Hausdorff, we obtain that $z = z'$.

Let $(X, \mathcal{T}(\mathcal{F}))$ be a gauge space and *f*, $g : X \to X$ are two giving mappings. For all *x*, $y \in X$ and $\lambda \in \mathcal{A}$, we denote

$$M_\lambda(gx, gy) = \max\left\{d_\lambda(gx, gy), d_\lambda(gx, fx), d_\lambda(gy, fy), \frac{d_\lambda(gy, fy) + d_\lambda(gy, fx)}{2}\right\}.$$

We shall prove the following result.

Theorem 3.4 *Let* $(X, \mathcal{F}, \preccurlyeq)$ *be an ordered complete gauge space and let f,* $g : X \to X$ *be two continuous mappings such that f is g-nondecreasing,* $f(X) \subseteq g(X)$ *and the pair {f, g} is compatible. Suppose that*

$$\psi_\lambda(d_\lambda(fx, fy)) \leq \psi_\lambda(M_\lambda(gx, gy)) - \varphi_\lambda(M_\lambda(gx, gy)) \tag{16}$$

for all $\lambda \in \mathcal{A}$ *, for all* $x, y \in X$ *for which* $gx \preccurlyeq gy$*. If there exists* x_0 *such that* $gx_0 \preccurlyeq fx_0$*, then f and g have a coincidence point.*

Proof. Similarly to the proof of Theorem3.1, we can construct a sequence $\{x_n\}$ in *X* such that

$$gx_{n+1} = fx_n, \quad n = 0, 1, \ldots$$

for which

$$gx_0 \preccurlyeq gx_1 \preccurlyeq gx_2 \preccurlyeq \cdots \preccurlyeq gx_n \preccurlyeq \cdots$$

Then from (16), for all $p, q \in \mathbb{N}$, for all $\lambda \in \mathcal{A}$, we have

$$\psi_\lambda(d_\lambda(fx_p, fx_q)) \le \psi_\lambda(M_\lambda(gx_p, gx_q)) - \varphi_\lambda(M_\lambda(gx_p, gx_q)). \tag{17}$$

We complete the proof in the following three steps.

Step 1. We will prove that

$$d_\lambda(fx_n, fx_{n+1}) \to 0 \quad \text{as } n \to +\infty, \quad \text{for all } \lambda \in \mathcal{A}. \tag{18}$$

Let $\lambda \in \mathcal{A}$. We distinguish two cases.

• *First case:* We suppose that there exists $m \in \mathbb{N}$ such that $d_\lambda(fx_m, fx_{m+1}) = 0$. Applying (17), we get that

$$\psi_\lambda(d_\lambda(fx_{m+1}, fx_{m+2})) \le \psi_\lambda(M_\lambda(gx_{m+1}, gx_{m+2})) - \varphi_\lambda(M_\lambda(gx_{m+1}, gx_{m+2})).$$

A simple computation gives us that

$$M_\lambda(gx_{m+1}, gx_{m+2})) = d_\lambda(fx_{m+1}, fx_{m+2}).$$

Thus, we get that

$$\psi_\lambda(d_\lambda(fx_{m+1}, fx_{m+2})) \le \psi_\lambda(d_\lambda(fx_{m+1}, fx_{m+2})) - \varphi_\lambda(d_\lambda(fx_{m+1}, fx_{m+2})),$$

which implies from (C2) that $d_\lambda(fx_{m+1}, fx_{m+2}) = 0$. Continuing this process, one can show that $d_\lambda(fx_n, fx_{n+1}) = 0$ for all $n \ge m$. Then our claim (18) holds.

• *Second case:* We suppose that

$$d_\lambda(fx_n, fx_{n+1}) > 0, \quad \text{for all } n \in N. \tag{19}$$

Applying (17), for all $n \ge 1$, we have

$$\psi_\lambda(d_\lambda(fx_n, fx_{n+1})) \le \psi_\lambda(M_\lambda(gx_n, gx_{n+1})) - \varphi_\lambda(M_\lambda(gx_n, gx_{n+1})). \tag{20}$$

A simple computation gives us that

$$M_\lambda(gx_n, gx_{n+1}) = \max\{d_\lambda(fx_{n-1}, fx_n), d_\lambda(fx_n, fx_{n+1})\}.$$

