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Common fixed points of Ćirić-type contractive mappings in two ordered generalized metric spaces

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Abstract
In this paper, using the setting of two ordered generalized metric spaces, a unique common fixed point of four mappings satisfying a generalized contractive condition is obtained. We also present an example to demonstrate the results presented herein.
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1 Introduction and preliminaries

The study of a unique common fixed point of given mappings satisfying certain contractive conditions has been at the center of rigorous research activity. Mustafa and Sims [1] generalized the concept of a metric in which a real number is assigned to every triplet of an arbitrary set. Based on the notion of generalized metric spaces, Mustafa *et al.* [2–5] obtained some fixed point theorems for some mappings satisfying different contractive conditions. The existence of common fixed points in generalized metric spaces was initiated by Abbas and Rhoades [6] (see also [7] and [8]). For further study of common fixed points in generalized metric spaces, we refer to [9–12] and references mentioned therein. Abbas *et al.* [13] showed the existence of coupled common fixed points in two generalized metric spaces (for more results on couple fixed points, see also [14–21]).

The existence of fixed points in ordered metric spaces has been initiated in 2004 by Ran and Reurings [22] and further studied by Nieto and Lopez [23]. Subsequently, several interesting and valuable results have appeared in this direction [24–30].

The aim of this paper is to study common fixed point of four mappings that satisfy the generalized contractive condition in two ordered generalized metric spaces.

In the sequel, $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{N}$ denote the set of real numbers, the set of nonnegative integers and the set of positive integers respectively. The usual order on $\mathbb{R}$ (respectively, on $\mathbb{R}^+$) will be indistinctly denoted by $\leq$ or by $\geq$.

In [1], Mustafa and Sims introduced the following definitions and results:

Definition 1.1 Let X be a nonempty set. Suppose that a mapping $G : X \times X \times X \to \mathbb{R}^+$ satisfies the following conditions:

(a) $G(x, y, z) = 0$ if $x = y = z$ for all $x, y, z \in X$;
(b) $0 < G(x, y, z)$ for all $x, y, z \in X$ with $x \neq y$;

(c) $G(x,x,y) \leq G(x,y,z)$ for all $x,y,z \in X$ with $y \neq z$;
(d) $G(x,y,z) = G(p\{x,y,z\})$, where p is a permutation of $x,y,z \in X$ (symmetry);
(e) $G(x,y,z) \leq G(x,a,a) + G(a,y,z)$ for all $x,y,z,a \in X$.

Then G is called a G-metric on X and (X,G) is called a G-metric space.

Definition 1.2 A sequence $\{x_n\}$ in a G-metric space X is called:

(1) a *G-Cauchy sequence* if, for any $\varepsilon > 0$, there exists $n_0 \in N$ (the set of natural numbers) such that, for all $n,m,l \geq n_0$, $G(x_n,x_m,x_l) < \varepsilon$;
(2) *G-convergent* if, for any $\varepsilon > 0$, there exist $x \in X$ and $n_0 \in N$ such that, for all $n,m \geq n_0$, $G(x,x_n,x_m) < \varepsilon$;
(3) A G-metric space X is said to be *G-complete* if every G-Cauchy sequence in X is G-convergent in X.

It is known that $\{x_n\}$ is G-convergent to a point $x \in X$ if and only if $G(x_m,x_n,x) \to 0$ as $n,m \to \infty$.

Proposition 1.3 [1] *Let X be a G-metric space. Then the following items are equivalent:*

(1) A sequence $\{x_n\}$ in X is G-convergent to a point $x \in X$;
(2) $G(x_n,x_m,x) \to 0$ as $n,m \to \infty$;
(3) $G(x_n,x_n,x) \to 0$ as $n \to \infty$;
(4) $G(x_n,x,x) \to 0$ as $n \to \infty$.

Definition 1.4 A G-metric on X is said to be *symmetric* if $G(x,y,y) = G(y,x,x)$ for all $x,y \in X$.

Proposition 1.5 *Every G-metric on X defines a metric d_G on X by*

$$d_G(x,y) = G(x,y,y) + G(y,x,x) \tag{1.1}$$

for all $x,y \in X$.

