Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

Fixed point theorems of (a, b)-monotone mappings in Hilbert spaces

Lai-Jiu Lin* and Sung-Yu Wang

*Correspondence:
maljlin@cc.ncue.edu.tw
Department of Mathematics,
National Changhua University of
Education, Changhua, 50058,
Taiwan

Abstract
We propose a new class of nonlinear mappings, called (a, b)-monotone mappings, and show that this class of nonlinear mappings contains nonspreading mappings, hybrid mappings, firmly nonexpansive mappings, and $(a_1, a_2, a_3, k_1, k_2)$-generalized hybrid mappings with $a_1 < 1$. We also give an example to show that a (a, b)-monotone mapping is not necessary to be a quasi-nonexpansive mapping. We establish an existence theorem of fixed points and the demiclosed principle for the class of (a, b)-monotone mappings. As a special case of our result, we give an existence theorem of fixed points for $(a_1, a_2, a_3, k_1, k_2)$-generalized hybrid mappings with $a_1 < 1$. We also consider Mann's type weak convergence theorem and CQ type strong convergence theorem for (a, b)-monotone mappings. We give an example of (a, b)-monotone mappings which assures the Mann's type weak convergence.

Keywords: fixed point; demiclosed principle; strong convergence; weak convergence; nonspreading mapping; hybrid mapping; nonexpansive mapping; (a, b)- monotone mapping; Mann's type iteration; CQ type iteration

1 Introduction

Let H be a real Hilbert space with a nonempty closed convex subset C. Let $T : C \to C$ be a self-mapping defined on C. We denote by $F(T)$ the set of fixed points of T. The mapping T is called quasi-nonexpansive if $F(T) \neq \emptyset$ and

$$\|Tx - Ty\| \leq \|x - y\| \quad \text{for all } x \in C \text{ and } y \in F(T).$$

Takahashi *et al.* [1–7] gave the following definitions of nonlinear mappings and studied the existence and convergence theorems of fixed points for these mappings.

Definition 1.1 A mapping $T : C \to C$ is called

(i) nonspreading [1] if for every $x, y \in C$,

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2,$$

(ii) TY [3] if for every $x, y \in C$,

$$2\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - y\|^2,$$

(iii) hybrid [4] if for every $x, y \in C$,

$$3\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - y\|^2 + \|Ty - x\|^2,$$

(iv) λ-hybrid ($\lambda \in \mathbb{R}$) [5] if for every $x, y \in C$,

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + 2\lambda\langle x - Tx, y - Ty\rangle,$$

(v) (α, β)-generalized hybrid ($\alpha, \beta \in \mathbb{R}$) [6] if for every $x, y \in C$,

$$\alpha\|Tx - Ty\|^2 + (1 - \alpha)\|Ty - x\|^2 \leq \beta\|Tx - y\|^2 + (1 - \beta)\|x - y\|^2,$$

(vi) α-nonexpansive ($\alpha \in (-\infty, 1)$) [7] if for every $x, y \in C$,

$$\|Tx - Ty\|^2 \leq \alpha\|Tx - y\|^2 + \alpha\|Ty - x\|^2 + (1 - 2\alpha)\|x - y\|^2.$$

It is obvious that the mappings mentioned in Definition 1.1 are quasi-nonexpansive. Recently, Lin *et al.* [8] gave the following definition of a new class of nonlinear mappings.

Definition 1.2 [8] Let $a_1 \in [0, 1]$, $a_2, a_3 \in [0, 1)$, $k_1, k_2 \in [0, 1 - a_2) \cap [0, 1 - a_3)$ and $a_1 + a_2 + a_3 = 1$. A mapping $T : C \to C$ is called a $(a_1, a_2, a_3, k_1, k_2)$-generalized hybrid mapping if for every $x, y \in C$,

$$\|Tx - Ty\|^2 \leq a_1\|x - y\|^2 + a_2\|Tx - y\|^2 + a_3\|Ty - x\|^2 + k_1\|x - Tx\|^2 + k_2\|y - Ty\|^2.$$

This class of mappings are not necessary to be quasi-nonexpansive and contains nonexpansive mappings, nonspreading mappings, hybrid mappings, and TY mappings. Lin *et al.* [8] studied weak and strong convergence theorems of $(a_1, a_2, a_3, k_1, k_2)$-generalized hybrid mappings, but existence theorems of fixed points for $(a_1, a_2, a_3, k_1, k_2)$-generalized hybrid mapping are not discussed in [8]. On the other hand, Aoyama and Kohsaka [7] characterized the existence of fixed points of α-nonexpansive mappings in uniformly convex Banach spaces.

Motivated by the literatures above, we study existence theorems of fixed points for the mappings mentioned in Definitions 1.1 and 1.2 in an unified method. Precisely, we propose a new class of nonlinear mappings in Hilbert spaces.

Definition 1.3 Let $a \in (\frac{1}{2}, \infty)$ and $b \in (-\infty, a)$. A mapping $T : C \to C$ is called an (a, b)-monotone mapping if for every $x, y \in C$,

$$\langle x - y, Tx - Ty\rangle \geq a\|Tx - Ty\|^2 + (1 - a)\|x - y\|^2 - b\|x - Tx\|^2 - b\|y - Ty\|^2$$

or equivalently,

$$\begin{aligned} &a\|Tx - Ty\|^2 + (1 - a)\|x - y\|^2 \\ &\quad\leq \frac{1}{2}\|x - Ty\|^2 + \frac{1}{2}\|y - Tx\|^2 + \left(b - \frac{1}{2}\right)\left(\|x - Tx\|^2 + \|y - Ty\|^2\right). \end{aligned}$$

Remark 1.1 Let C be a nonempty, closed, and convex subset of a Hilbert space, and let $\alpha > 0$. Recall that a mapping $T : C \to C$ is called α-inverse strongly monotone if

$$\langle x - y, Tx - Ty\rangle \geq \alpha \| Tx - Ty\|^2 \quad \text{for all } x, y \in C.$$

A firmly nonexpansive mapping is an α-inverse strongly monotone mapping with $\alpha = 1$. Note that a firmly nonexpansive mapping (1-inverse strongly monotone mapping with $\alpha = 1$) is a (1,0)-monotone mapping.

