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Fixed point theorems for multivalued contractive mappings and multivalued Caristi type mappings in cone metric spaces

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Abstract

In this paper, we establish a fixed-point theorem for multivalued contractive mappings in complete cone metric spaces. We generalize Caristi's fixed-point theorem to the case of multivalued mappings in complete cone metric spaces. We give examples to support our main results. Our results are extensions of the results obtained by Feng and Liu (J. Math. Anal. Appl. 317:103-112, 2006) to the case of cone metric spaces.

MSC: 47H10; 54H25

Keywords: fixed point; multivalued map; cone metric space

1 Introduction

Banach's contraction principle plays an important role in several branches of mathematics. Because of its importance for mathematical theory, it has been extended in many directions (see [10, 11, 14, 19, 21, 37, 46]); especially, the authors [36, 37, 39] generalized Banach's principle to the case of multivalued mappings. Feng and Liu gave a generalization of Nadler's fixed-point theorem. They proved the following theorem in [21].

Theorem 1.1 *Let (X,d) be a complete metric space and let $T : X \to 2^X$ be a multivalued map such that Tx is a closed subset of X for all $x \in X$. Let $I_b^x = \{y \in Tx : bd(x,y) \leq d(x,Tx)\}$, where $b \in (0,1)$.*

If there exists a constant $c \in (0,1)$ such that for any $x \in X$, there exists $y \in I_b^x$ satisfying

$$d(y,Ty) \leq cd(x,y),$$

then T has a fixed point in X, i.e., there exists $z \in X$ such that $z \in Tz$ provided $c < b$ and the function $d(x,Tx)$, $x \in X$ is lower semicontinuous.

Recently, in [22], the authors used the notion of a cone metric space to generalize the Banach contraction principle to the case of cone metric spaces. Since then, many authors [1–3, 7, 9, 13, 15, 18, 22–28, 32–34, 41, 43, 44, 48] obtained fixed-point theorems in cone metric spaces. The cone Banach space was first used in [4, 6]. Since then, the authors [29, 30] obtained fixed-point results in cone Banach spaces. The authors [8] proved a Caristi-type fixed-point theorem for single valued maps in cone metric spaces. The author [5] studied the structure of cone metric spaces.

Especially, the authors [16, 31, 35, 42, 45, 47] proved fixed point theorems for multivalued maps in cone metric spaces.

In this paper, we give a generalization of Theorem 1.1 to the case of cone metric spaces and we establish a Caristi-type fixed-point theorem for multivalued maps in cone metric spaces.

Consistent with Huang and Zhang [22], the following definitions will be needed in the sequel.

Let E be a topological vector space. A subset P of E is a *cone* if the following conditions are satisfied:

(1) P is nonempty, closed, and $P \neq \{\theta\}$,

(2) $ax + by \in P$, whenever $x, y \in P$ and $a, b \in \mathbb{R}$ $(a, b \geq 0)$,

(3) $P \cap (-P) = \{\theta\}$.

Given a cone $P \subset E$, we define a partial ordering $\preceq$ with respect to P by $x \preceq y$ if and only if $y - x \in P$. We write $x \prec y$ to indicate that $x \preceq y$ but $x \neq y$.

For $x, y \in P$, $x \ll y$ stand for $y - x \in \text{int}(P)$, where $\text{int}(P)$ is the interior of P.

If E is a normed space, a cone P is called *normal* whenever there exists a number $K > 0$ such that for all $x, y \in E$, $\theta \preceq x \preceq y$ implies $\|x\| \leq K\|y\|$.

A cone P is *minihedral* [20] if $\sup\{x, y\}$ exists for all $x, y \in E$. A cone P is *strongly minihedral* [20] if every upper bounded nonempty subset A of E, $\sup A$ exists in E. Equivalently, a cone P is strongly minihedral if every lower bounded nonempty subset A of E, $\inf A$ exists in E (see also [1, 8]).

