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Nonlinear algorithms approach to split common solution problems

Zhenhua He[1] and Wei-Shih Du[2*]

*Correspondence: wsdu@nknucc.nknu.edu.tw
[2]Department of Mathematics, National Kaohsiung Normal University, Kaohsiung, 824, Taiwan
Full list of author information is available at the end of the article

Abstract
In this paper, we introduce some new iterative algorithms for the split common solution problems for equilibrium problems and fixed point problems of nonlinear mappings. Some examples illustrating our results are also given.
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1 Introduction

Throughout this paper, we assume that H is a real Hilbert space with zero vector θ, whose inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively. Let K be a nonempty subset of H and T be a mapping from K into itself. The set of fixed points of T is denoted by $\mathcal{F}(T)$. The symbols $\mathbb{N}$ and $\mathbb{R}$ are used to denote the sets of positive integers and real numbers, respectively.

Let C and K be nonempty subsets of real Banach spaces E_1 and E_2, respectively. Let $A: E_1 \to E_2$ be a bounded linear mapping, T a mapping from C into itself with $\mathcal{F}(T) \neq \emptyset$ and f a bi-function from $K \times K$ into $\mathbb{R}$. The classical equilibrium problem is to find $x \in K$ such that

$$f(x,y) \geq 0, \quad \forall y \in K. \tag{1.1}$$

The symbol $EP(f)$ is used to denote the set of all solutions of the problem (1.1), that is,

$$EP(f) = \{u \in K : f(u,v) \geq 0, \forall v \in K\}.$$

The equilibrium problem contains optimization problems, variational inequalities problems, saddle point problems, the Nash equilibrium problems, fixed point problems, complementary problems, bilevel problems, and semi-infinite problems as special cases and have many applications in mathematical program with equilibrium constraint; for detail, one can refer to [1–4] and references therein.

In this paper, we study the following split common solution problem (SCSP) for equilibrium problems and fixed point problems of nonlinear mappings A, T and f:

(SCSP) Find $p \in C$ such that $p \in \mathcal{F}(T)$ and $u := Ap \in K$ which satisfies $f(u,v) \geq 0$, $\forall v \in K$. The solution set of (SCSP) is denoted by

$$\Gamma = \{p \in \mathcal{F}(T) : Ap \in EP(f)\}.$$

Many authors had proposed some methods to find the solution of the equilibrium problem (1.1). As a generalization of the equilibrium problem (1.1), finding a common solution for some equilibrium problems and fixed point problems of nonlinear operators, it has been considered in the same subset of the same space; see [5–15]. However, some equilibrium problems and fixed point problems of nonlinear mappings always belong to different subsets of spaces in general. So the split common solution is very important for the research on generalized equilibriums problems and fixed point problems.

Example 1.1 Let $E_1 = E_2 = \mathbb{R}$, $C := [1, +\infty)$ and $K := (-\infty, -2]$. Let $A(x) = -2x$ for all $x \in \mathbb{R}$ and $Tx = \frac{2x}{x+1}$ for all $x \in C$. Let $f : K \times K \to \mathbb{R}$ be define by $f(u,v) = 2(u - v)$ for all $u, v \in K$. Clearly, A is a bounded linear operator, $\mathcal{F}(T) = \{1\}$ and $A(1) = -2 \in EP(f)$. So $\Gamma = \{p \in \mathcal{F}(T) : Ap \in EP(f)\} \neq \emptyset$.

Example 1.2 Let $E_1 = \mathbb{R}^2$ with the norm $\|\alpha\| = (a_1^2 + a_2^2)^{\frac{1}{2}}$ for $\alpha = (a_1, a_2) \in \mathbb{R}^2$ and $E_2 = \mathbb{R}$ with the standard norm $|\cdot|$. Let $C := \{\alpha = (a_1, a_2) \in \mathbb{R}^2 | a_1^2 + a_2^2 \leq 1\}$ and $K := [-2, 2]$. Let $A\alpha = -2a_1$ for $\alpha = (a_1, a_2) \in E_1$ and $T\alpha = (a_1^2, a_2^2)$ for all $\alpha = (a_1, a_2) \in C$. Then $\mathcal{F}(T) = \{(0,1), (1,0), (0,0)\}$ and A is a bounded and linear operator from E_1 into E_2 with $\|A\| = 2$. Now define a bi-function f as $f(u,v) = v - u$ for all $u, v \in K$. Then f is a bi-function from $K \times K$ into $\mathbb{R}$ with $EP(f) = \{-2\}$.

Clearly, $p = (1,0) \in \mathcal{F}(T)$, $Ap = -2 \in EP(f)$. So $\Gamma = \{p \in \mathcal{F}(T) : Ap \in EP(f)\} \neq \emptyset$.

Remark 1.1 It is worth to mention that the split common solution problem in Example 1.1 lies in two different subsets of the same space and the split common solution problem in Example 1.2 lies in two different subsets of the different space. So, Examples 1.1 and 1.2 also show that the split common solution problem is meaningful.

In this paper, we introduce a weak convergence algorithm and a strong convergence algorithm for the split common solution problem when the nonlinear operator T is a quasi-nonexpansive mapping. Some strong and weak convergence theorems are established. We also give some examples to illustrate our results.

2 Preliminaries

We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ weakly converges to x and $x_n \to x$ will symbolize strong convergence as usual.

A Banach space $(X, \|\cdot\|)$ is said to satisfy Opial's condition, if for each sequence $\{x_n\}$ in X which converges weakly to a point $x \in X$, we have

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|, \quad \forall y \in X, y \neq x.$$

It is well known that any Hilbert space satisfies Opial's condition.

Let K be a nonempty subset of real Hilbert spaces H. Recall that a mapping $T : K \to K$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in K$ and quasi-nonexpansive if $\mathcal{F}(T) \neq \emptyset$ and $\|Tx - Tp\| \leq \|x - p\|$ for all $x \in K$, $p \in \mathcal{F}(T)$.