If $M_\lambda(gx_n, gx_{n+1}) = d_\lambda(fx_n, fx_{n+1})$, we get that

$$\psi_\lambda(d_\lambda(fx_n, fx_{n+1})) \le \psi_\lambda(d_\lambda(fx_n, fx_{n+1})) - \varphi_\lambda(d_\lambda(fx_n, fx_{n+1})),$$

which implies from (C2) that $d_\lambda(fx_n, fx_{n+1}) = 0$, that is a contradiction with (19). We deduce that $M_\lambda(gx_n, gx_{n+1}) = d_\lambda(fx_{n-1}, fx_n)$, that is, $d_\lambda(fx_n, fx_{n+1}) \le d_\lambda(fx_{n-1}, fx_n)$. So, it follows that $\{d_\lambda(fx_{n-1}, fx_n)\}$ is a decreasing sequence of non-negative real numbers. Hence, there is $r \ge 0$ such that

$$d_\lambda(fx_{n-1}, fx_n) \to r \quad \text{as } n \to +\infty. \tag{21}$$

On the other hand, from (20), we have

$$\psi_\lambda(d_\lambda(fx_n, fx_{n+1})) \le \psi_\lambda(d_\lambda(fx_{n-1}, fx_n)) - \varphi_\lambda(d_\lambda(fx_{n-1}, fx_n)).$$

Letting $n \to \infty$ in the above inequality and using the properties (C1) and (C2), we get that

$$\psi_\lambda(r) \le \psi_\lambda(r) - \varphi_\lambda(r),$$

which implies from (C2) that $r = 0$. Then our claim (18) holds.

Step 2. We will prove that $\{fx_n\}$ is a Cauchy sequence in the gauge space $(X, \mathcal{T}(\mathcal{F}))$. Suppose that $\{fx_n\}$ is not a Cauchy sequence. Then there exists $(\lambda, \varepsilon) \in \mathcal{A} \times (0, \infty)$ for which we can find two sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ such that for all positive integers k,

$$n(k) > m(k) > k, \quad d_\lambda(fx_{m(k)}, fx_{n(k)}) \ge \varepsilon, \quad d_\lambda(fx_{m(k)}, fx_{n(k)-1}) < \varepsilon.$$

As in the proof of Theorem 3.1, one can show that

$$\lim_{k\to\infty} d_\lambda(fx_{n(k)}, fx_{m(k)}) = \lim_{k\to\infty} d_\lambda(fx_{n(k)-1}, fx_{m(k)-1}) = \varepsilon. \tag{22}$$

Applying inequality (17) with $p = n(k)$ and $q = m(k)$, we get that

$$\psi_\lambda(d_\lambda(fx_{n(k)}, fx_{m(k)})) \le \psi_\lambda(M_\lambda(gx_{n(k)}, gx_{m(k)})) - \varphi_\lambda(M_\lambda(gx_{n(k)}, gx_{m(k)})). \tag{23}$$

On the other hand, we have

$$\begin{aligned} M_\lambda(gx_{n(k)}, gx_{m(k)}) &= \max\Big\{d_\lambda(gx_{n(k)}, gx_{m(k)}), d_\lambda(gx_{n(k)}, fx_{n(k)}), d_\lambda(gx_{m(k)}, fx_{m(k)}), \\ &\qquad \frac{d_\lambda(gx_{n(k)}, fx_{m(k)}) + d_\lambda(gx_{m(k)}, fx_{n(k)})}{2}\Big\} \\ &= \max\Big\{d_\lambda(fx_{n(k)-1}, fx_{m(k)-1}), d_\lambda(fx_{n(k)-1}, fx_{n(k)}), d_\lambda(fx_{m(k)-1}, fx_{m(k)}), \\ &\qquad \frac{d_\lambda(fx_{n(k)-1}, fx_{m(k)}) + d_\lambda(fx_{m(k)-1}, fx_{n(k)})}{2}\Big\}. \end{aligned}$$

Using the triangular inequality, we get that

$$\left|d_\lambda(fx_{n(k)-1}, fx_{m(k)}) - d_\lambda(fx_{n(k)}, fx_{m(k)})\right| \le d(fx_{n(k)-1}, fx_{n(k)})$$

and

$$\left|d_\lambda(fx_{m(k)-1}, fx_{n(k)}) - d_\lambda(fx_{n(k)-1}, fx_{m(k)-1})\right| \le d(fx_{n(k)-1}, fx_{n(k)}).$$