For a symmetric G-metric, we have

$$d_G(x,y) = 2G(x,y,y) \tag{1.2}$$

for all $x,y \in X$. However, if G is non-symmetric, then the following inequality holds:

$$\frac{3}{2}G(x,y,y) \leq d_G(x,y) \leq 3G(x,y,y) \tag{1.3}$$

for all $x,y \in X$. It is obvious that

$$G(x,x,y) \leq 2G(x,y,y)$$

for all $x,y \in X$.

Now, we give an example of a non-symmetric G-metric.

Table 1 ***G*-metric**

(x, y, z)	$G(x, y, z)$
(1, 1, 1), (2, 2, 2)	0
(1, 1, 2), (1, 2, 1), (2, 1, 1)	0.5
(1, 2, 2), (2, 1, 2), (2, 2, 1)	1.0

Example 1.6 Let $X = \{1, 2\}$ and $G : X \times X \times X \to \mathbb{R}^+$ be a mapping defined by Table 1.

Note that G satisfies all the axioms of a generalized metric, but $G(x, x, y) \neq G(x, y, y)$ for two distinct points $x, y \in X$.

Definition 1.7 Let f and g be self-mappings on a set X. If $w = fx = gx$ for some $x \in X$, then the point x is called a *coincidence point* of f and g and w is called a *point of coincidence* of f and g.

Definition 1.8 [31] Let f and g be self-mappings on a set X. Then f and g are said to be *weakly compatible* if they commute at every coincidence point.

Definition 1.9 [8] Let X be a G-metric space and f, g be self-mappings on X. Then f and g are said to be *R-weakly commuting* if there exists a positive real number R such that $G(fgx, fgx, gfx) \leq RG(fx, fx, gx)$ for all $x \in X$.

The maps f and g are R-weakly commuting on X if and only if they commute at their coincidence points.

Recall that two mappings f and g on a G-metric space X are said to be compatible if, for a sequence $\{x_n\}$ in X such that $\{fx_n\}$ and $\{gx_n\}$ are G-convergent to some $t \in X$,

$$\lim_{n\to\infty} G(fgx_n, fgx_n, gfx_n) = 0.$$

Definition 1.10 Let X be a nonempty set. Then $(X, \preceq, G)$ is called an *ordered generalized metric space* if the following conditions hold:

(a) G is a generalized metric on X;

(b) $\preceq$ is a partial order on X.

Definition 1.11 Let $(X, \preceq)$ be a partial ordered set. Then two points $x, y \in X$ are said to be *comparable* if $x \preceq y$ or $y \preceq x$.

Definition 1.12 [24] Let $(X, \preceq)$ be a partially ordered set. A self-mapping f on X is said to be *dominating* if $x \preceq fx$ for all $x \in X$.

Example 1.13 [24] Let $X = [0, 1]$ be endowed with usual ordering and $f : X \to X$ be a mapping defined by $fx = \sqrt[n]{x}$ for some $n \in \mathbb{N}$. Since $x \leq x^{\frac{1}{n}} = fx$ for all $x \in X$, f is a dominating mapping.

Definition 1.14 Let $(X, \preceq)$ be a partially ordered set. A self-mapping f on X is said to be *dominated* if $fx \preceq x$ for all $x \in X$.

Example 1.15 Let $X = [0, 1]$ be endowed with usual ordering and $f : X \to X$ be a mapping defined by $fx = x^n$ for some $n \in \mathbb{N}$. Since $fx = x^n \leq x$ for all $x \in X$, f is a dominated mapping.

Definition 1.16 A subset $\mathcal{K}$ of a partially ordered set X is said to be *well-ordered* if every two elements of $\mathcal{K}$ are comparable.