Example 2.1 Let $H = \mathbb{R}$ with the inner product defined by $\langle x, y\rangle = xy$ for all $x, y \in \mathbb{R}$ and the standard norm $|\cdot|$. Let $C := [0, +\infty)$ and $Tx = \frac{x^2+2}{1+x}$ for all $x \in C$. Obviously, $\mathcal{F}(T) = \{2\}$. It is easy to see that

$$|Tx - 2| = \frac{x}{1+x}|x-2| \le |x-2| \quad \text{for } x \in C$$

and

$$\left|T(0) - T\left(\frac{1}{3}\right)\right| = \frac{5}{12} > \left|0 - \frac{1}{3}\right|.$$

Hence, T is a continuous quasi-nonexpansive mapping but not nonexpansive.

Definition 2.1 (see [16]) Let K be a nonempty closed convex subset of a real Hilbert space H and T a mapping from K into K. The mapping T is said to be demiclosed if, for any sequence $\{x_n\}$ which weakly converges to y, and if the sequence $\{Tx_n\}$ strongly converges to z, then $Ty = z$.

Remark 2.1 In Definition 2.1, the particular case of demiclosedness at zero is frequently used in some iterative convergence algorithms, which is the particular case when $z = \theta$, the zero vector of H; for more detail, one can refer to [16].

The following concept of zero-demiclosedness was introduced in [17].

Definition 2.2 (see [17]) Let K be a nonempty, closed, and convex subset of a real Hilbert space and T a mapping from K into K. The mapping T is called *zero-demiclosed* if $\{x_n\}$ in K satisfying $\|x_n - Tx_n\| \to 0$ and $x_n \rightharpoonup z \in K$ implies $Tz = z$.

The following result was essentially proved in [17], but we give the proof for the sake of completeness.

Proposition 2.1 *Let K be a nonempty, closed, and convex subset of a real Hilbert space with zero vector θ and T a mapping from K into K. Then the following statements hold.*

- *(a) T is zero-demiclosed if and only if $I - T$ is demiclosed at θ;*
- *(b) If T is a nonexpansive mappings and there is a bounded sequence $\{x_n\} \subset H$ such that $\|x_n - Tx_n\| \to 0$ as $n \to 0$, then T is zero-demiclosed.*

Proof Obviously, the conclusion (a) holds. To see (b), since $\{x_n\}$ is bounded, there is a subsequence $\{x_{n_k}\} \subset \{x_n\}$ and $z \in H$ such that $x_{n_k} \rightharpoonup z$. One can claim $Tz = z$. Indeed, if $Tz \neq z$, it follows from the Opial's condition that

$$\begin{aligned}
\liminf_{k\to\infty} \|x_{n_k} - z\| &< \liminf_{k\to\infty} \|x_{n_k} - Tz\| \\
&\le \liminf_{k\to\infty} \big\{ \|x_{n_k} - Tx_{n_k}\| + \|Tx_{n_k} - Tz\| \big\} \\
&= \liminf_{k\to\infty} \|Tx_{n_k} - Tz\| \\
&\le \liminf_{k\to\infty} \|x_{n_k} - z\|,
\end{aligned}$$

which is a contradiction. So $Tz = z$ and hence T is zero-demiclosed. □

Example 2.2 Let H, C, and T be the same as in Example 2.1. Let $\{x_n\}$ be a sequence in C. If $x_n \to z$ and $x_n - Tx_n \to 0$, then $z \in F(T) = \{2\}$. Indeed, since T is continuous, we have $Tz = z$ and $z \in F(T) = \{2\}$. Hence, T is zero-demiclosed.

Example 2.3 Let $H = \mathbb{R}$ with the inner product defined by $\langle x, y\rangle = xy$ for all $x, y \in \mathbb{R}$ and the standard norm $|\cdot|$. Let $C := [0, +\infty)$. Let T be a mapping from C into C defined by

$$Tx = \begin{cases} \frac{2x}{x^2+1}, & x \in (1, +\infty), \\ 0, & x \in [0, 1]. \end{cases}$$

Then T is a discontinuous quasi-nonexpansive mapping but not zero-demiclosed.

Proof Obviously, $\mathcal{F}(T) = \{0\}$, and T is a quasi-nonexpansive operator. On the other hand, let $x_n = 1 + \frac{1}{n}$ for all $n \in \mathbb{N}$, then it is not hard to prove that $x_n \to 1$, $x_n - Tx_n \to 0$ and $1 \notin \mathcal{F}(T)$. So T is not zero-demiclosed. □

Let H_1 and H_2 be two Hilbert spaces. Let $A : H_1 \to H_2$ and $B : H_2 \to H_1$ be two bounded linear operators. B is called the *adjoint operator* (or *adjoint*) of A, if for all $z \in H_1$, $w \in H_2$, B satisfies $\langle Az, w\rangle = \langle z, Bw\rangle$. It is known that the adjoint operator of a bounded linear operator on a Hilbert space always exists and is bounded linear and unique. Moreover, it is not hard to show that if B is an adjoint operator of A, then $\|A\| = \|B\|$.