Letting $k \to \infty$ in the above inequalities and using (18), (22), we get that

$$\lim_{k\to\infty} d_\lambda(fx_{n(k)-1}, fx_{m(k)}) = \lim_{k\to\infty} d_\lambda(fx_{m(k)-1}, fx_{n(k)}) = \varepsilon. \tag{24}$$

Now, combining (18), (22), and (24), we obtain

$$M_\lambda(gx_{n(k)}, gx_{m(k)}) \to \varepsilon \quad \text{as } k \to \infty. \tag{25}$$

Letting $k \to \infty$ in (23), using (22), (25) and the properties of functions ψ_λ and ϕ_λ, we get that

$$\psi_\lambda(\varepsilon) \le \psi_\lambda(\varepsilon) - \varphi_\lambda(\varepsilon),$$

which implies that $\varepsilon = 0$, a contradiction. Finally, we deduce that $\{fx_n\}$ is a Cauchy sequence.

Step 3. Existence of a coincidence point.

Since $\{fx_n\}$ is a Cauchy sequence in the complete gauge space $(X, \mathcal{T}(\mathcal{F}))$, then there exists a $z \in X$ such that $fx_n \xrightarrow{\mathcal{F}} z$. The rest part of the proof is similar to that of Theorem 3.1.

Theorem 3.5 *Let* $(X, \mathcal{F}, \preccurlyeq)$ *be an ordered complete gauge space satisfying the assumption (H). Let f, g* : $X \to X$ *be two mappings such that f is g-nondecreasing,* $f(X) \subseteq g(X)$ *and* $g(X)$ *is closed. Suppose that*

$$\psi_\lambda(d_\lambda(fz, fy)) \leq \psi_\lambda(M_\lambda(gx, gy)) - \varphi_\lambda(M_\lambda(gx, gy))$$

for all $\lambda \in \mathcal{A}$ *, for all x, y* $\in X$ *for which* $gx \preccurlyeq gy$. *If there exists* x_0 *such that* $gx_0 \preccurlyeq fx_0$, *then f and g have a coincidence point.*

Proof. It is similar to the proof of Theorem 3.2.

Using the same technique of the proof of Theorem 3.3, we deduce from Theorems 3.4 and 3.5 the following fixed point result.

Theorem 3.6 *Let* $(X, \mathcal{F}, \preccurlyeq)$ *be an ordered complete gauge space and* $f : X \to X$ *be a nondecreasing mapping. Suppose that*

$$\psi_\lambda(d_\lambda(fx, fy)) \leq (\psi_\lambda - \varphi_\lambda)\left(\max\left\{d_\lambda(x, y), d_\lambda(x, fx), d_\lambda(y, fy), \frac{d_\lambda(x, fy) + d_\lambda(y, fx)}{2}\right\}\right)$$

for all $\lambda \in \mathcal{A}$ *, for all x, y* $\in X$ *with* $x \preccurlyeq y$. *Also suppose either*

(I) *f is continuous or*

(II) *If* $\{x_n\} \subset X$ *is a nondecreasing sequence with* $x_n \xrightarrow{\mathcal{F}} z \in X$ *, then* $x_n \preccurlyeq z$ *for all n.*

If there exists x_0 *such that* $x_0 \preccurlyeq fx_0$, *then f has a fixed point. Moreover, if* $(X, \preccurlyeq)$ *is directed, we obtain the uniqueness of the fixed point of f.*

4 Some consequences

In this section, we present some fixed point theorems of integral-type on ordered gauge spaces, deduced from our previous obtained results.

Let Γ be the set of functions $a : [0, \infty) \to [0, \infty)$ satisfying

(i) a is locally integrable on $[0, \infty)$.

(ii) *For* all $\varepsilon > 0$, we have $\int_0^\varepsilon a(t)dt > 0$.