If E is a normed space, a strongly minihedral cone P is *continuous* whenever, for any bounded chain $\{x_\alpha : \alpha \in \Gamma\}$, $\inf\{\|x_\alpha - \inf\{x_\alpha : \alpha \in \Gamma\}\| : \alpha \in \Gamma\} = 0$ and $\sup\{\|x_\alpha - \sup\{x_\alpha : \alpha \in \Gamma\}\| : \alpha \in \Gamma\} = 0$.

From now on, we assume that E is a normed space, $P \subset E$ is a solid cone (that is, $\text{int}(P) \neq \emptyset$), and $\preceq$ is a partial ordering with respect to P.

Let X be a nonempty set. A mapping $d : X \times X \to E$ is called *cone metric* [22] on X if the following conditions are satisfied:

(1) $\theta \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$,

(2) $d(x, y) = d(y, x)$ for all $x, y \in X$,

(3) $d(x, y) \preceq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Let (X, d) be a cone metric space, and let $\{x_n\} \subset X$ be a sequence. Then

$\{x_n\}$ is *convergent* [22] to a point $x \in X$ (denoted by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$) if for any $c \in \text{int}(P)$, there exists N such that for all $n > N$, $d(x_n, x) \ll c$.

$\{x_n\}$ is *Cauchy* [22] if for any $c \in \text{int}(P)$, there exists N such that for all $n, m > N$, $d(x_n, x_m) \ll c$. A cone metric space (X, d) is called *complete* [22] if every Cauchy sequence is convergent.

Remark 1.1 (1) If $\lim_{n\to\infty} d(x_n, x) = \theta$, then $\lim_{n\to\infty} x_n = x$. The converse is true if E is a normed space and P is a normal cone.

(2) If $\lim_{n,m\to\infty} d(x_n, x_m) = \theta$, then $\{x_n\}$ is a Cauchy sequence in X. If E is a normed space and P is a normal cone, then $\{x_n\}$ is a Cauchy sequence in X if and only if $\lim_{n,m\to\infty} d(x_n, x_m) = \theta$.

We denote by $N(X)$ (resp. $B(X)$, $C(X)$, $CB(X)$) the set of nonempty (resp. bounded, closed, closed and bounded) subsets of a cone metric space or a metric space.

The following definitions are found in [16].

Let $s(p) = \{q \in E : p \preceq q\}$ for $p \in E$, and $s(a,B) = \bigcup_{b\in B} s(d(a,b))$ for $a \in X$ and $B \in N(X)$.

For $A, B \in B(X)$, we denote

$$s(A,B) = \left(\bigcap_{a\in A} s(a,B)\right) \cap \left(\bigcap_{b\in B} s(b,A)\right).$$

Lemma 1.1 ([16]) *Let (X,d) be a cone metric space, and let $P \subset E$ be a cone.*

(1) Let $p,q \in E$. If $p \preceq q$, then $s(q) \subset s(p)$.

(2) Let $x \in X$ and $A \in N(X)$. If $\theta \in s(x,A)$, then $x \in A$.

(3) Let $q \in P$ and let $A, B \in B(X)$ and $a \in A$. If $q \in s(A,B)$, then $q \in s(a,B)$.

Remark 1.2 Let (X,d) be a cone metric space. If $E = \mathbb{R}$ and $P = [0,\infty)$, then (X,d) is a metric space. Moreover, for $A, B \in CB(X)$, $H(A,B) = \inf s(A,B)$ is the Hausdorff distance induced by d.

Remark 1.3 Let (X,d) be a cone metric space. Then $s(\{a\},\{b\}) = s(d(a,b))$ for $a,b \in X$.

Lemma 1.2 ([16, 40]) *If $u_n \in E$ with $u_n \to \theta$, then for each $c \in \mathrm{int}(P)$ there exists N such that $u_n \ll c$ for all $n > N$.*

2 Fixed-point theorems for multivalued contractive mappings

Let (X,d) be a cone metric space, and let $A \in N(X)$.