Example 2.4 Let $H_1 = \mathbb{R}^3$ with the norm $\|\alpha\| = (a_1^2 + a_2^2 + a_3^2)^{\frac{1}{2}}$ for $\alpha = (a_1, a_2, a_3) \in \mathbb{R}^2$ and $H_2 = \mathbb{R}^4$ with the norm $\|\gamma\| = (c_1^2 + c_2^2 + c_3^2 + c_4^2)^{\frac{1}{2}}$ for $\gamma = (c_1, c_2, c_3, c_4) \in \mathbb{R}^4$. Let $\langle\alpha, \beta\rangle = a_1b_1 + a_2b_2 + a_3b_3$ and $\langle\gamma, \eta\rangle = c_1d_1 + c_2d_2 + c_3d_3 + c_4d_4$ denote the inner product of H_1 and H_2, respectively, where $\alpha = (a_1, a_2, a_3)$, $\beta = (b_1, b_2, b_3) \in H_1$, $\gamma = (c_1, c_2, c_3, c_4)$, $\eta = (d_1, d_2, d_3, d_4) \in H_2$. Let $A\alpha = (a_3, a_1 + a_2, a_1 - a_2, a_3)$ for $\alpha = (a_1, a_2, a_3) \in H_1$. Then A is a bounded linear operator from H_1 into H_2 with $\|A\| = \sqrt{2}$. For $\gamma = (c_1, c_2, c_3, c_4) \in H_2$, let $B\gamma = (c_2 + c_3, c_2 - c_3, c_1 + c_4)$. Then B is a bounded linear operator from H_2 into H_1 with $\|B\| = \sqrt{2}$. Moreover, for any $\alpha = (a_1, a_2, a_3) \in H_1$ and $\gamma = (c_1, c_2, c_3, c_4) \in H_2$, $\langle A\alpha, \gamma\rangle = \langle\alpha, B\gamma\rangle$, so B is an adjoint operator of A.

Let K be a closed and convex subset of a real Hilbert space H. For each point $x \in H$, there exists a unique nearest point in K, denoted by P_Kx, such that $\|x - P_Kx\| \le \|x - y\|$, $\forall y \in K$. The mapping P_K is called the *metric projection* from H onto K. It is well known that P_K has the following characterizations:

(i) $\langle x - y, P_Kx - P_Ky\rangle \ge \|P_Kx - P_Ky\|^2$ for every $x, y \in H$.

(ii) for $x \in H$, and $z \in K$, $z = P_K(x) \Longleftrightarrow \langle x - z, z - y\rangle \ge 0$, $\forall y \in K$.

(iii) $\|y - P_K(x)\|^2 + \|x - P_K(x)\|^2 \le \|x - y\|^2$ for all $x \in H$ and $y \in K$.

The following lemmas are crucial in our proofs.

Lemma 2.1 (see [1]) *Let K be a nonempty, closed, and convex subset of H and F be a bifunction of $K \times K$ into $\mathbb{R}$ satisfying the following conditions.*

(A1) $F(x, x) = 0$ for all $x \in K$;

(A2) F is monotone, that is, $F(x, y) + F(y, x) \le 0$ for all $x, y \in K$;

(A3) for each $x, y, z \in K$, $\limsup_{t\downarrow 0} F(tz + (1 - t)x, y) \le F(x, y)$;

(A4) for each $x \in K$, $y \mapsto F(x, y)$ is convex and lower semicontinuous.

Let $r > 0$ and $x \in H$. Then there exists $z \in K$ such that $F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0$, for all $y \in K$.

Lemma 2.2 (see [3]) *Let K be a nonempty, closed, and convex subset of H and let F be a bi-function of $K \times K$ into $\mathbb{R}$ satisfying (A1)-(A4). For $r > 0$, define a mapping $T_r^F : H \to K$ as follows:*

$$T_r^F(x) = \left\{ z \in K : F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in K \right\} \tag{2.1}$$

for all $x \in H$. Then the following hold:

(i) T_r^F is single-valued and $F(T_r^F) = EP(F)$ for any $r > 0$ and $EP(F)$ is closed and convex;
(ii) T_r^F is firmly nonexpansive, that is, for any $x, y \in H$,
$\|T_r^F x - T_r^F y\|^2 \leq \langle T_r^F x - T_r^F y, x - y\rangle$.

Lemma 2.3 (see, *e.g.*, [9]) *Let H be a real Hilbert space. Then the following hold.*

(a) $\|x + y\|^2 \leq \|y\|^2 + 2\langle x, x + y\rangle$ and $\|x - y\|^2 = \|x\|^2 + \|y\|^2 - 2\langle x, y\rangle$ for all $x, y \in H$;
(b) $\|\alpha x + (1 - \alpha)y\|^2 = \alpha\|x\|^2 + (1 - \alpha)\|y\|^2 - \alpha(1 - \alpha)\|x - y\|^2$ for all $x, y \in H$ and $\alpha \in [0, 1]$.

The following result is simple, but it is very useful in this paper; see also [18].

Lemma 2.4 *Let the mapping T_r^F be defined as (2.1). Then for $r, s > 0$ and $x, y \in H$,*

$$\left\| T_r^F(x) - T_s^F(y)\right\| \leq \|x - y\| + \frac{|s - r|}{s}\left\| T_s^F(y) - y\right\|.$$

In particular, $\|T_r^F(x) - T_r^F(y)\| \leq \|x - y\|$ for any $r > 0$ and $x, y \in H$, that is T_r^F is nonexpansive for any $r > 0$.

Proof For $r, s > 0$ and $x, y \in H$, by (i) of Lemma 2.2, $T_r^F(x) = z_1$ and $T_s^F(y) = z_2$ for some $z_1, z_2 \in K$. By the definition of T_r^F, we have

$$F(z_1, u) + \frac{1}{r}\langle u - z_1, z_1 - x\rangle \geq 0, \quad \forall u \in K \tag{2.2}$$

and

$$F(z_2, u) + \frac{1}{s}\langle u - z_2, z_2 - y\rangle \geq 0, \quad \forall u \in K. \tag{2.3}$$