Theorem 4.1 *Let* $(X, \mathcal{F}, \preccurlyeq)$ *be an ordered complete gauge space and let f, g* : $X \to X$ *be two continuous mappings such that f is g-nondecreasing,* $f(X) \subseteq g(X)$ *and the pair {f, g} is compatible. Suppose that*

$$\int_0^{d_\lambda(fx,fy)} a_\lambda(t)dt \leq \int_0^{d_\lambda(gx,gy)} a_\lambda(t)dt - \int_0^{d_\lambda(gx,gy)} b_\lambda(t)dt$$

for all $\lambda \in \mathcal{A}$ *, for all x, y* $\in X$ *for which* $gx \preccurlyeq gy$, *where* $a_\lambda, b_\lambda \in \Gamma$ *for all* $\lambda \in \mathcal{A}$ *. If there exists* x_0 *such that* $gx_0 \preccurlyeq fx_0$, *then f and g have a coincidence point.*

Proof. It follows from Theorem 3.1, by taking for all $\lambda \in \mathcal{A}$,

$$\psi_\lambda(t) = \int_0^t a_\lambda(s)ds \quad \text{and} \quad \varphi_\lambda(t) = \int_0^t b_\lambda(s)ds, \quad t \geq 0.$$

It is clear that for all $\lambda \in \mathcal{A}$, the functions ψ_λ and φ_λ satisfy conditions (C1) and (C2).

Theorem 4.2 *Let $(X, \mathcal{F}, \preccurlyeq)$ be an ordered complete gauge space satisfying the assumption (H). Let f, g : X → X be two mappings such that f is g-nondecreasing, $f(X) \subseteq g(X)$ and $g(X)$ is closed. Suppose that*

$$\int_0^{d_\lambda(fx,fy)} a_\lambda(t)dt \le \int_0^{d_\lambda(gx,gy)} a_\lambda(t)dt - \int_0^{d_\lambda(gx,gy)} b_\lambda(t)dt$$

for all $\lambda \in \mathcal{A}$, for all $x, y \in X$ for which $gx \preccurlyeq gy$, where $a_\lambda, b_\lambda \in \Gamma$ for all $\lambda \in \mathcal{A}$. If there exists x_0 such that $gx_0 \preccurlyeq fx_0$, then f and g have a coincidence point.

Proof. It follows from Theorem 3.2.

Theorem 4.3 *Let $(X, \mathcal{F}, \preccurlyeq)$ be an ordered complete gauge space and $f : X \to X$ be a nondecreasing mapping. Suppose that*

$$\int_0^{d_\lambda(fx,fy)} a_\lambda(t)dt \le \int_0^{d_\lambda(x,y)} a_\lambda(t)dt - \int_0^{d_\lambda(x,y)} b_\lambda(t)dt$$

for all $\lambda \in \mathcal{A}$, for all $x, y \in X$ with $x \preccurlyeq y$, where $a_\lambda, b_\lambda \in \Gamma$ for all $\lambda \in \mathcal{A}$. Also suppose either

(I) *f is continuous or*

(II) *If $\{x_n\} \subset X$ is a nondecreasing sequence with $x_n \xrightarrow{\mathcal{F}} z \in X$, then $x_n \preccurlyeq z$ for all n.*

If there exists x_0 such that $x_0 \preccurlyeq fx_0$, then f has a fixed point. Moreover, if $(X, \preccurlyeq)$ is directed, we obtain the uniqueness of the fixed point of f.

Proof. It follows from Theorem 3.3.

Theorem 4.4 *Let $(X, \mathcal{F}, \preccurlyeq)$ be an ordered complete gauge space and let f, g : X → X be two continuous mappings such that f is g-nondecreasing, $f(X) \subseteq g(X)$ and the pair {f, g} is compatible. Suppose that*

$$\int_0^{d_\lambda(fx,fy)} a_\lambda(t)dt \le \int_0^{M_\lambda(gx,gy)} a_\lambda(t)dt - \int_0^{M_\lambda(gx,gy)} b_\lambda(t)dt$$

for all $\lambda \in \mathcal{A}$, for all $x, y \in X$ for which $gx \preccurlyeq gy$, where $a_\lambda, b_\lambda \in \Gamma$ for all $\lambda \in \mathcal{A}$. If there exists x_0 such that $gx_0 \preccurlyeq fx_0$, then f and g have a coincidence point.

Proof. It follows from Theorem 3.4.