Next, we give an example to show that a (a, b)-monotone mapping is not necessary to be a quasi-nonexpansive mapping.

Example 1.1 Let $H = \mathbb{R}^2$. Let $\phi : \mathbb{R} \times H \to H$ and $T : H \to H$ be defined by

$$\begin{aligned} &\phi(\alpha, x) = \big(r\cos(\theta + \alpha), r\sin(\theta + \alpha)\big), \\ &Tx = \frac{5}{4}\phi\left(\frac{3}{4}\pi, x\right) = \frac{5}{4}\left(r\cos\left(\theta + \frac{3}{4}\pi\right), r\sin\left(\theta + \frac{3}{4}\pi\right)\right) \end{aligned}$$

for all $\alpha \in \mathbb{R}$ and for all $x = (r\cos\theta, r\sin\theta) \in H$. Then the following statements hold:

(i) T is a $(4, 3)$-monotone mapping;
(ii) T is not a quasi-nonexpansive mapping.

Proof It's obvious that $F(T) = \{0\}$. We first prove part (i). For each $x \in H$,

$$Tx = \frac{5}{4}\phi\left(\frac{3}{4}\pi, x\right) = \frac{5\sqrt{2}}{8}\phi(\pi, x) + \frac{5\sqrt{2}}{8}\phi\left(\frac{\pi}{2}, x\right).$$

Then for each $x, y \in H$, we have

(1) $\langle x - y, Tx - Ty\rangle = -\frac{5\sqrt{2}}{8}\|x - y\|^2$,
(2) $\| Tx - Ty\|^2 = \frac{25}{16}\|x - y\|^2$,
(3)

$$\|x - Tx\|^2 = \|x\|^2 + \| Tx\|^2 - 2\|x\| \| Tx\| \cos\frac{3}{4}\pi = \left(\frac{41}{16} + \frac{5\sqrt{2}}{4}\right)\|x\|^2.$$

Then

$$\begin{aligned} &\langle x - y, Tx - Ty\rangle - a\| Tx - Ty\|^2 - (1 - a)\|x - y\|^2 + b\|x - Tx\|^2 + b\|y - Ty\|^2 \\ &= -\frac{5\sqrt{2}}{8}\|x - y\|^2 - \frac{25}{16}a\|x - y\|^2 - (1 - a)\|x - y\|^2 + b\|x - Tx\|^2 + b\|y - Ty\|^2 \\ &= \left(\frac{41}{16} + \frac{5\sqrt{2}}{4}\right)b\big(\|x\|^2 + \|y\|^2\big) - \left(\frac{25}{16}a + \frac{5\sqrt{2}}{8} + 1 - a\right)\|x - y\|^2. \end{aligned}$$

By parallelogram law, we have

$$\begin{aligned} &\langle x - y, Tx - Ty\rangle - a\| Tx - Ty\|^2 - (1 - a)\|x - y\|^2 + b\|x - Tx\|^2 + b\|y - Ty\|^2 \\ &\geq \left[\left(\frac{41}{32} + \frac{5\sqrt{2}}{8}\right)b - \left(\frac{25}{16}a + \frac{5\sqrt{2}}{8} + 1 - a\right)\right]\|x - y\|^2. \end{aligned}$$

Take $a = 4$ and $b = 3$. Then

$$\begin{aligned}
&\langle x-y, Tx-Ty\rangle - a\|Tx-Ty\|^2 - (1-a)\|x-y\|^2 + b\|x-Tx\|^2 + b\|y-Ty\|^2 \\
&\geq \left[3\left(\frac{41}{32} + \frac{5\sqrt{2}}{8}\right) - \left(4\cdot\frac{25}{16} + \frac{5\sqrt{2}}{8} + 1 - 4\right)\right]\|x-y\|^2 \\
&= \left(\frac{19}{32} + \frac{10\sqrt{2}}{8}\right)\|x-y\|^2 \geq 0.
\end{aligned}$$

Then T is a $(4,3)$-monotone mapping. Next we want to prove part (ii). Since $\|Tx - T0\| = \frac{5}{4}\|x-0\|$, T is not a quasi-nonexpansive mapping. The proof of part (ii) is complete. □

Remark 1.2 Since T in Example 1.1 is not a quasi-nonexpansive mapping, T is not nonspreading, TY, hybrid, λ-hybrid, (α,β)-generalized hybrid, and α-nonexpansive. This example shows that an (a,b)-monotone mapping is not necessary to be a quasi-nonexpansive mapping, TY mapping, hybrid mapping, λ-hybrid mapping, (α,β)-generalized hybrid mapping, and α-nonexpansive mapping.

In this paper, we first show that the class of (a,b)-monotone mappings contains nonspreading mappings, hybrid mappings, TY mappings, firmly nonexpansive mappings, and (a_1,a_2,a_3,k_1,k_2)-generalized hybrid mappings with $a_1 < 1$. We also give an example to show that this class of mappings are not necessary to be quasi-nonexpansive mappings. We establish an existence theorem of fixed points and the demiclosed principle for the class of (a,b)-monotone mappings. As a special case of our result, we give an existence theorem of fixed points for (a_1,a_2,a_3,k_1,k_2)-generalized hybrid mappings with $a_1 < 1$. We also consider Mann's type weak convergence theorem and CQ type strong convergence theorem for (a,b)-monotone mappings. An example of (a,b)-monotone mappings is given to show the Mann's type weak convergence.

2 Preliminaries

In this paper, we use the following notations:

(i) $\rightharpoonup$ for weak convergence and $\to$ for strong convergence.

(ii) $\omega_w(x_n) = \{x : \exists x_{n_j} \rightharpoonup x\}$ denotes the weak ω-limit set of $\{x_n\}$.

Let us recall some known results, which will be used later.