So, combining (2.2), (2.3), and (A2), we get

$$\frac{1}{r}\langle z_2 - z_1, z_1 - x\rangle + \frac{1}{s}\langle z_1 - z_2, z_2 - y\rangle \geq 0,$$

or

$$\left\langle z_2 - z_1, \frac{z_1 - x}{r}\right\rangle - \left\langle z_2 - z_1, \frac{z_2 - y}{s}\right\rangle \geq 0,$$

or

$$\left\langle z_2 - z_1, \frac{s}{r}(z_1 - x)\right\rangle - \langle z_2 - z_1, z_2 - y\rangle \geq 0,$$

or

$$\left\langle z_2 - z_1, z_1 - x - \frac{r}{s}(z_2 - y)\right\rangle \geq 0,$$

or

$$\left\langle z_2 - z_1, z_1 - z_2 + z_2 - x - \frac{r}{s}(z_2 - y)\right\rangle \geq 0,$$

which implies

$$\|z_2 - z_1\|^2 \leq \left\langle z_2 - z_1, z_2 - x - \frac{r}{s}(z_2 - y)\right\rangle \leq \|z_2 - z_1\| \left\| z_2 - x - \frac{r}{s}(z_2 - y)\right\|,$$

and hence

$$\begin{aligned} \left\| T_r^F(x) - T_s^F(y)\right\| &= \|z_2 - z_1\| \\ &\leq \left\| z_2 - x - \frac{r}{s}(z_2 - y)\right\| \\ &\leq \|y - x\| + \left\| \left(1 - \frac{r}{s}\right)(z_2 - y)\right\| \\ &= \|y - x\| + \frac{|s - r|}{s}\left\| T_s^F(y) - y\right\|. \end{aligned}$$

In particular, the last inequality show that for any $r > 0$, T_r^F is nonexpansive. The proof is completed. □

3 Main results

In this section, we first introduce a weak convergence iterative algorithms for the split common solution problem.

Theorem 3.1 *Let H_1 and H_2 be two real Hilbert spaces and $C \subset H_1$ and $K \subset H_2$ be two nonempty closed convex sets. Let $T : C \to C$ be zero-demiclosed quasi-nonexpansive mappings and $f : K \times K \to \mathbb{R}$ be bi-functions with $\Gamma = \{p \in F(T) : Ap \in EP(f)\} \neq \emptyset$. Let $A : H_1 \to H_2$ be a bounded linear operator with its adjoint B.*

Given $x_1 \in C$ and $\eta \in (0,1)$. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated by

$$\begin{cases} u_n = T_{r_n}^f A x_n, \\ x_{n+1} = (1 - \alpha_n) y_n + \alpha_n T y_n, \\ y_n = P_C(x_n + \varepsilon B(T_{r_n}^f - I) A x_n), \quad \forall n \in \mathbb{N}, \end{cases} \tag{3.1}$$

where $\{r_n\} \subset (0, +\infty)$ with $\liminf_{n\to\infty} r_n > 0$, $\varepsilon \in (0, \frac{1}{\|B\|^2})$ is a constant, P_C is a projection operator from H_1 into C and $\{\alpha_n\}$ satisfies $\alpha_n \in [\eta, 1 - \eta]$ for $n \in \mathbb{N}$. Then $x_n \rightharpoonup p \in \Gamma$ and $u_n \rightharpoonup Ap \in EP(f)$.

Proof Let $x^* \in \Gamma$. Then $Ax^* \in EP(f)$. For each $n \in \mathbb{N}$, by Lemmas 2.2 and 2.3, we have

$$\begin{aligned}\|T_{r_n}^f Ax_n - T_{r_n}^f Ax^*\|^2 &\leq \langle T_{r_n}^f Ax_n - T_{r_n}^f Ax^*, Ax_n - Ax^*\rangle \\ &= \frac{1}{2}\left\{\|T_{r_n}^f Ax_n - Ax^*\|^2 + \|Ax_n - Ax^*\|^2 - \|T_{r_n}^f Ax_n - Ax_n\|^2\right\}.\end{aligned}$$

So,

$$\|T_{r_n}^f Ax_n - Ax^*\|^2 \leq \|Ax_n - Ax^*\|^2 - \|T_{r_n}^f Ax_n - Ax_n\|^2 \quad \text{for any } n \in \mathbb{N}. \tag{3.2}$$

By (b) of Lemma 2.3 and (3.2), for each $n \in \mathbb{N}$, we get

$$\begin{aligned}&2\varepsilon\langle x_n - x^*, B(T_{r_n}^f - I)Ax_n\rangle \\ &\quad= 2\varepsilon\langle A(x_n - x^*) + (T_{r_n}^f - I)Ax_n - (T_{r_n}^f - I)Ax_n, (T_{r_n}^f - I)Ax_n\rangle \\ &\quad= 2\varepsilon\left(\frac{1}{2}\|T_{r_n}^f Ax_n - Ax^*\|^2 + \frac{1}{2}\|(T_{r_n}^f - I)Ax_n\|^2 - \frac{1}{2}\|Ax_n - Ax^*\|^2 - \|(T_{r_n}^f - I)Ax_n\|^2\right) \\ &\quad\leq 2\varepsilon\left(\frac{1}{2}\|(T_{r_n}^f - I)Ax_n\|^2 - \|(T_{r_n}^f - I)Ax_n\|^2\right) = -\varepsilon\|(T_{r_n}^f - I)Ax_n\|^2.\end{aligned} \tag{3.3}$$

Note that for any $n \in \mathbb{N}$,

$$\|B(T_{r_n}^f - I)Ax_n\|^2 \leq \|B\|^2\|(T_{r_n}^f - I)Ax_n\|^2, \tag{3.4}$$

so it follows from (3.1), (3.3), and (3.4) that

$$\begin{aligned}&\|x_{n+1} - x^*\|^2 \\ &\quad= (1-\alpha_n)\|y_n - x^*\|^2 + \alpha_n\|Ty_n - x^*\|^2 - (1-\alpha_n)\alpha_n\|y_n - Ty_n\|^2 \\ &\quad\leq \|y_n - x^*\|^2 - \eta^2\|y_n - Ty_n\|^2 \\ &\quad= \|P_C(x_n + \varepsilon B(T_{r_n}^f - I)Ax_n) - P_C x^*\|^2 - \eta^2\|y_n - Ty_n\|^2 \\ &\quad\leq \|x_n + \varepsilon B(T_{r_n}^f - I)Ax_n - x^*\|^2 - \eta^2\|y_n - Ty_n\|^2 \\ &\quad= \|x_n - x^*\|^2 + \|\varepsilon B(T_{r_n}^f - I)Ax_n\|^2 + 2\varepsilon\langle x_n - x^*, B(T_{r_n}^f - I)Ax_n\rangle - \eta^2\|y_n - Ty_n\|^2 \\ &\quad\leq \|x_n - x^*\|^2 + \varepsilon^2\|B\|^2\|(T_{r_n}^f - I)Ax_n\|^2 - \varepsilon\|(T_{r_n}^f - I)Ax_n\|^2 - \eta^2\|y_n - Ty_n\|^2 \\ &\quad= \|x_n - x^*\|^2 - \varepsilon(1 - \varepsilon\|B\|^2)\|(T_{r_n}^f - I)Ax_n\|^2 - \eta^2\|y_n - Ty_n\|^2.\end{aligned} \tag{3.5}$$