A function $h : X \to 2^P - \{\emptyset\}$ defined by $h(x) = s(x,A)$ is called *sequentially lower semicontinuous* if for any $c \in \mathrm{int}(P)$, there exists $n_0 \in \mathbb{N}$ such that $h(x_n) \subset h(x) - c$ for all $n \geq n_0$, whenever $\lim_{n\to\infty} x_n = x$ for any sequence $\{x_n\} \subset X$ and $x \in X$.

Let $T : X \to C(X)$ be a multivalued mapping. We define a function $h : X \to 2^P - \{\emptyset\}$ as $h(x) = s(x, Tx)$.

For a $b \in (0,1]$, let $J_b^x = \{y \in Tx : s(x,Tx) \subset s(bd(x,y))\}$.

Theorem 2.1 *Let (X,d) be a complete cone metric space and let $T : X \to C(X)$ be a multivalued map. If there exists a constant $c \in (0,1)$ such that for any $x \in X$ there exists $y \in J_b^x$ satisfying*

$$cd(x,y) \in s(y, Ty) \tag{2.1}$$

then T has a fixed point in X provided $c < b$ and h is sequentially lower semicontinuous.

Proof Let $x_0 \in X$. Then there exists $x_1 \in J_b^{x_0}$ such that $cd(x_0,x_1) \in s(x_1, Tx_1)$. For x_1, there exists $x_2 \in J_b^{x_1}$ such that $cd(x_1,x_2) \in s(x_2, Tx_2)$.

Continuing this process, we can find a sequence $\{x_n\} \subset X$ such that

$$x_{n+1} \in J_b^{x_n}$$

and

$$cd(x_n,x_{n+1}) \in s(x_{n+1}, Tx_{n+1}) \tag{2.2}$$

for all $n = 0,1,\ldots$.

We now show that $\{x_n\}$ is a Cauchy sequence in X.

Since $x_{n+2} \in J_b^{x_{n+1}}$, $s(x_{n+1}, Tx_{n+1}) \subset s(bd(x_{n+1}, x_{n+2}))$.

From (2.2), we have $cd(x_n, x_{n+1}) \in s(bd(x_{n+1}, x_{n+2}))$. Thus, $bd(x_{n+1}, x_{n+2}) \preceq cd(x_n, x_{n+1})$. Hence,

$$d(x_{n+1}, x_{n+2}) \preceq kd(x_n, x_{n+1})$$

for all $n = 0, 1, \ldots$, where $k = \frac{c}{b}$.

So we have

$$d(x_n, x_{n+1}) \preceq kd(x_{n-1}, x_n) \preceq k^2 d(x_{n-2}, x_{n-1}) \preceq \cdots \preceq k^n d(x_0, x_1).$$

For $m > n$, we have

$$\begin{aligned} &d(x_n, x_m) \\ &\quad \preceq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\ &\quad \preceq \left(k^n + k^{n+1} + \cdots + k^{m-1}\right) d(x_0, x_1) \preceq \frac{k^n}{1-k} d(x_0, x_1). \end{aligned}$$

By Lemma 1.2, $\{x_n\}$ is a Cauchy sequence in X. It follows from the completeness of X that there exists $z \in X$ such that $\lim_{n\to\infty} x_n = z$.

We now show that $z \in Tz$.

Suppose that $z \notin Tz$.

Since Tz is closed, there exists $c \in \operatorname{int}(P)$ such that $d(z, y) \ll c$ implies $y \notin Tz$.

But since h is sequentially lower semicontinuous, there exists N such that $d(x_N, x_{N+1}) \ll \frac{c}{2}$ and $s(x_N, Tx_N) \subset s(z, Tz) - \frac{c}{2}$.

Thus, there exists $y \in Tz$ such that $d(z, y) - \frac{c}{2} \preceq d(x_N, x_{N+1})$. Hence, $d(z, y) \preceq d(x_N, x_{N+1}) + \frac{c}{2} \ll c$, which is a contradiction. □

Remark 2.1 By Remark 1.1, Theorem 2.1 generalizes Theorem 1.1 ([12, Theorem 3.1]).