Proposition 2.1 [8] *Let C be a nonempty, closed, and convex subset of a Hilbert space H. A mapping $T : C \to C$ be a mapping.*

(i) If T is a nonexpansive mapping, then T is a $(1,0,0,0,0)$-generalized hybrid mapping;

(ii) If T is a nonspreading mapping, then T is a $(0,\frac{1}{2},\frac{1}{2},0,0)$-generalized hybrid mapping;

(iii) If T is a hybrid mapping, then T is a $(\frac{1}{3},\frac{1}{3},\frac{1}{3},0,0)$-generalized hybrid mapping;

(iv) If T is a TY mapping, then T is a $(\frac{1}{2},\frac{1}{2},0,0,0)$-generalized hybrid mapping;

(v) If T is an (α,β)-generalized hybrid mapping with $\alpha \geq 1 \geq \beta \geq 0$ and $\alpha > \beta$, then T is a $(\frac{1-\beta}{\alpha},\frac{\beta}{\alpha},1-\frac{1}{\alpha},0,0)$-generalized hybrid mapping.

Lemma 2.1 [3] *Let C be a nonempty, closed and convex subset of a Hilbert space H. Let $T : C \to C$ be a mapping. Suppose that there exist $x \in C$ and a Banach limit μ such that*

$\{T^n x\}$ is bounded and

$$\mu_n \| T^n x - Ty \|^2 \le \mu_n \| T^n x - y \|^2 \quad \text{for all } y \in C.$$

Then T has a fixed point.

Lemma 2.2 [9] *Let H be a real Hilbert space. Let C be a closed convex subset of H, let $w, x, y \in H$ and let a be a real number. The set*

$$D := \{ v \in C : \|y - v\|^2 \le \|x - v\|^2 + \langle w, v \rangle + a \}$$

is closed and convex.

Lemma 2.3 *Let K be a closed convex subset of a real Hilbert space H and let P_K be the metric projection from H onto K. Let $x \in H$ and $z \in K$. Then $z = P_K x$ if and only if*

$$\langle x - z, y - z \rangle \le 0 \quad \text{for all } y \in K.$$

Lemma 2.4 [9] *Let K be a closed convex subset of a real Hilbert space H. Let $\{x_n\}$ be a sequence in H and $u \in H$. Let $q = P_K u$. Suppose that $\omega_w(x_n) \subseteq K$ and*

$$\|x_n - u\| \le \|u - q\| \quad \text{for all } n \in \mathbb{N}.$$

Then $x_n \to q$.

Lemma 2.5 *Let H be a real Hilbert space. Then*

$$\|u - v\|^2 = \|u\|^2 - \|v\|^2 - 2\langle u - v, v \rangle, \quad \text{for all } u, v \in H.$$

Theorem 2.1 [10] *Let H be a Hilbert space and let $\{x_n\}$ be a bounded sequence in H. Then $\{x_n\}$ is weakly convergent if and only if each weakly convergent subsequence of $\{x_n\}$ has the same weak limit, that is, for $x \in H$,*

$$x_n \rightharpoonup x \quad \Leftrightarrow \quad (x_{n_i} \rightharpoonup y \Rightarrow x = y).$$

3 Fixed point theorem of (a, b)-monotone mappings

Proposition 3.1 *Let C be a nonempty, closed, and convex subset of a Hilbert space H. If $T : C \to C$ is a $(a_1, a_2, a_3, k_1, k_2)$-generalized hybrid mapping with $a_1 < 1$, then T is a $(\frac{1}{1-a_1}, \frac{k}{1-a_1} + \frac{1}{2})$-monotone mapping, where $k = \max\{k_1, k_2\}$.*

Proof If T is an $(a_1, a_2, a_3, k_1, k_2)$-generalized hybrid mapping with $a_1 < 1$, then for every $x, y \in C$,

$$\begin{aligned} &\frac{1}{1-a_1} \| Tx - Ty \|^2 + \frac{-a_1}{1-a_1} \|x - y\|^2 \\ &\quad \le \frac{a_2}{1-a_1} \| Tx - y \|^2 + \frac{a_3}{1-a_1} \| Ty - x \|^2 + \frac{k}{1-a_1} \|x - Tx\|^2 + \frac{k}{1-a_1} \|y - Ty\|^2 \end{aligned}$$

and

$$\begin{aligned}&\frac{1}{1-a_1}\|Tx-Ty\|^2+\frac{-a_1}{1-a_1}\|x-y\|^2\\&\quad\le\frac{a_2}{1-a_1}\|Ty-x\|^2+\frac{a_3}{1-a_1}\|Tx-y\|^2+\frac{k}{1-a_1}\|x-Tx\|+\frac{k}{1-a_1}\|y-Ty\|^2,\end{aligned}$$

where $k=\max\{k_1,k_2\}$.

Note that

$$\begin{aligned}&\left(\frac{a_2}{1-a_1}\|y-Tx\|^2+\frac{a_3}{1-a_1}\|x-Ty\|^2\right)+\left(\frac{a_2}{1-a_1}\|x-Ty\|^2+\frac{a_3}{1-a_1}\|y-Tx\|^2\right)\\&\quad=\frac{a_2+a_3}{1-a_1}\|x-Ty\|^2+\frac{a_2+a_3}{1-a_1}\|y-Tx\|^2=\|y-Tx\|^2+\|x-Ty\|^2.\end{aligned}$$

We have

$$\begin{aligned}&\frac{1}{1-a_1}\|Tx-Ty\|^2+\frac{-a_1}{1-a_1}\|x-y\|^2\\&\quad\le\frac{1}{2}\|Tx-y\|^2+\frac{1}{2}\|Ty-x\|^2+\frac{k}{1-a_1}\|x-Tx\|^2+\frac{k}{1-a_1}\|y-Ty\|^2.\end{aligned}$$

Without loss of generality, we may assume that $a_2\ge a_3$.