Since $\varepsilon \in (0, \frac{1}{\|B\|^2})$, $\varepsilon(1 - \varepsilon\|B\|^2) > 0$, by (3.5), we obtain

$$\|x_{n+1} - x^*\| \leq \|y_n - x^*\| \leq \|x_n - x^*\| \tag{3.6}$$

and

$$\begin{aligned}&\eta^2\|y_n - Ty_n\|^2 + \varepsilon(1 - \varepsilon\|B\|^2)\|(T_{r_n}^f - I)Ax_n\|^2 \\ &\quad\leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \quad \text{for any } n \in \mathbb{N}.\end{aligned} \tag{3.7}$$

The inequality (3.6) implies that $\lim_{n\to\infty}\|x_n - x^*\|$ exists. Further, from (3.6) and (3.7), we get

$$\lim_{n\to\infty}\|x_n - x^*\| = \lim_{n\to\infty}\|y_n - x^*\|, \tag{3.8}$$

$$\lim_{n\to\infty}\|y_n - Ty_n\| = 0 \tag{3.9}$$

and

$$\lim_{n\to\infty}\|(T^f_{r_n} - I)Ax_n\| = 0. \tag{3.10}$$

From (3.1) and (3.10), we have

$$\begin{aligned}\|y_n - x_n\| &= \|P_C(x_n + \varepsilon B(T^f_{r_n} - I)Ax_n) - P_C x_n\| \\ &\le \varepsilon\|B(T^f_{r_n} - I)Ax_n\| \to 0 \quad \text{as } n\to\infty.\end{aligned} \tag{3.11}$$

Since $\lim_{n\to\infty}\|x_n - x^*\|$ exists, $\{x_n\}$ is bounded and hence $\{x_n\}$ has a weakly convergence subsequence $\{x_{n_j}\}$. Assume that $x_{n_j} \rightharpoonup p$ for some $p \in C$. Then $Ax_{n_j} \rightharpoonup Ap \in K$, $y_{n_j} \rightharpoonup p$ and $T^f_{r_{n_j}}Ax_{n_j} \rightharpoonup Ap$ by (3.10) and (3.11).

We argue $p \in \Gamma$. Since T is a zero-demiclosed mapping, by (3.9) and $y_{n_j} \rightharpoonup p$, we obtain $p \in \mathcal{F}(T)$. Applying Lemma 2.2, $EP(f) = \mathcal{F}(T^f_r)$ for any $r > 0$. We claim $T^f_r Ap = Ap$. If $T^f_r Ap \neq Ap$, since $Ax_n - T^f_{r_n}Ax_n = (I - T^f_{r_n})Ax_n \to 0$ as $n \to \infty$ from (3.10) and applying Opial's condition, we have

$$\begin{aligned}&\liminf_{j\to\infty}\|Ax_{n_j} - Ap\| \\ &\quad < \liminf_{j\to\infty}\|Ax_{n_j} - T^f_r Ap\| \\ &\quad = \liminf_{j\to\infty}\|Ax_{n_j} - T^f_{r_{n_j}}Ax_{n_j} + T^f_{r_{n_j}}Ax_{n_j} - T^f_r Ap\| \\ &\quad \le \liminf_{j\to\infty}\{\|Ax_{n_j} - T^f_{r_{n_j}}Ax_{n_j}\| + \|T^f_{r_{n_j}}Ax_{n_j} - T^f_r Ap\|\} \\ &\quad = \liminf_{j\to\infty}\|T^f_{r_{n_j}}Ax_{n_j} - T^f_r Ap\| \\ &\quad = \liminf_{j\to\infty}\|T^f_r Ap - T^f_{r_{n_j}}Ax_{n_j}\| \\ &\quad \le \liminf_{j\to\infty}\left(\|Ax_{n_j} - Ap\| + \frac{|r_{n_j} - r|}{r_{n_j}}\|T^f_{r_{n_j}}Ax_{n_j} - Ax_{n_j}\|\right) \quad \text{(by Lemma 2.4)} \\ &\quad = \liminf_{j\to\infty}\|Ax_{n_j} - Ap\|,\end{aligned}$$

which lead to a contradiction. So $Ap \in F(T^f_r) = EP(f)$, and hence we show $p \in \Gamma$.

Now, we prove $\{x_n\}$ converges weakly to $p \in \Gamma$. Otherwise, if there exists other subsequence of $\{x_n\}$ which is denoted by $\{x_{n_l}\}$ such that $x_{n_l} \rightharpoonup q \in \Gamma$ with $q \neq p$. Then, by Opial's condition,

$$\liminf_{l\to\infty}\|x_{n_l} - q\| < \liminf_{l\to\infty}\|x_{n_l} - p\| < \liminf_{l\to\infty}\|x_{n_l} - q\|.$$

This is a contradiction. Hence, $\{x_n\}$ converges weakly to an element $p \in \Gamma$.