Corollary 2.2 *Let (X, d) be a complete cone metric space and let $T : X \to C(X)$ be a multivalued map. If there exists a constant $c \in (0, 1)$ such that for any $x \in X$, $y \in Tx$*

$$cd(x, y) \in s(y, Ty)$$

then T has a fixed point in X provided h is sequentially lower semicontinuous.

By Lemma 1.1(3), we have the following result, which is Nadler's fixed-point theorem in the cone metric space.

Corollary 2.3 *Let (X, d) be a complete cone metric space, and let $T : X \to CB(X)$ be a multivalued map. If there exists a constant $c \in (0, 1)$, such that*

$$cd(x, y) \in s(Tx, Ty)$$

for all $x \in X$, $y \in Tx$, then T has a fixed point in X provided h is sequentially lower semicontinuous.

By Remark 1.1, we have the following corollaries.

Corollary 2.4 ([21]) *Let (X,d) be a complete metric space and let $T : X \to C(X)$ be a multivalued map. If there exists a constant $c \in (0,1)$ such that*

$$d(y,Ty) \leq cd(x,y)$$

for all $x \in X$, $y \in Tx$, then T has a fixed point in X provided h is sequentially lower semicontinuous.

Corollary 2.5 *Let (X,d) be a complete metric space and let $T : X \to CB(X)$ be a multivalued map. If there exists a constant $c \in (0,1)$ such that*

$$H(Tx,Ty) \leq cd(x,y)$$

for all $x \in X$, $y \in Tx$, then T has a fixed point in X provided h is sequentially lower semicontinuous.

The following example illustrates our main theorem.

Example 2.1 Let $X = \{f \in L^1[0,1] : f(x) \geq 0\}$, $E = C[0,1]$ and $P = \{f \in E : f \geq 0 \text{ a.e.}\}$. Define $d : X \times X \to E$ by $d(f,g)(t) = \int_0^t |f(x) - g(x)|\,dx$, where $0 \leq t \leq 1$. Then d is a complete cone metric on X. Consider a mapping $T : X \to CB(X)$ defined by

$$(Tf)(x) = \{a(f), a(f) + 2f\},$$

where $a(f) \in X$ is defined by $a(f)(x) = \int_0^x y(f(y) + 1)\,dy$.

Obviously, $h(f) = s(f, Tf)$ is sequentially lower semicontinuous.

For any $f \in X$, we can prove $a(f) \in I_1^f$. To see this, we compute for $0 \leq t \leq 1$

$$\begin{aligned} &d\big(f, a(f) + 2f\big)(t) \\ &\quad = \int_0^t \big|a(f)(x) + f(x)\big|\,dx \\ &\quad = \int_0^t \big(a(f)(x) + f(x)\big)\,dx \\ &\quad \geq \int_0^t \big(a(f)(x) - f(x)\big)\,dx \\ &\quad = d\big(f, a(f)\big)(t). \end{aligned}$$

Since $(Tf)(x) = \{a(f), a(f) + 2f\}$, we have $s(f, Tf) \subset s(d(f, a(f)))$, and hence we obtain $a(f) \in I_1^f$.

Put $a(f) = g$. Then we have $a(a(f)) = a(g) \in T(a(f))$ and for $0 \leq t \leq 1$

$$\begin{aligned} &d\big(a(f), a\big(a(f)\big)\big)(t) \\ &\quad = d\big(a(f), a(g)\big)(t) \end{aligned}$$