Theorem 4.5 *Let $(X, \mathcal{F}, \preccurlyeq)$ be an ordered complete gauge space satisfying the assumption (H). Let f, g : X → X be two mappings such that f is g-nondecreasing, $f(X) \subseteq g(X)$ and $g(X)$ is closed. Suppose that*

$$\int_0^{d_\lambda(fx,fy)} a_\lambda(t)dt \le \int_0^{M_\lambda(gx,gy)} a_\lambda(t)dt - \int_0^{M_\lambda(gx,gy)} b_\lambda(t)dt$$

for all $\lambda \in \mathcal{A}$, for all $x, y \in X$ for which $gx \preccurlyeq gy$, where $a_\lambda, b_\lambda \in \Gamma$ for all $\lambda \in \mathcal{A}$. If there exists x_0 such that $gx_0 \preccurlyeq fx_0$, then f and g have a coincidence point.

Proof. It follows from Theorem 3.5.

Theorem 4.6 *Let* $(X, \mathcal{F}, \preccurlyeq)$ *be an ordered complete gauge space and* $f : X \to X$ *be a nondecreasing mapping. Suppose that*

$$\int_0^{d_\lambda(fx,fy)} a_\lambda(t)dt \le \int_0^{M_\lambda(x,y)} a_\lambda(t)dt - \int_0^{M_\lambda(x,y)} b_\lambda(t)dt$$

for all $\lambda \in \mathcal{A}$ *, for all* $x, y \in X$ *with* $x \preccurlyeq y$*, where* $a_\lambda, b_\lambda \in \Gamma$ *for all* $\lambda \in \mathcal{A}$ *. Also suppose either*

(I) *f is continuous or*

(II) *If* $\{x_n\} \subset X$ *is a nondecreasing sequence with* $x_n \xrightarrow{\mathcal{F}} z \in X$ *, then* $x_n \preccurlyeq z$ *for all* n*.*

If there exists x_0 *such that* $x_0 \preccurlyeq fx_0$*, then* f *has a fixed point. Moreover, if* $(X, \preccurlyeq)$ *is directed, we obtain the uniqueness of the fixed point of* f*.*

Proof. It follows from Theorem 3.6.

5 Applications

In this section, we present some examples of nonlinear integral equations, where our obtained results can be applied.

Consider the integral equation

$$x(t) = \int_0^t k(t, s, x(s))\, ds + h(t), \quad t \ge 0, \tag{26}$$

where $k : [0, \infty) \times [0, \infty) \times \mathbb{R} \to \mathbb{R}$ and $h : [0, \infty) \to \mathbb{R}$.

Previously, we consider the space $X = C([0, \infty), \mathbb{R})$ of real continuous functions defined on $[0, \infty)$. For each positive integer $n \ge 1$, we define the map $\|\cdot\|_n : X \to [0, \infty)$ by

$$\|x\|_n = \max_{0 \le t \le n} |x(t)|, \quad \text{for all } x \in X.$$

This map is a semi-norm on X. Define now,

$$d_n(x, y) = \|x - y\|_n, \quad \text{for all } n \ge 1, \quad x, y \in X.$$

Then $\mathcal{F} = \{d_n\}_{n \ge 1}$ is a separating family of pseudo-metrics on X. The gauge space $(X, \mathcal{T}(\mathcal{F}))$ with respect to the family $\mathcal{F}$ is complete. Consider on X the partial order $\preccurlyeq$ defined by

$$x, y \in X, \quad x \preccurlyeq y \Leftrightarrow x(t) \le y(t) \text{ for all } t \ge 0.$$

For any increasing sequence $\{x_n\}$ in X converging to some $z \in X$ we have $x_n(t) \le z(t)$ for any $t \ge 0$. Also, for every $x, y \in X$, there exists $c(x, y) \in X$ which is comparable to x and y.

We shall prove the following result.