Finally, we prove $\{u_n\} \equiv \{T^f_{r_n} Ax_n\}$ converges weakly to $Ap \in EP(f)$. Since $x_n \rightharpoonup p$, we have $Ax_n \rightharpoonup Ap$ as $n \to \infty$. Thus, by (3.10), we obtain $u_n \rightharpoonup Ap \in EP(f)$ as $n \to \infty$. The proof is completed. □

Corollary 3.1 *Let H_1 and H_2 be two real Hilbert spaces. Let $T : H_1 \to H_1$ be a zero-demiclosed quasi-nonexpansive mapping with $\mathcal{F}(T) \neq \emptyset$ and $f : H_2 \times H_2 \to \mathbb{R}$ be a bi-function with $EP(f) \neq \emptyset$. Let $A : H_1 \to H_2$ be a bounded linear operator with its adjoint B. Given $\eta \in (0,1)$. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated by*

$$\begin{cases} x_1 \in H_1, \\ u_n = T^f_{r_n} Ax_n, \\ x_{n+1} = (1-\alpha_n)y_n + \alpha_n Ty_n, \\ y_n = x_n + \varepsilon B(T^f_{r_n} - I)Ax_n, \quad \forall n \in \mathbb{N}, \end{cases} \tag{3.12}$$

where $\varepsilon \in (0, \frac{1}{\|B\|^2})$ and $\{r_n\} \subset (0, +\infty)$ with $\liminf_{n\to\infty} r_n > 0$. Suppose $\Gamma = \{p \in \mathcal{F}(T) : Ap \in EP(f)\} \neq \emptyset$ and the control coefficient sequence $\{\alpha_n\}$ satisfies $\alpha_n \in [\eta, 1-\eta]$ for $n \in \mathbb{N}$. Then the sequence $\{x_n\}$ converges weakly to an element $p \in \Gamma$ and $\{u_n\}$ weakly to $Ap \in EP(f)$.

Next, we introduce a strong convergence algorithm for the split common solution problem.

Theorem 3.2 *Let $C \subset H_1$ and $K \subset H_2$ be two nonempty, closed, and convex sets, $T : C \to C$ zero-demiclosed quasi-nonexpansive mappings and $f : K \times K \to \mathbb{R}$ a bi-function with $\Gamma = \{p \in \mathcal{F}(T) : Ap \in EP(f)\} \neq \emptyset$. Let $A : H_1 \to H_2$ be a bounded linear operator with the adjoint B. Given $x_1 \in C$, $C_1 = C$ and $\eta \in (0,1)$. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated by*

$$\begin{cases} u_n = T^f_{r_n} Ax_n, \\ y_n = (1-\alpha_n)z_n + \alpha_n Tz_n, \\ z_n = P_C(x_n + \varepsilon B(T^f_{r_n} - I)Ax_n)), \\ C_{n+1} = \{v \in C_n : \|y_n - v\| \le \|z_n - v\| \le \|x_n - v\|\}, \\ x_{n+1} = P_{C_{n+1}}(x_1), \quad n \in \mathbb{N}, \end{cases} \tag{3.13}$$

where $\{r_n\} \subset (0, +\infty)$ with $\liminf_{n\to\infty} r_n > 0$, P_C is a projection operator from H_1 into C and $\varepsilon \in (0, \frac{1}{\|B\|^2})$ is a constant, $\{\alpha_n\}$ satisfies $\alpha_n \in [\eta, 1-\eta]$ for $n \in \mathbb{N}$, then $x_n \to p \in \Gamma$ and $u_n \to Ap \in EP(f)$.

Proof First, we claim $\Gamma \subset C_n$ for $n \in \mathbb{N}$. In fact, let $p \in \Gamma$. Following the same argument as in Theorem 3.1, we have

$$2\varepsilon\langle x_n - p, B(T^f_{r_n} - I)Ax_n\rangle \le -\varepsilon\|(T^f_{r_n} - I)Ax_n\|^2, \tag{3.14}$$

and

$$\|B(T^f_{r_n} - I)Ax_n\|^2 \le \|B\|^2\|(T^f_{r_n} - I)Ax_n\|^2 \quad \text{for any } n \in \mathbb{N}. \tag{3.15}$$

By (3.13), (3.14), and (3.15), we get

$$\begin{aligned}
&\|y_n - p\|^2 \\
&\quad \le \|z_n - p\|^2 - (1-\alpha_n)\alpha_n \|z_n - Tz_n\|^2 \\
&\quad \le \big\|x_n + \varepsilon B\big(T^f_{r_n} - I\big)Ax_n - p\big\|^2 - \eta^2 \|z_n - Tz_n\|^2 \\
&\quad = \|x_n - p\|^2 + \big\|\varepsilon B\big(T^f_{r_n} - I\big)Ax_n\big\|^2 + 2\varepsilon\big\langle x_n - p, B\big(T^f_{r_n} - I\big)Ax_n\big\rangle - \eta^2\|z_n - Tz_n\|^2 \\
&\quad \le \|x_n - p\|^2 + \varepsilon^2\|B\|^2\big\|\big(T^f_{r_n} - I\big)Ax_n\big\|^2 - \varepsilon\big\|\big(T^f_{r_n} - I\big)Ax_n\big\|^2 - \eta^2\|z_n - Tz_n\|^2 \\
&\quad \le \|x_n - p\|^2 - \varepsilon\big(1 - \varepsilon\|B\|^2\big)\big\|\big(T^f_{r_n} - I\big)Ax_n\big\|^2 - \eta^2\|z_n - Tz_n\|^2 \\
&\quad \text{for any } n \in \mathbb{N}.
\end{aligned} \tag{3.16}$$

Notice $\varepsilon \in (0, \frac{1}{\|B\|^2})$, $\varepsilon(1 - \varepsilon\|B\|^2) > 0$. It follows from (3.16) that

$$\|y_n - p\| \le \|z_n - p\| \le \|x_n - p\| \quad \text{for all } n \in \mathbb{N},$$

and hence $p \in C_n$ for all $n \in \mathbb{N}$. Hence, $\Gamma \subset C_n$ and $C_n \neq \emptyset$ for all $n \in \mathbb{N}$.