$$
\begin{aligned}
&= \int_0^t \left|a(f)(x) - a(g)(x)\right| dx \\
&= \int_0^t \left| \int_0^x y\bigl(f(y)+1\bigr)\,dy - \int_0^x y\bigl(g(y)+1\bigr)\,dy \right| dx \\
&= \int_0^t \left| \int_0^x y\bigl(f(y)-g(y)\bigr)\,dy \right| dx \\
&\le \int_0^t \int_0^x y\left|f(y)-g(y)\right| dy\,dx \\
&= \int_0^t \int_y^t y\left|f(y)-g(y)\right| dx\,dy \\
&= \int_0^t (t-y)y\left|f(y)-g(y)\right| dy \\
&\le \int_0^t \frac{t^2}{4}\left|f(y)-g(y)\right| dy \\
&\le \frac{1}{4}\int_0^t \left|f(y)-g(y)\right| dy \\
&= \frac{1}{4} d(f,g)(t).
\end{aligned}
$$

Thus, we have $g \in J_1^f$, and $\frac{1}{4}d(f,g) \in s(g, Tg)$.

Therefore, all conditions of Theorem 2.1 are satisfied and T has a fixed point $f^*(x) = e^{\frac{x^2}{2}} - 1$.

3 Fixed-point theorems for multivalued Caristi type mappings

Let (X,d) be a cone metric space with a preordering $\sqsubseteq$.

A sequence $\{x_n\}$ of points in X is called $\sqsubseteq$-*decreasing* if $x_{n+1} \sqsubseteq x_n$ for all $n \geq 0$. The set $S(x) = \{y \in X : y \sqsubseteq x\}$ is $\sqsubseteq$-*complete* if every decreasing Cauchy sequence in $S(x)$ converges in it.

A function $f : X \to E$ is called *lower semicontinuous from above* if, for every sequence $\{x_n\} \subset X$ conversing to some point $x \in X$ and satisfying $fx_{n+1} \preceq fx_n$ for all $n \in \mathbb{N}$, we have $fx \preceq \lim_{n\to\infty} fx_n$.

Lemma 3.1 *Let (X,d) be a cone metric space, and let $T : X \to N(X)$ be a multivalued mapping. Suppose that $\phi : X \to E$ is a function and $\eta : P \to P$ is a nondecreasing, continuous, and subadditive function such that $\eta(t) = 0$ if and only if $t = 0$.*

We define a relation $\preceq_\eta$ on X as follows:

$$y \preceq_\eta x \quad \text{if and only if } \phi(x) - \phi(y) \in s\bigl(\eta\bigl(d(x,y)\bigr)\bigr). \tag{3.1}$$

Then $\preceq_\eta$ is a partial order on X.

Proof The proof follows by using the cone metric axioms, properties (2) and (3) for the cone, and (3.1). □

Lemma 3.2 ([17]) *Let $P \subset E$ be a strongly minihedral and continuous cone, and let $(X, \sqsubseteq)$ be a preordered set. Suppose that a mapping $\psi : X \to E$ satisfies the following conditions:*

(1) $x \sqsubseteq y$ and $x \neq y$ imply $\psi(x) \prec \psi(y)$;
(2) for every $\sqsubseteq$-decreasing sequence $\{x_n\} \subset X$, there exists $y \in X$ such that $y \sqsubseteq x_n$ for all $n \in \mathbb{N}$;
(3) ψ is bounded from below.

Then, for each $x \in X$, $S(x)$ has a minimal element in $S(x)$, where $S(x) = \{y \in X : y \sqsubseteq x\}$.

Theorem 3.1 *Let (X, d) be a cone metric space such that P is strongly minihedral and continuous, and let $T : X \to N(X)$ be a multivalued mapping and $\phi : X \to E$ be a mapping bounded from below. Suppose that, for each $x \in X$, $S(x) = \{y \in X : y \preceq_\eta x\}$ is $\preceq_\eta$-complete, where $\preceq_\eta$ is a partial ordering on X defined as (3.1).*

If for any $x \in X$, there exists $y \in Tx$ satisfying

$$\phi(x) - \phi(y) \in s\big(\eta\big(d(x,y)\big)\big),$$

then T has a fixed point in X.

Proof We define a partial ordering $\preceq_\eta$ on X as (3.1).