Theorem 5.1 *Suppose that*

(i) $k : [0, \infty) \times [0, \infty) \times \mathbb{R} \to \mathbb{R}$ *and* $h : [0, \infty) \to \mathbb{R}$ *are continuous;*

(ii) $k(t, s, \cdot): \mathbb{R} \to \mathbb{R}$ *is increasing for each* $t, s \ge 0$*;*

(iii) for each $t, s \ge 0$, $u, v \in \mathbb{R}$, $u \le v$, *we have*

$$|k(t, s, u) - k(t, s, v)| \le \gamma(t, s)\sqrt{\ln[(v - u)^2 + 1]},$$

where $\gamma : [0, \infty) \times [0, \infty) \to [0, \infty)$ *is continuous, the function* $t \mapsto \int_0^t \gamma(t,s)\,ds$ *is bounded on* $[0, \infty)$ and

$$\sup_{t\geq 0} \int_0^t \gamma(t,s)\,ds \leq 1;$$

(iv) there exists $x_0 \in C([0, \infty), \mathbb{R})$ *such that*

$$x_0(t) \leq \int_0^t k(t,s,x_0(s))\,ds + h(t), \quad \text{for any } t \geq 0.$$

Then the integral equation (26) has a unique solution $x^* \in C([0, \infty), \mathbb{R})$.

Proof. Consider the operator $f : X \to X$ given by

$$fx(t) = \int_0^t k(t,s,x(s))\,ds + h(t), \quad t \geq 0, x \in X.$$

It is clear that f is well defined since k and h are continuous functions.

From condition (ii), for every $x, y \in X$ with $x \preccurlyeq y$, we have

$$k(t,s,x(s)) \leq k(t,s,y(s)), \quad \text{for all } t,s \geq 0,$$

which implies that

$$\int_0^t k(t,s,x(s))\,ds + h(t) \leq \int_0^t k(t,s,y(s))\,ds + h(t), \quad \text{for all } t \geq 0,$$

that is, $fx \preccurlyeq fy$. This proves that f is a nondecreasing operator.

Taking into account (iii), for each $x, y \in X$ with $x \preccurlyeq y$, for all $t \in [0, n]$, $n \geq 1$, we have

$$\begin{aligned} |fx(t) - fy(t)| &\leq \int_0^t |k(t,s,y(s)) - k(t,s,x(s))|\,ds \\ &\leq \int_0^t \gamma(t,s)\sqrt{\ln[(y(s)-x(s))^2+1]}\,ds \\ &\leq \sqrt{\ln[(d_n(x,y))^2+1]}\int_0^t \gamma(t,s)\,ds \\ &\leq \sqrt{\ln[(d_n(x,y))^2+1]}. \end{aligned}$$

Then, for all $n \geq 1$, we have

$$d_n(fx,fy) \leq \sqrt{\ln[(d_n(x,y))^2+1]}, \quad \text{for all } x,y \in X, \quad x \preccurlyeq y.$$

Hence, for all $n \geq 1$, we have

$$\psi_n(d_n(fx,fy)) \leq \psi_n(d_n(fx,fy)) - \varphi_n(d_n(fx,fy)), \quad \text{for all } x,y \in X, \quad x \preccurlyeq y,$$

where $\psi_n(t) = t^2$ and $\phi_n(t) = t^2 - \ln(t^2 + 1)$. Obviously, ψ_n, ϕ_n satisfy the conditions (C1) and (C2). Moreover, from (iv), there exists $x_0 \in X$ such that $x_0 \preccurlyeq fx_0$.

Now, applying Theorem 3.3, we obtain that f has a unique fixed point $x^* \in X$, that is, $x^* \in C([0, \infty), \mathbb{R})$ is the unique solution to (26).

Consider now the integral equation

$$x(t) = \int_{-t^2}^{t^2} k(t, s, x(s))\, ds + h(t), \quad t \in \mathbb{R}, \tag{27}$$

where $k : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$.

We consider the space $X = C(\mathbb{R}, \mathbb{R})$ of real continuous functions defined on $\mathbb{R}$. For each positive integer $n \geq 1$, we define the map $\|\cdot\|_n : X \to [0, \infty)$ by

$$\|x\|_n = \max_{-n \leq t \leq n} |x(t)|, \quad \text{for all } x \in X.$$

This map is a semi-norm on X. Define now,

$$d_n(x, y) = \|x - y\|_n, \quad \text{for all } n \geq 1, \quad x, y \in X.$$

Then $\mathcal{F} = \{d_n\}_{n \geq 1}$ is a separating family of pseudo-metrics on X. The gauge space $(X, \mathcal{T}(\mathcal{F}))$ with respect to the family $\mathcal{F}$ is complete. As before, consider on X the partial order $\preccurlyeq$ defined by

$$x, y \in X, \quad x \preccurlyeq y \Leftrightarrow x(t) \leq y(t) \text{ for all } t \in \mathbb{R}.$$

For any increasing sequence $\{x_n\}$ in X converging to some $z \in X$ we have $x_n(t) \leq z(t)$ for any $t \in \mathbb{R}$. Also, for every $x, y \in X$, there exists $c(x, y) \in X$ which is comparable to x and y. We shall prove the following result.