Now, we prove C_n is a closed convex set for each $n \in \mathbb{N}$. It is not hard to verify that C_n is closed for each $n \in \mathbb{N}$, so it suffices to verify that C_n is convex for each $n \in \mathbb{N}$. Indeed, let $w_1, w_2 \in C_{n+1}$. For any $\gamma \in (0,1)$, since

$$\begin{aligned}
&\big\|y_n - \big(\gamma w_1 + (1-\gamma)w_2\big)\big\|^2 \\
&\quad = \big\|\gamma(y_n - w_1) + (1-\gamma)(y_n - w_2)\big\|^2 \\
&\quad = \gamma\|y_n - w_1\|^2 + (1-\gamma)\|y_n - w_2\|^2 - \gamma(1-\gamma)\|w_1 - w_2\|^2 \\
&\quad \le \gamma\|z_n - w_1\|^2 + (1-\gamma)\|z_n - w_2\|^2 - \gamma(1-\gamma)\|w_1 - w_2\|^2 \\
&\quad = \big\|z_n - \big(\gamma w_1 + (1-\gamma)w_2\big)\big\|^2,
\end{aligned}$$

we have $\|y_n - (\gamma w_1 + (1-\gamma)w_2)\| \le \|z_n - (\gamma w_1 + (1-\gamma)w_2)\|$. Similarly, we also have $\|z_n - (\gamma w_1 + (1-\gamma)w_2)\| \le \|x_n - (\gamma w_1 + (1-\gamma)w_2)\|$, which implies $\gamma w_1 + (1-\gamma)w_2 \in C_{n+1}$. Hence, we show that C_{n+1} is a convex set for each $n \in \mathbb{N}$.

Notice that $C_{n+1} \subset C_n$ and $x_{n+1} = P_{C_{n+1}}(x_1) \subset C_n$, then $\|x_{n+1} - x_1\| \le \|x_n - x_1\|$ for $n \in \mathbb{N}$ with $n \ge 2$. It follows that $\lim_{n\to\infty}\|x_n - x_1\|$ exists. Hence $\{x_n\}$ is bounded, which yields that $\{z_n\}$ and $\{y_n\}$ are bounded. For any $k, n \in \mathbb{N}$ with $k > n$, from $x_k = P_{C_k}(x_1) \subset C_n$ and the character (iii) of the projection operator P, we have

$$\|x_n - x_k\|^2 + \|x_1 - x_k\|^2 = \big\|x_n - P_{C_k}(x_1)\big\|^2 + \big\|x_1 - P_{C_k}(x_1)\big\|^2 \le \|x_n - x_1\|^2. \tag{3.17}$$

Since $\lim_{n\to\infty}\|x_n - x_1\|$ exists, by (3.17), we have $\lim_{n\to\infty}\|x_n - x_k\| = 0$, which implies that $\{x_n\}$ is a Cauchy sequence.

Let $x_n \to p$. One claim $p \in \Gamma$. Firstly, by $x_{n+1} = P_{C_{n+1}}(x_0) \in C_{n+1} \subset C_n$, from (3.13) we have

$$\|y_n - x_n\| \le \|y_n - x_{n+1}\| + \|x_{n+1} - x_n\| \le 2\|x_{n+1} - x_n\| \to 0 \quad \text{as } n \to \infty \tag{3.18}$$

and

$$\|z_n - x_n\| \le \|z_n - x_{n+1}\| + \|x_{n+1} - x_n\| \le 2\|x_{n+1} - x_n\| \to 0 \quad \text{as } n \to \infty. \tag{3.19}$$

Setting $\rho = \varepsilon(1 - \varepsilon\|B\|^2)$, from (3.16) again, we have

$$\begin{aligned}\rho\big\|\big(T^f_{r_n} - I\big)Ax_n\big\|^2 + \eta^2\|z_n - Tz_n\|^2 &\le \|x_n - p\|^2 - \|y_n - p\|^2 \\ &\le \|x_n - y_n\|\big\{\|x_n - p\| + \|y_n - p\|\big\}.\end{aligned}$$

So

$$\lim_{n\to\infty}\|Tz_n - z_n\| = 0 \tag{3.20}$$

and

$$\lim_{n\to\infty}\big\|\big(T^f_{r_n} - I\big)Ax_n\big\| = 0. \tag{3.21}$$

Let $r > 0$. Since $x_n \to p$ as $n \to \infty$, Lemma 2.4 and equation (3.21) imply that

$$\begin{aligned}\big\|T^f_r Ap - Ap\big\| &\le \big\|T^f_r Ap - T^f_{r_n}Ax_n\big\| + \big\|T^f_{r_n}Ax_n - Ax_n\big\| + \|Ax_n - Ap\| \\ &\le 2\|Ax_n - Ap\| + \left(1 + \frac{|r_n - r|}{r_n}\right)\big\|T^f_{r_n}Ax_n - Ax_n\big\| \to 0 \quad \text{as } n \to \infty.\end{aligned}$$

So $T^f_r Ap = Ap$, which say that $Ap \in \mathcal{F}(T^f_r) = EP(f)$. On the other hand, since $x_n - z_n \to 0$ by (3.19) and $x_n \to p$, we have $z_n \to p$. Notice that T is zero-demiclosed quasi-nonexpansive mappings, by (3.20), $Tp = p$, namely, $p \in \mathcal{F}(T)$. So $p \in \Gamma$. From (3.21), we also have $\{u_n\} \equiv \{T^f_{r_n}Ax_n\}$ converges strongly to $Ap \in EP(f)$. The proof is completed. □