If $x \preceq_\eta y$ and $x \neq y$, then $0 \prec d(y,x)$ and $\phi(y) - \phi(x) \in s(\eta(d(y,x)))$, and so $0 \prec \eta(d(y,x)) \preceq \phi(y) - \phi(x)$. Hence, $\phi(x) \prec \phi(y)$.

Let $\{x_n\}$ be a $\preceq_\eta$-decreasing sequence in X. Then $x_{n+1} \in S(x_n)$ for all $n \geq 0$, and $\{\phi(x_n)\}$ is bounded from below, because ϕ is bounded from below. Hence, $\{\phi(x_n)\}$ is bounded. Since P is strongly minihedral, $u = \inf \phi(x_n)$ exists in E. Also, since P is continuous, $\inf\{\|\phi(x_n) - u\| : n \in \mathbb{N}\} = 0$. Hence, $\lim_{n\to\infty} \phi(x_n) = u$ and $u \preceq \phi(x_n)$ for all $n \geq 0$.

For $m > n$, since $x_m \preceq_\eta x_n$, $\phi(x_n) - \phi(x_m) \in s(\eta(d(x_n,x_m)))$. Hence $\eta(d(x_n,x_m)) \preceq \phi(x_n) - \phi(x_m) \preceq \phi(x_n) - u$. Thus, $\lim_{n,m\to\infty} \eta(d(x_n,x_m)) = \theta$. Since η is continuous, $\eta(\lim_{n,m\to\infty} d(x_n,x_m)) = \theta$. So $\lim_{n,m\to\infty} d(x_n,x_m) = \theta$.

Hence, $\{x_n\}$ is a $\preceq_\eta$-decreasing Cauchy sequence in $S(x_0)$. Since $S(x_n)$ is $\preceq_\eta$-complete and $x_{n+1} \in S(x_n)$ for all $n \geq 0$, there exists $x \in S(x_n)$ such that $\lim_{n\to\infty} x_n = x$. Thus, $x \preceq_\eta x_n$ for all $n \geq 0$.

By Lemma 3.2, $S(x_0)$ has a minimal element $\overline{x}$ in $S(x_0)$. By assumption, there exists $y_0 \in T\overline{x}$ such that $\phi(\overline{x}) - \phi(y_0) \in s(\eta(d(\overline{x}, y_0)))$. Hence, $y_0 \preceq_\eta \overline{x}$. Since $\overline{x}$ is minimal element in $S(x_0)$, $y_0 = \overline{x}$. Thus, $\overline{x} \in T\overline{x}$. □

Corollary 3.2 *Let (X, d) be a cone metric space such that P is strongly minihedral and continuous, and let $T : X \to N(X)$ be a multivalued mapping and $\phi : X \to E$ be a mapping bounded from below. Suppose that, for each $x \in X$, $S(x) = \{y \in X : y \preceq_\eta x\}$ is $\preceq_\eta$-complete, where $\preceq_\eta$ is a partial ordering on X defined as (3.1).*

If for any $x \in X$ and for any $y \in Tx$,

$$\phi(x) - \phi(y) \in s\big(\eta\big(d(x,y)\big)\big),$$

then there exists $x_0 \in X$ such that $Tx_0 = \{x_0\}$.

Theorem 3.3 *Let (X, d) be a complete cone metric space such that P is strongly minihedral and continuous. Suppose that $T : X \to N(X)$ is a multivalued mapping and $\phi : X \to E$ is lower semicontinuous from above and bounded from below.*

If for any $x \in X$, there exists $y \in Tx$ satisfying

$$\phi(x) - \phi(y) \in s(\eta(d(x,y))),$$

then T has a fixed point in X.

Proof We define a partial ordering $\preceq_\eta$ on X as (3.1). It suffices to show that, for each $x_0 \in X$, $S(x_0)$ is $\preceq_\eta$-complete.