Since $k<\min\{1-a_2,1-a_3\}$, we have that

$$\begin{aligned}\frac{k}{1-a_1}+\frac{1}{2}-\frac{1}{1-a_1}&=\frac{1}{2(1-a_1)}\bigl(2k+(1-a_1)-2\bigr)\\&<\frac{1}{2(1-a_1)}\bigl(2(1-a_2)+(1-a_1)-2\bigr)=\frac{1}{2(1-a_1)}(1-a_1-2a_2)\\&\le\frac{1}{2(1-a_1)}(1-a_1-a_2-a_3)=0,\end{aligned}$$

that is, $\frac{k}{1-a_1}+\frac{1}{2}<\frac{1}{1-a_1}$. Take $a=\frac{1}{1-a_1}\ge 1>\frac{1}{2}$ and $b=\frac{k}{1-a_1}+\frac{1}{2}<a$, we see that T is an (a,b)-monotone mapping. □

The following proposition follows immediately from Propositions 2.1 and 3.1.

Proposition 3.2 *Let C be a nonempty, closed, and convex subset of a Hilbert space H. A mapping $T:C\to C$ be a mapping.*

(i) *If T is a nonspreading mapping, then T is a $(1,\frac{1}{2})$-monotone mapping;*
(ii) *If T is a hybrid mapping, then T is a $(\frac{3}{2},\frac{1}{2})$-monotone mapping;*
(iii) *If T is a TY mapping, then T is a $(2,\frac{1}{2})$-monotone mapping;*
(vi) *If T is an (α,β)-generalized hybrid mapping with $\alpha\ge 1\ge\beta\ge 0$, $\alpha>\beta$ and $\alpha+\beta>1$, then T is an $(\frac{\alpha}{\alpha+\beta-1},\frac{1}{2})$-monotone mapping.*

Proposition 3.3 *Let C be a closed convex subset of a Hilbert space. Let T be a (a,b)-monotone mapping defined on C. Then*

$$\|x-p\|^2\ge\|Tx-p\|^2+\frac{1-2b}{2a-1}\|x-Tx\|^2\quad\textit{for all } x\in C \textit{ and } p\in F(T).$$

Proof Since T is a (a,b)-monotone mapping, we have that for each $x \in C$ and $p \in F(T)$,

$$\langle x-p, Tx-p\rangle \geq a\|Tx-p\|^2 + (1-a)\|x-p\|^2 - b\|x-Tx\|^2,$$

that is,

$$\begin{aligned}&\frac{1}{2}\left(\|x-p\|^2 + \|p-Tx\|^2 - \|x-Tx\|^2\right)\\&\quad\geq a\|Tx-p\|^2 + (1-a)\|x-p\|^2 - b\|x-Tx\|^2.\end{aligned}$$

Then

$$(2a-1)\|x-p\|^2 \geq (2a-1)\|Tx-p\|^2 + (1-2b)\|x-Tx\|^2,$$

that is,

$$\|x-p\|^2 \geq \|Tx-p\|^2 + \frac{1-2b}{2a-1}\|x-Tx\|^2. \qquad \square$$

Now we give a demiclosed principle of (a,b)-monotone mappings:

Theorem 3.1 *Let C be a closed convex subset of a Hilbert space. Let T be a (a,b)-monotone mapping defined on C. If a sequence $\{x_n\} \subseteq C$ with $x_n \rightharpoonup x^*$ and $\|x_n - Tx_n\| \to 0$. Then $x^* = Tx^*$.*

Proof Since T is a (a,b)-monotone mapping, we have that

$$\begin{aligned}&\left\langle x_n - x^*, Tx_n - Tx^*\right\rangle\\&\quad\geq a\left\|Tx_n - Tx^*\right\|^2 + (1-a)\left\|x_n - x^*\right\|^2 - b\|x_n - Tx_n\|^2 - b\left\|x^* - Tx^*\right\|^2,\end{aligned}$$

that is,

$$\begin{aligned}&b\|x_n - Tx_n\|^2 + b\left\|x^* - Tx^*\right\|^2\\&\quad\geq a\left\langle Tx_n - Tx^*, Tx_n - Tx^* - x_n + x^*\right\rangle\\&\qquad+ (1-a)\left\langle x_n - x^*, x_n - x^* - Tx_n + Tx^*\right\rangle\\&\quad= a\left\langle Tx_n - Tx^*, Tx_n - x_n\right\rangle + a\left\langle Tx_n - Tx^*, x^* - Tx^*\right\rangle\\&\qquad+ (1-a)\left\langle x_n - x^*, x_n - Tx_n\right\rangle + (1-a)\left\langle x_n - x^*, Tx^* - x^*\right\rangle\\&\quad= a\left\langle Tx_n - Tx^*, Tx_n - x_n\right\rangle\\&\qquad+ a\left\langle Tx_n - x_n, x^* - Tx^*\right\rangle + a\left\langle x_n - x^*, x^* - Tx^*\right\rangle + a\left\langle x^* - Tx^*, x^* - Tx^*\right\rangle\\&\qquad+ (1-a)\left\langle x_n - x^*, x_n - Tx_n\right\rangle + (1-a)\left\langle x_n - x^*, Tx^* - x^*\right\rangle.\end{aligned}$$

Since $x_n \rightharpoonup x^*$ and $\|x_n - Tx_n\| \to 0$, $\{x_n\}$ and $\{Tx_n\}$ are bounded. Taking limit on the inequality above, we have

$$b\left\|x^* - Tx^*\right\|^2 \geq a\left\|x^* - Tx^*\right\|^2.$$

Since $b < a$, we have that $\|x^* - Tx^*\| = 0$, that is, $x^* = Tx^*$. $\square$

Corollary 3.1 [2–4] *Let C be a closed convex subset of a Hilbert space. Let T be a self-mapping defined on C and satisfies one of the following:*

(i) T is a nonspreading mapping;
(ii) T is a hybrid mapping;
(iii) T is a TY mapping.

If a sequence $\{x_n\} \subseteq C$ with $x_n \rightharpoonup x^*$ and $\|x_n - Tx_n\| \to 0$, then $x^* = Tx^*$.