2 Common fixed point theorems

In [32], Kannan proved a fixed point theorem for a single valued self-mapping T on a metric space X satisfying the following property:

$$d(Tx,Ty)\leq h\{d(x,Tx)+d(y,Ty)\}$$

for all $x,y\in X$, where $h\in[0,\frac{1}{2})$. If a self-mapping T on a metric space X satisfies the following property:

$$d(Tx,Ty)\leq ad(x,y)+bd(x,Tx)+cd(y,Ty)+e[d(x,Ty)+d(y,Tx)]$$

for all $x,y\in X$, where $a,b,c,e\geq 0$ with $a+b+c+2e<1$, then T has a unique fixed point provided that X is T-orbitally complete (for related definitions and results, we refer to [33]).

Afterwards, Ćirić [34] obtained a fixed point result for a mapping satisfying the following property:

$$d(Tx,Ty)\leq q\max\left\{d(x,y),d(x,Tx),d(y,Ty),\frac{d(x,Ty)+d(y,Tx)}{2}\right\}$$

for all $x,y\in X$, where $0\leq q<1$.

In this section, we show the existence of a unique common fixed point of four mappings satisfying Ćirić-type contractive condition in the framework of two ordered generalized metric spaces.

Now, we start with the following result:

Theorem 2.1 *Let $(X,\preceq)$ be a partially ordered set and G_1, G_2 be two G-metrics on X such that $G_2(x,y,z)\leq G_1(x,y,z)$ for all $x,y,z\in X$ with a complete metric G_1 on X. Suppose that f, g, S and T are self-mappings on X satisfying the following properties:*

$$\begin{aligned} G_1(fx,fx,gy)\leq k\max\{&G_2(Sx,Sx,Ty),G_2(fx,fx,Sx),G_2(gy,gy,Ty),\\ &[G_2(fx,fx,Ty)+G_2(gy,gy,Sx)]/2\}\end{aligned}\tag{2.1}$$

and

$$\begin{aligned} G_1(fx,gy,gy)\leq k\max\{&G_2(Sx,Ty,Ty),G_2(fx,Sx,Sx),G_2(gy,Ty,Ty),\\ &[G_2(fx,Ty,Ty)+G_2(gy,Sx,Sx)]/2\}\end{aligned}\tag{2.2}$$

for all comparable $x,y\in X$, where $k\in[0,1)$. Suppose that $f(X)\subseteq T(X)$ and $g(X)\subseteq S(X)$, f, g are dominated mappings and S, T are dominating mappings. If, for any nonincreasing sequence $\{x_n\}$ in X with $y_n\preceq x_n$ for all $n\in\mathbb{N}$, $y_n\to u$ implies that $u\preceq x_n$ and either

(a) f, S are compatible, f or S is continuous and g, T are weakly compatible

or

(b) g, T are compatible, g or T is continuous and f, S are weakly compatible,

then f, g, S and T have a common fixed point. Moreover, the set of common fixed points of f, g, S and T is well-ordered if and only if f, g, S and T have one and only one common fixed point.

Proof Let x_0 be an arbitrary point in X. Since $f(X) \subseteq T(X)$ and $g(X) \subseteq S(X)$, we can define the sequences $\{x_n\}$ and $\{y_n\}$ in X by

$$y_{2n} = gx_{2n} = Sx_{2n+1}, \quad y_{2n+1} = fx_{2n+1} = Tx_{2n+2}$$

for all $n \geq 0$. By the given assumptions, we have

$$\begin{aligned} &x_{2n+2} \preceq Tx_{2n+2} = fx_{2n+1} \preceq x_{2n+1}, \\ &x_{2n+1} \preceq Sx_{2n+1} = gx_{2n} \preceq x_{2n}. \end{aligned}$$

Thus, for all $n \geq 0$, we have $x_{n+1} \preceq x_n$. Suppose that $G_1(y_{2n}, y_{2n+1}, y_{2n+1}) > 0$ for all $n \geq 0$. If not, then, for some $m \geq 0$, $y_m = y_{m+1}$. Indeed, if $m = 2k$, then $y_{2k} = y_{2k+1}$ and from (2.1), it follows that