Corollary 3.2 *Let H_1 and H_2 be two real Hilbert spaces. Let $T : H_1 \to H_2$ be a zero-demiclosed quasi-nonexpansive mappings with $F(T) \neq \emptyset$ and $f : H \times H \to \mathbb{R}$ be a bi-function with $EP(f) \neq \emptyset$. Let $A : H_1 \to H_2$ be a bounded linear operator with the adjoint B. Given $x_1 \in H_1$, $C_1 = H_1$, and $\eta \in (0,1)$. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated by*

$$\begin{cases} u_n = T^f_{r_n}Ax_n, \\ y_n = (1 - \alpha_n)z_n + \alpha_n Tz_n, \\ z_n = x_n + \varepsilon B(T^f_{r_n} - I)Ax_n, \\ C_{n+1} = \{v \in C_n : \|y_n - v\| \le \|z_n - v\| \le \|x_n - v\|\}, \\ x_{n+1} = P_{C_{n+1}}(x_1), \quad n \in \mathbb{N}, \end{cases} \tag{3.22}$$

where $\{r_n\} \subset (0, +\infty)$ with $\liminf_{n\to\infty} r_n > 0$, and $\varepsilon \in (0, \frac{1}{\|B\|^2})$ is a constant. Suppose that $\Gamma = \{p \in \mathcal{F}(T) : Ap \in EP(f)\} \neq \emptyset$ and the control coefficient sequence $\{\alpha_n\}$ satisfies $\alpha_n \in [\eta, 1 - \eta]$ for $n \in \mathbb{N}$, then the sequence $\{x_n\}$ converges strongly to an element $p \in \Gamma$ and $\{u_n\}$ converges strongly to $Ap \in EP(f)$.

Example 3.1 Let $H_1 = H_2 = \mathbb{R}$ with the inner product defined by $\langle x, y\rangle = xy$ for all $x, y \in \mathbb{R}$ and the standard norm $|\cdot|$. Let $C := [0, +\infty)$ and $Tx = \frac{x^2+2}{1+x}$ for all $x \in C$. From Examples 2.1 and 2.2, we know that T is an zero-demiclosed quasi-nonexpansive mapping with $F(T) = \{2\}$.

Let $K := [-\infty, 0]$ and $f_1(x, y) = (y - x)(x + 4)$ for all $x, y \in K$, then f satisfies the condition (A1)-(A4) and $EP(f) = \{-4\}$. Let $Ax = -2x$ for all $x \in \mathbb{R}$, then A is a bounded linear operator with B (the adjoint of A) = A and $\|A\| = \|B\| = 2$.

Obviously, $\Gamma = \{p \in F(T) : Ap \in EP(f)\} = \{2\} = \mathcal{F}(T)$, so $\Gamma \neq \emptyset$. Let $x_1 \in C$, $\{x_n\}$ and $\{u_n\}$ be sequences generated by

$$\begin{cases} u_n = T^f_{r_n} Ax_n, \\ x_{n+1} = (1 - \alpha_n)y_n + \alpha_n Ty_n, \\ y_n = P_C(x_n + \frac{1}{8}B(T^f_{r_n} - I)Ax_n), \quad \forall n \in \mathbb{N}, \end{cases} \tag{3.23}$$

where, $r_n = 1$ and $\alpha_n \in (0, 1)$ for all $n \in \mathbb{N}$, P_C is a projection operator from H_1 into C. Then the sequence $\{x_n\}$ converges strongly to $2 \in \Gamma$ and $\{u_n\}$ converges strongly to $A(2) = -4 \in EP(f)$.

Proof

(i) Firstly, for given $r_n = 1$ for $n \in \mathbb{N}$, we prove that for any $\{x_n\} \subset C$, there exists a unique sequence $\{u_n\}_{n\in\mathbb{N}} \equiv \{-x_n - 2\}_{n\in\mathbb{N}}$ in K such that

$$f(u_n, v) + \langle v - u_n, u_n - Ax_n\rangle \geq 0, \quad \forall v \in K, n \in \mathbb{N}. \tag{3.24}$$

Because (3.24) is equivalent with

$$\begin{aligned}(v - u_n)\big(u_n + 4 + (u_n + 2x_n)\big) \\ = (v - u_n)\big(u_n + 4 + (u_n - Ax_n)\big) \geq 0, \quad \forall v \in K, n \in \mathbb{N},\end{aligned} \tag{3.25}$$

while (3.25) is true if and only if $u_n = -(x_n + 2)$ for all $n \in \mathbb{N}$. So the conclusion is true.

(ii) Secondly, it is not hard to compute $B(T^f_{r_n} - I)Ax_n = B(u_n - Ax_n) = -2(x_n - 2)$ for all $n \in \mathbb{N}$. Hence,

$$x_n + \frac{1}{8}B\big(T^f_{r_n} - I\big)Ax_n = \frac{3}{4}x_n + \frac{1}{2} \in C \quad \text{for all } n \in \mathbb{N}.$$

(iii) By (i) and (ii), for $x_1 \in C$, we can rewrite the algorithm (3.23) as follows:

$$x_{n+1} = (1 - \alpha_n)y_n + \alpha_n Ty_n, \quad y_n = \frac{3}{4}x_n + \frac{1}{2} \tag{3.26}$$

and

$$u_n = T^f_{r_n} Ax_n = -(x_n + 2), \quad \forall n \in \mathbb{N}. \tag{3.27}$$

As in Example 2.1, we easily obtain $|Ty_n - 2| \leq |y_n - 2|$. Hence, from (3.26) and (3.27), we get

$$\begin{aligned} |x_{n+1} - 2| &\leq (1 - \alpha_n)|y_n - 2| + \alpha_n |Ty_n - 2| \\ &\leq |y_n - 2| = \frac{3}{4}|x_n - 2| \\ &\leq \cdots \\ &\leq \left(\frac{3}{4}\right)^n |x_1 - 2|, \quad \forall n \in \mathbb{N}, \end{aligned}$$

which shows $x_n \to 2 \in \mathcal{F}(T) = \Gamma$. Since $u_n = -(x_n + 2)$, $n \in \mathbb{N}$, we obtain $u_n \to -4 = A(2) \in EP(f)$.

□

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both authors contributed equally and significantly in writing this paper. Both authors read and approved the final manuscript.

Author details
[1]Department of Mathematics, Honghe University, Yunnan, 661100, China. [2]Department of Mathematics, National Kaohsiung Normal University, Kaohsiung, 824, Taiwan.

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