Theorem 3.2 *Let C be a closed convex subset of a Hilbert space. Let T be a (a,b)-monotone mapping defined on C. If $F(T)$ is nonempty, then $F(T)$ is closed and convex.*

Proof First, we show that $F(T)$ is closed. For each $x \in \overline{F(T)}$, there exists a sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq F(T)$ with $x_n \to x$. Since $x_n \to x$ and $x_n \in F(T)$ for all $n \in \mathbb{N}$, we have that $x_n \rightharpoonup x$ and $\|x_n - Tx_n\| = 0$ for all $n \in \mathbb{N}$. By Theorem 3.1, $x = Tx$. Next, we want to show that $F(T)$ is a convex subset of C. Take any $u, v \in F(T)$ and $t \in [0,1]$. Let $z_t := tu + (1-t)v$. By Proposition 3.3, we have

$$
\begin{aligned}
&\|Tz_t - z_t\|^2 \\
&\quad = t\|Tz_t - u\|^2 + (1-t)\|Tz_t - v\|^2 - t(1-t)\|u - v\|^2 \\
&\quad \le t\left(\|z_t - u\|^2 + \frac{2b-1}{2a-1}\|Tz_t - z_t\|^2\right) \\
&\qquad + (1-t)\left(\|z_t - v\|^2 + \frac{2b-1}{2a-1}\|Tz_t - z_t\|^2\right) - t(1-t)\|u - v\|^2 \\
&\quad = t(1-t)^2\|u - v\|^2 + (1-t)t^2\|u - v\|^2 \\
&\qquad + \frac{2b-1}{2a-1}\|Tz_t - z_t\|^2 - t(1-t)\|u - v\|^2 \\
&\quad = \frac{2b-1}{2a-1}\|Tz_t - z_t\|^2.
\end{aligned}
$$

Since $\frac{2b-1}{2a-1} < 1$, we have that $z_t = Tz_t$. □

Theorem 3.3 *Let C be a nonempty subset of a Hilbert space H. Let $T : C \to C$ be a (a,b)-monotone mapping with $b \in (-\infty, \frac{1}{2}]$. Suppose that $\{T^n x\}$ is bounded for some $x \in C$. Then $\mu_n\|T^n x - Ty\|^2 \le \mu_n\|T^n x - y\|^2$ for all Banach limits μ and for all $y \in C$.*

Proof Let μ be a Banach limit and let $y \in C$ be given. Since T is a (a,b)-monotone mapping with $b \in (-\infty, \frac{1}{2}]$, we have that

$$
\begin{aligned}
&\langle T^n x - y, T^{n+1}x - Ty\rangle \\
&\quad \ge a\|T^{n+1}x - Ty\|^2 + (1-a)\|T^n x - y\|^2 - \frac{1}{2}\|T^n x - T^{n+1}x\|^2 - \frac{1}{2}\|y - Ty\|^2,
\end{aligned}
$$

that is,

$$
\frac{1}{2}\|T^n x - Ty\|^2 + \frac{1}{2}\|y - T^{n+1}x\| \ge a\|T^{n+1}x - Ty\|^2 + (1-a)\|T^n x - y\|^2.
$$

Then

$$\mu_n\|Ty - T^n x\|^2 + \mu_n\|y - T^n x\|^2 \geq 2a\mu_n\|Ty - T^n x\|^2 + 2(1-a)\mu_n\|y - T^n x\|^2.$$

Hence $(2a-1)\mu_n\|y - T^n x\|^2 \geq (2a-1)\mu_n\|Ty - T^n x\|^2$. Since $a > \frac{1}{2}$, we have that

$$\mu_n\|y - T^n x\|^2 \geq \mu_n\|Ty - T^n x\|^2. \qquad \square$$

As a direct consequence of Theorem 3.3 and Lemma 2.1, we have the following existence theorem of fixed points for (a,b)-monotone mappings.

Theorem 3.4 *Let C be a nonempty, closed, and convex subset of a Hilbert space H. Let $T : C \to C$ be a (a,b)-monotone mapping with $b \in (-\infty, \frac{1}{2}]$. Then $F(T) \neq \emptyset$ if and only if there exists $x \in C$ such that $\{T^n x\}$ is bounded.*

Corollary 3.2 [1, 3, 4] *Let C be a closed convex subset of a Hilbert space. Let T be a self-mapping defined on C and satisfies one of the following:*
(i) T is a nonspreading mapping;
(ii) T is a hybrid mapping;
(iii) T is a TY mapping.
Then $F(T) \neq \emptyset$ if and only if there exists $x \in C$ such that $\{T^n x\}$ is bounded.

4 Convergence theorems

In this section, we first prove a weak convergence theorem of Mann's type for (a,b)-monotone mappings in a Hilbert space.

Theorem 4.1 *Let C be a nonempty, closed, and convex subset of a real Hilbert space H. Let $T : C \to C$ be a (a,b)-monotone mapping satisfies $F(T) \neq \emptyset$. If a sequence $\{\alpha_n\}_{n\in\mathbb{N}} \subseteq (0,1)$ with $\alpha_n > \frac{2b-1}{2a-1}$ and $\liminf_{n\to\infty}(1-\alpha_n)(\alpha_n + \frac{1-2b}{2a-1}) > 0$, then for each $x_1 \in C$, the sequence $\{x_n\}$ with $x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tx_n$ for all $n \in \mathbb{N}$ weakly converges to some fixed point of T.*