$$\begin{aligned} &G_1(y_{2k+1}, y_{2k+1}, y_{2k+2}) \\ &\quad = G_1(fx_{2k+1}, fx_{2k+1}, gx_{2k+2}) \\ &\quad \leq k \max\bigl\{G_2(Sx_{2k+1}, Sx_{2k+1}, Tx_{2k+2}), G_2(fx_{2k+1}, fx_{2k+1}, Sx_{2k+1}), \\ &\qquad G_2(gx_{2k+2}, gx_{2k+2}, Tx_{2k+2}), \\ &\qquad \bigl[G_2(fx_{2k+1}, fx_{2k+1}, Tx_{2k+2}) + G_2(gx_{2k+2}, gx_{2k+2}, Sx_{2k+1})\bigr]/2\bigr\} \\ &\quad = k \max\bigl\{G_2(y_{2k}, y_{2k}, y_{2k+1}), G_2(y_{2k+1}, y_{2k+1}, y_{2k}), G_2(y_{2k+2}, y_{2k+2}, y_{2k+1}), \\ &\qquad \bigl[G_2(y_{2k+1}, y_{2k+1}, y_{2k+1}) + G_2(y_{2k+2}, y_{2k+2}, y_{2k})\bigr]/2\bigr\} \\ &\quad \leq k \max\bigl\{G_2(y_{2k}, y_{2k}, y_{2k+1}), G_2(y_{2k+1}, y_{2k+1}, y_{2k}), G_2(y_{2k+2}, y_{2k+2}, y_{2k+1}), \\ &\qquad \bigl[G_2(y_{2k+2}, y_{2k+2}, y_{2k+1}) + G_2(y_{2k+1}, y_{2k+1}, y_{2k})\bigr]/2\bigr\} \\ &\quad \leq k \max\bigl\{G_1(y_{2k}, y_{2k}, y_{2k+1}), G_1(y_{2k+1}, y_{2k+1}, y_{2k}), G_1(y_{2k+2}, y_{2k+2}, y_{2k+1}), \\ &\qquad \bigl[G_1(y_{2k+2}, y_{2k+2}, y_{2k+1}) + G_1(y_{2k+1}, y_{2k+1}, y_{2k})\bigr]/2\bigr\} \\ &\quad = kG_1(y_{2k+2}, y_{2k+2}, y_{2k+1}). \end{aligned} \tag{2.3}$$

Again, from (2.2), it follows that

$$\begin{aligned} &G_1(y_{2k+1}, y_{2k+2}, y_{2k+2}) \\ &\quad = G_1(fx_{2k+1}, gx_{2k+2}, gx_{2k+2}) \\ &\quad \leq k \max\bigl\{G_2(Sx_{2k+1}, Tx_{2k+2}, Tx_{2k+2}), G_2(fx_{2k+1}, Sx_{2k+1}, Sx_{2k+1}), \\ &\qquad G_2(gx_{2k+2}, Tx_{2k+2}, Tx_{2k+2}), \\ &\qquad \bigl[G_2(fx_{2k+1}, Tx_{2k+2}, Tx_{2k+2}) + G_2(gx_{2k+2}, Sx_{2k+1}, Sx_{2k+1})\bigr]/2\bigr\} \end{aligned}$$

$$
\begin{aligned}
&= k\max\bigl\{G_2(y_{2k},y_{2k+1},y_{2k+1}),G_2(y_{2k+1},y_{2k},y_{2k}),G_2(y_{2k+2},y_{2k+1},y_{2k+1}),\\
&\quad \bigl[G_2(y_{2k+1},y_{2k+1},y_{2k+1})+G_2(y_{2k+2},y_{2k},y_{2k})\bigr]/2\bigr\}\\
&\le k\max\bigl\{G_2(y_{2k},y_{2k+1},y_{2k+1}),G_2(y_{2k+1},y_{2k},y_{2k}),G_2(y_{2k+2},y_{2k+1},y_{2k+1}),\\
&\quad \bigl[G_2(y_{2k+2},y_{2k+1},y_{2k+1})+G_2(y_{2k+1},y_{2k},y_{2k})\bigr]/2\bigr\}\\
&\le k\max\bigl\{G_1(y_{2k},y_{2k},y_{2k+1}),G_2(y_{2k+1},y_{2k},y_{2k}),G_1(y_{2k+2},y_{2k+1},y_{2k+1}),\\
&\quad \bigl[G_1(y_{2k+2},y_{2k+1},y_{2k+1})+G_1(y_{2k+1},y_{2k},y_{2k})\bigr]/2\bigr\}\\
&= kG_1(y_{2k+2},y_{2k+1},y_{2k+1}).
\end{aligned}
\tag{2.4}
$$