Let $x_0 \in X$ be a fixed, and let $\{x_n\}$ be a $\preceq_\eta$-decreasing Cauchy sequence in $S(x_0)$. Then it is a $\preceq_\eta$-decreasing Cauchy sequence in X. Hence, $\phi(x_{n+1}) \preceq \phi(x_n)$ for all $n \in \mathbb{N}$. Since X is complete, there exists $x \in X$ such that $\lim_{n\to\infty} x_n = x$. Since ϕ is lower semicontinuous from above, $\phi(x) \preceq \lim_{n\to\infty} \phi(x_n)$. Thus, $\phi(x) \preceq \phi(x_n)$ for all $n \in \mathbb{N}$. Since $x_m \preceq_\eta x_n$ for $m > n$, we obtain

$$\phi(x_n) - \phi(x_m) \in s(\eta(d(x_n, x_m))).$$

Hence,

$$\eta(d(x_n, x_m)) \preceq \phi(x_n) - \phi(x_m) \preceq \phi(x_n) - \phi(x).$$

Letting $m \to \infty$ in above inequality, we have $\eta(d(x_n, x)) \preceq \phi(x_n) - \phi(x)$ because η and d are continuous. Hence, $\phi(x_n) - \phi(x) \in s(\eta(d(x_n, x)))$.

Thus, we have $x \preceq_\eta x_n$, and so $x \preceq_\eta x_n \preceq_\eta x_0$. Hence, $x \in S(x_0)$, and hence $S(x_0)$ is $\preceq_\eta$-complete. From Theorem 3.3, T has a fixed point in X. □

Corollary 3.4 *Let (X,d) be a complete cone metric space such that P is strongly minihedral and continuous. Suppose that $T : X \to N(X)$ is a multivalued mapping and $\phi : X \to E$ is lower semicontinuous from above and bounded from below.*

If for any $x \in X$ and for any $y \in Tx$,

$$\phi(x) - \phi(y) \in s(\eta(d(x,y))),$$

then there exists $x_0 \in X$ such that $Tx_0 = \{x_0\}$.

We now give an example to support Theorem 3.3.

Example 3.1 Let $X = L^\infty[0,1]$, and let $E = \mathbb{R}^2$ and $P = \{(x,y) : x, y \geq 0\}$. We define $d : X \times X \to E$ by $d(f,g) = (\|f-g\|_\infty, \|f-g\|_p)$, where $1 \leq p < \infty$. Then (X,d) is a complete cone metric space, and P is strongly minihedral and continuous.

Let $\eta(s) = s$ for all $s \in P$.

We define a multivalued mapping $T : X \to N(X)$ by

$$Tf = \Big\{ g \in X : -2f(x) \leq g(x) \leq \frac{1}{2} f(x) \text{ if } f(x) \geq 0 \text{ and } \\ \frac{1}{2} f(x) \leq g(x) \leq -2f(x) \text{ if } f(x) < 0 \Big\}$$

and we define a mapping $\phi : X \to P$ by

$$\phi(f) = \left(\|f\|_\infty, \|f\|_p\right).$$

Then ϕ is lower semicontinuous from above and bounded from below.

For any $f \in X$, put $g(x) = \frac{1}{2}f(x) \in Tf$. Then we have $\eta(d(f,g)) = (\frac{1}{2}\|f\|_\infty, \frac{1}{2}\|f\|_p) = \phi(f) - \phi(g)$, and so $\phi(f) - \phi(g) \in s(\eta(d(f,g)))$.

Thus, all conditions of Theorem 3.3 are satisfied and T has a fixed point $f^*(x) = 0$.

Remark 3.1 Theorem 3.3 (resp. Corollary 3.4) is a generalization of Theorem 4.2 (resp. Corollary 4.3) in [21], and also results in [38, 50] to the case of cone metric spaces.

If $\eta(t) = t$ in Theorem 3.3 (resp. Corollary 3.4), then we have generalizations of the results in [12, 36, 49] to the case of cone metric spaces.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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