Proof We first show that there exists a sequence $\{\alpha_n\}$ satisfies our assumptions. Since $b < a$, $2a > 1$, we have $\frac{2b-1}{2a-1} < 1$, there exists a constant $\alpha \in \mathbb{R}$ such that $\frac{2b-1}{2a-1} < \alpha < 1$. If we take $\alpha_n = \alpha$ for all $n \in \mathbb{N}$, then $\{\alpha_n\} \subseteq (0,1)$ such that $\alpha_n > \frac{2b-1}{2a-1}$ and $\liminf_{n\to\infty}(1-\alpha_n)(\alpha_n + \frac{1-2b}{2a-1}) > 0$. Since T is a (a,b)-monotone mapping, by Proposition 3.3, we have that for each $p \in F(T)$ and $x \in C$,

$$\|x - p\|^2 \geq \|Tx - p\|^2 + \frac{1-2b}{2a-1}\|x - Tx\|^2.$$

Since $\alpha_n > \frac{2b-1}{2a-1}$, we have that

$$\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&\quad= \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|Tx_n - p\|^2 - \alpha_n(1-\alpha_n)\|x_n - Tx_n\|^2 \\
&\quad\leq \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|x_n - p\|^2 - (1-\alpha_n)\frac{1-2b}{2a-1}\|x_n - Tx_n\|^2 \\
&\qquad - \alpha_n(1-\alpha_n)\|x_n - Tx_n\|^2
\end{aligned}$$

$$
\begin{aligned}
&= \|x_n - p\|^2 - (1-\alpha_n)\left(\alpha_n + \frac{1-2b}{2a-1}\right)\|x_n - Tx_n\|^2 \\
&\leq \|x_n - p\|^2.
\end{aligned}
$$

Then $\lim_{n\to\infty}\|x_n - p\|$ exists and sequence $\{x_n\}$ is bounded. Further, from the inequality above, we have that

$$(1-\alpha_n)\left(\alpha_n + \frac{1-2b}{2a-1}\right)\|x_n - Tx_n\|^2 \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2.$$

Since $\liminf_{n\to\infty}(1-\alpha_n)(\alpha_n + \frac{1-2b}{2a-1}) > 0$, we have $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$.

Therefore, $\lim_{n\to\infty}\|x_{n+1} - x_n\| = \lim_{n\to\infty}(1-\alpha_n)\|x_n - Tx_n\| = 0$. Since $\{x_n\}$ is bounded, there exist a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ and a point $x^* \in C$ such that $x_{n_j} \rightharpoonup x^*$. Since T is a (a,b)-monotone mapping, by Theorem 3.1, we have $x^* = Tx^*$.

For each subsequence $\{x_{n_k}\}$ of $\{x_n\}$ with $x_{n_k} \rightharpoonup u$ for some $u \in C$, we follow the same argument as above, we see that $u = Tu$. We have to show that $u = x^*$. Otherwise, if $u \neq x^*$, then by Optial condition,

$$
\begin{aligned}
\liminf_{j\to\infty}\left\|x_{n_j} - x^*\right\| &< \liminf_{j\to\infty}\|x_{n_j} - u\| \\
&= \lim_{n\to\infty}\|x_n - u\| \\
&= \liminf_{k\to\infty}\|x_{n_k} - u\| \\
&< \liminf_{k\to\infty}\left\|x_{n_k} - x^*\right\| \\
&= \lim_{n\to\infty}\left\|x_n - x^*\right\| = \liminf_{j\to\infty}\left\|x_{n_j} - x^*\right\|.
\end{aligned}
$$

This leads to a contradiction. Therefore $u = x^*$. By Theorem 2.1, we have that $x_n \rightharpoonup x^*$. □

Example 4.1 Let H, ϕ, T be the same as in Example 1.1. For any fixed $x_1 \in H$, take a sequence $\{x_n\}$ as in Theorem 4.1 with $\alpha_n = \frac{3}{4}$ for all $n \in \mathbb{N}$, that is,

$$x_{n+1} = \frac{3}{4}x_n + \frac{1}{4}Tx_n.$$

Then

$$x_{n+1} = \frac{3}{4}x_n + \frac{5}{16}\left(\frac{\sqrt{2}}{2}\phi(\pi, x_n) + \frac{\sqrt{2}}{2}\phi\left(\frac{\pi}{2}, x_n\right)\right) = \left(\frac{3}{4} - \frac{5\sqrt{2}}{32}\right)x_n + \frac{5\sqrt{2}}{32}\phi\left(\frac{\pi}{2}, x_n\right)$$

and hence

$$\|x_{n+1}\| \leq \left(\frac{3}{4} - \frac{5\sqrt{2}}{32}\right)\|x_n\| + \frac{5\sqrt{2}}{32}\|x_n\| \leq \frac{3}{4}\|x_n\|.$$

Therefore, $x_n \to 0 \in F(T)$, and hence $x_n \rightharpoonup 0$.

Corollary 4.1 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T: C \to C$ be a mapping with $F(T) \neq \emptyset$ and satisfies one of the following:*

(i) T is a nonspreading mapping;
(ii) T is a hybrid mapping;
(iii) T is a TY mapping.

If a sequence $\{\alpha_n\}_{n\in\mathbb{N}} \subseteq (0,1)$ satisfies $\liminf_{n\to\infty}(1-\alpha_n)(\alpha_n) > 0$, then for each $x_1 \in C$, the sequence $\{x_n\}$ with $x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tx_n$ for all $n \in \mathbb{N}$ weakly converges to some fixed point of T.

Proof Since T is an (a,b)-monotone mapping with $b = \frac{1}{2}$, we have $\frac{2b-1}{2a-1} = 0$. Then Corollary 4.1 follows from Theorem 4.1. □

Corollary 4.2 *Let C be a nonempty, closed, and convex subset of real Hilbert space H. Let $T : C \to C$ be a mapping with $F(T) \neq \emptyset$ and satisfies one of the following:*
(i) T is a nonspreading mapping;
(ii) T is a hybrid mapping;
(iii) T is a TY mapping.

Then for each $x_1 \in C$, the sequence $\{x_n\}$ with $x_{n+1} = \frac{1}{2}x_n + \frac{1}{2}Tx_n$ for all $n \in \mathbb{N}$ weakly converges to some fixed point of T.

Proof Take $\alpha_n = \frac{1}{2}$ for all $n \in \mathbb{N}$. Then Corollary 4.2 follows from Corollary 4.1. □

Next we prove a strong convergence theorem by hybrid method for (a,b)-monotone mappings in a Hilbert space.