Thus (2.3) and (2.4) imply that

$$G_1(y_{2k+2},y_{2k+1},y_{2k+1}) \le k^2 G_1(y_{2k+2},y_{2k+1},y_{2k+1})$$

and so $y_{2k+1} = y_{2k+2}$ since $k^2 < 1$.

Similarly, if $m = 2k+1$, then one can easily obtain $y_{2k+2} = y_{2k+3}$. Thus $\{y_n\}$ becomes a constant sequence and y_{2n} serves as the common fixed point of f, g, S and T.

Suppose that $G_1(y_{2n},y_{2n+1},y_{2n+1}) > 0$ for all $n \ge 0$.

If $n \in \mathbb{N}$ is even, then $n = 2k$ for some $k \in \mathbb{N}$; then it follows from (2.1) that

$$
\begin{aligned}
&G_1(y_{n+1},y_{n+1},y_n)\\
&\quad= G_1(y_{2k+1},y_{2k+1},y_{2k})\\
&\quad= G_1(fx_{2k+1},fx_{2k+1},gx_{2k})\\
&\quad\le k\max\bigl\{G_2(Sx_{2k+1},Sx_{2k+1},Tx_{2k}),G_2(fx_{2k+1},fx_{2k+1},Sx_{2k+1}),\\
&\qquad G_2(gx_{2k},gx_{2k},Tx_{2k}),\bigl[G_2(fx_{2k+1},fx_{2k+1},Tx_{2k})+G_2(gx_{2k},gx_{2k},Sx_{2k+1})\bigr]/2\bigr\}\\
&\quad= k\max\bigl\{G_2(y_{2k},y_{2k},y_{2k-1}),G_2(y_{2k+1},y_{2k+1},y_{2k}),\\
&\qquad G_2(y_{2k},y_{2k},y_{2k-1}),\bigl[G_2(y_{2k+1},y_{2k+1},y_{2k-1})+G_2(y_{2k},y_{2k},y_{2k})\bigr]/2\bigr\}\\
&\quad\le k\max\bigl\{G_2(y_{2k},y_{2k},y_{2k-1}),G_2(y_{2k+1},y_{2k+1},y_{2k}),\\
&\qquad \bigl[G_2(y_{2k+1},y_{2k+1},y_{2k})+G_2(y_{2k},y_{2k},y_{2k-1})\bigr]/2\bigr\}\\
&\quad\le k\max\bigl\{G_1(y_n,y_n,y_{n-1}),G_1(y_{n+1},y_{n+1},y_n)\bigr\},
\end{aligned}
$$

which implies that

$$G_1(y_{n+1},y_{n+1},y_n) \le kG_1(y_n,y_n,y_{n-1}).$$

If $n \in \mathbb{N}$ is odd, then $n = 2k+1$ for some $k \in \mathbb{N}$. Again, it follows from (2.1) that

$$
\begin{aligned}
&G_1(y_{n+1},y_{n+1},y_n)\\
&\quad= G_1(y_{2k+2},y_{2k+2},y_{2k+1})\\
&\quad= G_1(fx_{2k+2},fx_{2k+2},gx_{2k+1})\\
&\quad\le k\max\bigl\{G_2(Sx_{2k+2},Sx_{2k+2},Tx_{2k+1}),G_2(fx_{2k+2},fx_{2k+2},Sx_{2k+2}),
\end{aligned}
$$