Theorem 5.2 *Suppose that*

(i) $k : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *and* $h : \mathbb{R} \to \mathbb{R}$ *are continuous;*

(ii) $k(t, s, \cdot): \mathbb{R} \to \mathbb{R}$ *is increasing for each* $t, s \in \mathbb{R}$;

(iii) for each $t, s \in \mathbb{R}$, $u, v \in \mathbb{R}$, $u \leq v$, *we have*

$$|k(t, s, u) - k(t, s, v)| \leq \gamma(t, s)\sqrt{\ln[(v - u)^2 + 1]},$$

where $\gamma : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ *is continuous, the function* $t \mapsto \int_{-t^2}^{t^2} \gamma(t, s)\, ds$ *is bounded on* $\mathbb{R}$ *and*

$$\sup_{t \in \mathbb{R}} \int_{-t^2}^{t^2} \gamma(t, s)\, ds \leq 1;$$

(iv) there exists $x_0 \in C(\mathbb{R}, \mathbb{R})$ *such that*

$$x_0(t) \leq \int_0^t k(t, s, x_0(s))\, ds + h(t), \quad \text{for any } t \in \mathbb{R}.$$

Then the integral equation (27) has a unique solution $x^* \in C(\mathbb{R}, \mathbb{R})$.

Proof. Consider the operator $f: X \to X$ given by

$$fx(t) = \int_{-t^2}^{t^2} k(t, s, x(s))\, ds + h(t), \quad t \in R, x \in X.$$

From condition (ii), for every $x, y \in X$ with $x \preccurlyeq y$, we have

$$k(t, s, x(s)) \le k(t, s, y(s)), \quad \text{for all } t, s \in R,$$

which implies that

$$\int_{-t^2}^{t^2} k(t, s, x(s))\, ds + h(t) \le \int_{-t^2}^{t^2} k(t, s, y(s))\, ds + h(t), \quad \text{for all } t \in R,$$

that is, $fx \preccurlyeq fy$. This proves that f is a nondecreasing operator.

Taking into account (iii), for each $x, y \in X$ with $x \preccurlyeq y$, for all $t \in [-n, n]$, $n \ge 1$, we have

$$\begin{aligned}
\left|fx(t) - fy(t)\right| &\le \int_{-t^2}^{t^2} \left|k(t, s, y(s)) - k(t, s, x(s))\right|\, ds \\
&\le \int_{-t^2}^{t^2} \gamma(t, s)\sqrt{\ln[(y(s) - x(s))^2 + 1]}\, ds \\
&\le \sqrt{\ln[(d_n(x, y))^2 + 1]} \int_{-t^2}^{t^2} \gamma(t, s)\, ds \\
&\le \sqrt{\ln[(d_n(x, y))^2 + 1]}.
\end{aligned}$$

Then, for all $n \ge 1$, we have

$$d_n(fx, fy) \le \sqrt{\ln[(d_n(x, y))^2 + 1]}, \quad \text{for all } x, y \in X, \quad x \preccurlyeq y.$$

Hence, for all $n \ge 1$, we have

$$\psi_n(d_n(fx, fy)) \le \psi_n(d_n(fx, fy)) - \varphi_n(d_n(fx, fy)), \quad \text{for all } x, y \in X, \quad x \preccurlyeq y,$$

where $\psi_n(t) = t^2$ and $\phi_n(t) = t^2 - \ln(t^2 + 1)$. Moreover, from (iv), there exists $x_0 \in X$ such that $x_0 \preccurlyeq fx_0$.

Now, applying Theorem 3.3, we obtain that f has a unique fixed point $x^* \in X$ that is, $x^* \in C(\mathbb{R}, \mathbb{R})$ is the unique solution to (27).

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Authors' contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

Competing interests

The authors declare that they have no competing interests.

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