Theorem 4.2 *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $T : C \to C$ be a (a,b)-monotone mapping with $F(T) \neq \emptyset$. Suppose that $\{x_n\}$ is a sequence generated by the following scheme:*

$$\begin{cases} x_0 \in C \quad \textit{chosen arbitrarily}; \\ y_n = t_n x_n + (1-t_n)Tx_n; \\ C_n = \{v \in C : \|y_n - v\|^2 \leq \|x_n - v\|^2 + (1-t_n)\frac{2b-1}{2a-1}\|x_n - Tx_n\|^2\}; \\ Q_n = \{v \in C : \langle x_n - v, x_n - x_0\rangle \leq 0\}; \\ x_{n+1} = P_{C_n\cap Q_n}x_0, \quad \textit{where } P_{C_n\cap Q_n} \textit{ is the metric projection from } H \textit{ onto } C_n \cap Q_n. \end{cases}$$

If the sequence $\{t_n\}_{n\in\mathbb{N}} \subseteq (0,1)$ satisfies $\limsup_{n\to\infty}(\frac{2b-1}{2a-1} + t_n) < 1$, then $x_n \to P_{F(T)}x_0$.

Proof By Lemma 2.2, we see that C_n is closed and convex for all $n \in \mathbb{N}$. For any $p \in F(T)$, by Proposition 3.3, we have

$$\begin{aligned} \|y_n - p\|^2 &= \left\|t_n(x_n - p) + (1-t_n)(Tx_n - p)\right\|^2 \\ &= t_n\|x_n - p\|^2 + (1-t_n)\|Tx_n - p\|^2 - t_n(1-t_n)\|x_n - Tx_n\|^2 \\ &\leq t_n\|x_n - p\|^2 + (1-t_n)\|Tx_n - p\|^2 \\ &\leq t_n\|x_n - p\|^2 + (1-t_n)\left(\|x_n - p\|^2 + \frac{2b-1}{2a-1}\|x_n - Tx_n\|^2\right) \\ &= \|x_n - p\|^2 + (1-t_n)\frac{2b-1}{2a-1}\|x_n - Tx_n\|^2. \end{aligned}$$

Hence, $p \in C_n$. Then we have that $F(T) \subseteq C_n$ for all $n \geq 0$.

Next, we show that $F(T) \subseteq Q_n$ for all $n \geq 0$. We prove this by induction. For $n = 0$, we have $F(T) \subseteq C = Q_0$. Assume that $F(T) \subseteq Q_n$. Since x_{n+1} is the projection of x_0 onto $C_n \cap Q_n$, by Lemma 2.3, we have $\langle x_{n+1} - z, x_0 - x_{n+1}\rangle \geq 0$ for all $z \in C_n \cap Q_n$. As $F(T) \subseteq C_n \cap Q_n$ by the induction assumption, the last inequality holds, in particular, for all $z \in F(T)$. This together with the definition of Q_{n+1} implies that $F(T) \subseteq Q_{n+1}$. Hence $F(T) \subseteq Q_n$ for all $n \in \mathbb{N}$. Then the sequence $\{x_n\}$ is well defined.

The definition of Q_n and Lemma 2.3 imply that $x_n = P_{Q_n}x_0$, which in turn implies that $\|x_n - x_0\| \leq \|p - x_0\|$ for all $p \in F(T)$, in particular, $\{x_n\}$ is bounded and $\|x_n - x_0\| \leq \|q - x_0\|$ with $q = P_{F(T)}x_0$.

That $x_{n+1} \in Q_n$ asserts that

$$\langle x_n - x_{n+1}, x_n - x_0\rangle \leq 0.$$

It follows from Lemma 2.5 and the inequality above that

$$\begin{aligned}\|x_{n+1} - x_n\|^2 &= \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2 - 2\langle x_{n+1} - x_n, x_n - x_0\rangle \\ &\leq \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2.\end{aligned}$$

The last inequality implies that $\{\|x_n - x_0\|\}$ is increasing. Since $\{x_n\}$ is bounded, we have that $\lim_{n\to\infty} \|x_n - x_0\|$ exists and $\|x_{n+1} - x_n\| \to 0$. Since $x_{n+1} \in C_n$,

$$\begin{aligned}&\|y_n - x_{n+1}\|^2 \\ &\quad\leq \|x_n - x_{n+1}\|^2 + (1 - t_n)\frac{2b - 1}{2a - 1}\|x_n - Tx_n\|^2.\end{aligned}$$

Note that

$$\begin{aligned}&\|y_n - x_{n+1}\|^2 \\ &\quad= t_n\|x_n - x_{n+1}\|^2 + (1 - t_n)\|Tx_n - x_{n+1}\|^2 - t_n(1 - t_n)\|x_n - Tx_n\|^2.\end{aligned}$$

Then

$$\begin{aligned}&\|Tx_n - x_{n+1}\|^2 \leq \|x_n - x_{n+1}\|^2 + \left(\frac{2b - 1}{2a - 1} + t_n\right)\|x_n - Tx_n\|^2, \\ &\|Tx_n - x_{n+1}\|^2 \\ &\quad\geq \left(\|x_n - Tx_n\| - \|x_n - x_{n+1}\|\right)^2 \\ &\quad= \|x_n - Tx_n\|^2 - 2\|x_n - Tx_n\|\|x_n - x_{n+1}\| + \|x_n - x_{n+1}\|^2.\end{aligned}$$

Then

$$\begin{aligned}&\|x_n - Tx_n\|^2 - 2\|x_n - Tx_n\|\|x_n - x_{n+1}\| \\ &\quad\leq \left(\frac{2b - 1}{2a - 1} + t_n\right)\|x_n - Tx_n\|^2.\end{aligned}$$

Without loss of generality, we may assume that $x_n \neq Tx_n$ for all $n \in \mathbb{N}$. Otherwise, $x_n \in F(T)$ for some $n \in \mathbb{N}$ and we complete the proof. Therefore,

$$\|x_n - Tx_n\| - 2\|x_n - x_{n+1}\| \leq \left(\frac{2b-1}{2a-1} + t_n\right)\|x_n - Tx_n\|.$$