$$\begin{aligned}
&G_2(gx_{2k+1}, gx_{2k+1}, Tx_{2k+1}),\\
&\big[G_2(fx_{2k+2}, fx_{2k+2}, Tx_{2k+1}) + G_2(gx_{2k+1}, gx_{2k+1}, Sx_{2k+2})\big]/2\big\}\\
&= k \max\big\{G_2(y_{2k+1}, y_{2k+1}, y_{2k}), G_2(y_{2k+2}, y_{2k+2}, y_{2k+1}),\\
&G_2(y_{2k+1}, y_{2k+1}, y_{2k}), \big[G_2(y_{2k+2}, y_{2k+2}, y_{2k}) + G_2(y_{2k+1}, y_{2k+1}, y_{2k+1})\big]/2\big\}\\
&\le k \max\big\{G_2(y_{2k+1}, y_{2k+1}, y_{2k}), G_2(y_{2k+2}, y_{2k+2}, y_{2k+1}),\\
&\big[G_2(y_{2k+2}, y_{2k+2}, y_{2k+1}) + G_2(y_{2k+1}, y_{2k+1}, y_{2k})\big]/2\big\}\\
&\le k \max\big\{G_1(y_{2k+1}, y_{2k+1}, y_{2k}), G_1(y_{2k+2}, y_{2k+2}, y_{2k+1})\big\}\\
&= k \max\big\{G_1(y_n, y_n, y_{n-1}), G_1(y_{n+1}, y_{n+1}, y_n)\big\},
\end{aligned}$$

that is,

$$G_1(y_{n+1}, y_{n+1}, y_n) \le kG_1(y_n, y_n, y_{n-1})$$

for all $n \in \mathbb{N}$. Continuing the above process, we have

$$G_1(y_{n+1}, y_{n+1}, y_n) \le k^n G_1(y_1, y_1, y_0)$$

for all $n \in \mathbb{N}$. Thus, for all $n, m \in \mathbb{N}$ with $m > n$, we have

$$\begin{aligned}
&G_1(y_m, y_m, y_n)\\
&\quad\le G_1(y_n, y_{n+1}, y_{n+1}) + G_1(y_{n+1}, y_{n+2}, y_{n+2}) + \cdots + G_1(y_{m-1}, y_m, y_m)\\
&\quad\le k^n G_1(y_0, y_1, y_1) + k^{n+1} G_1(y_0, y_1, y_1) + \cdots + k^{m-1} G_1(y_0, y_1, y_1)\\
&\quad= k^n G_1(y_0, y_1, y_1) \sum_{i=0}^{m-n-1} k^i\\
&\quad\le \frac{k^n}{1-k} G_1(y_0, y_1, y_1)
\end{aligned}$$

and so $G_1(y_n, y_m, y_m) \to 0$ as $m, n \to \infty$. Hence $\{y_n\}$ is a G-Cauchy sequence in X. Since X is G_1-complete, there exists a point $z \in X$ such that $\lim_{n\to\infty} y_n = z$. Consequently, we have

$$\lim_{n\to\infty} y_{2n+1} = \lim_{n\to\infty} fx_{2n+1} = \lim_{n\to\infty} Tx_{2n+2} = z$$

and

$$\lim_{n\to\infty} y_{2n} = \lim_{n\to\infty} gx_{2n} = \lim_{n\to\infty} Sx_{2n+1} = z.$$

If S is continuous and $\{f, S\}$ is compatible, then

$$\begin{aligned}
&\lim_{n\to\infty} S^2 x_{2n+1} = Sz,\\
&\lim_{n\to\infty} fSx_{2n+1} = \lim_{n\to\infty} Sfx_{2n+1} = Sz.
\end{aligned}$$

Since $Sx_{2n+1} = gx_{2n} \preceq x_{2n}$, (2.1) gives

$$\begin{aligned} &G_1(fSx_{2n+1}, fSx_{2n+1}, gx_{2n}) \\ &\quad \le k \max\{G_2(SSx_{2n+1}, SSx_{2n+1}, Tx_{2n}), G_2(fSx_{2n+1}, fSx_{2n+1}, SSx_{2n+1}), \\ &\qquad G_2(gx_{2n}, gx_{2n}, Tx_{2n}), \\ &\qquad [G_2(fSx_{2n+2}, fSx_{2n+2}, Tx_{2n}) + G_2(gx_{2n}, gx_{2n}, SSx_{2n+1})]/2\}. \end{aligned}$$