Hence,

$$\limsup_{n\to\infty} \|x_n - Tx_n\| \leq \limsup_{n\to\infty}\left(\frac{2b-1}{2a-1} + t_n\right)\|x_n - Tx_n\|.$$

By the choice of $\{t_n\}$, $\limsup_{n\to\infty}(\frac{2b-1}{2a-1} + t_n) < 1$. Therefore, $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. Consequently, $\omega_w(x_n) \subseteq F(T)$ by Theorem 3.1. Hence, applying Lemma 2.4 (to $u := x_0$ and $K := F(T)$), one can conclude that $x_n \to q$. □

Corollary 4.3 *Let C be a nonempty, closed, and convex subset of a real Hilbert space H. Let $T : C \to C$ be a mapping with $F(T) \neq \emptyset$ and satisfies one of the following conditions:*
(i) T is a nonspreading mapping;
(ii) T is a hybrid mapping;
(iii) T is a TY mapping.
Suppose that $\{x_n\}$ is a sequence generated by the following scheme:

$$\begin{cases} x_0 \in C \quad \textit{chosen arbitrarily;} \\ y_n = t_n x_n + (1 - t_n)Tx_n; \\ C_n = \{v \in C : \|y_n - v\|^2 \leq \|x_n - v\|^2 + (1 - t_n)\frac{2b-1}{2a-1}\|x_n - Tx_n\|^2\}; \\ Q_n = \{v \in C : \langle x_n - v, x_n - x_0\rangle \leq 0\}; \\ x_{n+1} = P_{C_n\cap Q_n}x_0, \quad \textit{where } P_{C_n\cap Q_n} \textit{ is the metric projection from } H \textit{ onto } C_n \cap Q_n. \end{cases}$$

If the sequence $\{t_n\}_{n\in\mathbb{N}} \subseteq (0,1)$ satisfies $\limsup_{n\to\infty} t_n < 1$. Then $x_n \to P_{F(T)}x_0$.

Proof Since T is a (a,b)-monotone mapping with $b = \frac{1}{2}$, we have $\frac{2b-1}{2a-1} = 0$, then Corollary 4.3 follows from Theorem 4.2. □

Corollary 4.4 *Let C be a nonempty, closed, and convex subset of a real Hilbert space H. Let $T : C \to C$ be a mapping with $F(T) \neq \emptyset$ and satisfies one of the following conditions:*
(i) T is a nonspreading mapping;
(ii) T is a hybrid mapping;
(iii) T is a TY mapping.
Suppose that $\{x_n\}$ is a sequence generated by the following scheme:

$$\begin{cases} x_0 \in C \quad \textit{chosen arbitrarily;} \\ y_n = \frac{1}{2}x_n + \frac{1}{2}Tx_n; \\ C_n = \{v \in C : \|y_n - v\|^2 \leq \|x_n - v\|^2\}; \\ Q_n = \{v \in C : \langle x_n - v, x_n - x_0\rangle \leq 0\}; \\ x_{n+1} = P_{C_n\cap Q_n}x_0, \quad \textit{where } P_{C_n\cap Q_n} \textit{ is the metric projection from } H \textit{ onto } C_n \cap Q_n. \end{cases}$$

Then $x_n \to P_{F(T)}x_0$.

Proof Take $t_n = \frac{1}{2}$ for all $n \in \mathbb{N}$, then Corollary 4.4 follows from Corollary 4.3. □

Competing interests
The authors declare no competing interests, except Prof. LJL was supported by the National Science Council of Republic of China while he worked on the publish.

Authors' contributions
LJL responsible for the problem resign, coordinator, discussion, revised the manuscript and submission, SYW carried out this problem, complete the draft the manuscript. All the authors read and approved the manuscript.

Received: 12 November 2011 Accepted: 26 July 2012 Published: 7 August 2012

References

1. Kohsaka, F, Takahashi, W: Fixed point theorems for a class of nonlinear mappings related to maximal monotone operators in Banach spaces. Arch. Math. **91**, 166-177 (2008)
2. Iemoto, S, Takahashi, W: Approximating common fixed points of nonexpansive mappings and nonspreading mappings in a Hilbert space. Nonlinear Anal. **71**, 2082-2089 (2009)
3. Takahashi, W, Yao, JC: Fixed point theorems and ergodic theorems for nonlinear mappings in Hilbert spaces. Taiwan. J. Math. **15**, 457-472 (2011)
4. Takahashi, W: Fixed point theorems for new nonlinear mappings in a Hilbert space. J. Nonlinear Convex Anal. **11**, 79-88 (2010)
5. Aoyama, K, Iemoto, S, Kohsaka, F, Takahashi, W: Fixed point and ergodic theorems for λ-hybrid mappings in Hilbert spaces. J. Nonlinear Convex Anal. **11**, 335-343 (2010)
6. Kocourek, P, Takahashi, W, Yao, JC: Fixed point theorems and weak convergence theorems for generalized hybrid mappings in Hilbert spaces. Taiwan. J. Math. **14**, 2497-2511 (2010)
7. Aoyama, K, Kohsaka, F: Fixed point theorem for α-nonexpansive mappings in Banach spaces. Nonlinear Anal. **74**, 4387-4391 (2011)
8. Lin, LJ, Yu, ZT, Chuang, CS: Generalized ergodic theorems and convergence theorems for generalized nonlinear mappings in Hilbert spaces. Preprint
9. Carlos, M-Y, Xu, H-K: Strong convergence of the CQ method for fixed point iteration processes. Nonlinear Anal. **64**, 2400-2411 (2006)
10. Takahashi, W: Introduction to Nonlinear and Convex Analysis. Yokohoma Publishers, Yokohoma (2009)

doi:10.1186/1687-1812-2012-131
Cite this article as: Lin and Wang: **Fixed point theorems of** (a, b)**-monotone mappings in Hilbert spaces.** *Fixed Point Theory and Applications* 2012 **2012**:131.