Taking the limit as $n \to \infty$, we obtain

$$\begin{aligned} G_1(Sz, Sz, z) &\le k \max\{G_2(Sz, Sz, z), G_2(Sz, Sz, Sz), G_2(z, z, z), \\ &\qquad [G_2(Sz, Sz, z) + G_2(z, z, Sz)]/2\} \\ &\le k \max\{G_1(Sz, Sz, z), [G_1(Sz, Sz, z) + G_1(z, z, Sz)]/2\} \\ &= \frac{k}{2}[G_1(Sz, Sz, z) + G_1(z, z, Sz)], \end{aligned}$$

which further implies that

$$G_1(Sz, Sz, z) \le hG_1(z, z, Sz), \tag{2.5}$$

where $h = \frac{k}{2-k}$. Obviously, $0 \le h < 1$.

Similarly, we obtain

$$G_1(Sz, z, z) \le hG_1(z, Sz, Sz). \tag{2.6}$$

From (2.5) and (2.6), we have

$$G_1(Sz, Sz, z) \le h^2 G_1(z, Sz, Sz)$$

and so $Sz = z$ since $0 \le h^2 < 1$. Since $gx_{2n} \preceq x_{2n}$ and $gx_{2n} \to z$ as $n \to \infty$ implies $z \preceq x_{2n}$, it follows from (2.1) that

$$\begin{aligned} &G_1(fz, fz, gx_{2n}) \\ &\quad \le k \max\{G_2(Sz, Sz, Tx_{2n}), G_2(fz, fz, Sz), G_2(gx_{2n}, gx_{2n}, Tx_{2n}), \\ &\qquad [G_2(fz, fz, Tx_{2n}) + G_2(gx_{2n}, gx_{2n}, Sz)]/2\} \\ &\quad = k \max\{G_2(z, z, Tx_{2n}), G_2(fz, fz, z), G_2(gx_{2n}, gx_{2n}, Tx_{2n}), \\ &\qquad [G_2(fz, fz, Tx_{2n}) + G_2(gx_{2n}, gx_{2n}, z)]/2\}, \end{aligned}$$

which, taking the limit as $n \to \infty$, gives

$$\begin{aligned} G_1(fz, fz, z) &\le k \max\{G_2(z, z, z), G_2(fz, fz, z), G_2(z, z, z), \\ &\qquad [G_2(fz, fz, z) + G_2(z, z, z)]/2\} \\ &\le kG_1(fz, fz, z). \end{aligned} \tag{2.7}$$

Similarly, we obtain

$$G_1(fz,z,z) \leq kG_1(z,fz,fz). \tag{2.8}$$

Therefore, by using the above two inequalities, we have $fz = z$.

Since $f(X) \subseteq T(X)$, there exists a point $v \in X$ such that $fz = Tv$. Since $v \preceq Tv = fz \preceq z$, it follows from (2.1) that

$$\begin{aligned} G_1(fz,fz,gv) &\leq k\max\bigl\{G_2(Sz,Sz,Tv), G_2(fz,fz,Sz), G_2(gv,gv,Tv), \\ &\quad \bigl[G_2(fz,fz,Tv) + G_2(gv,gv,Sz)\bigr]/2\bigr\} \\ &= k\max\bigl\{G_2(fz,fz,fz), G_2(fz,fz,fz), G_2(gv,gv,fz), \\ &\quad \bigl[G_2(fz,fz,fz) + G_2(gv,gv,fz)\bigr]/2\bigr\} \\ &\leq kG_1(fz,gv,gv). \end{aligned} \tag{2.9}$$

Similarly, we get

$$G_1(fz,gv,gv) \leq kG_1(fz,fz,gv). \tag{2.10}$$

Thus (2.9) and (2.10) imply $fz = gv$. Since g and T are weakly compatible, we have $gz = gfz = gTv = Tgv = Tfz = Tz$, and so z is the coincidence point of g and T.

Now, from (2.1), we have

$$\begin{aligned} G_1(z,z,gz) &= G_1(fz,fz,gz) \\ &\leq k\max\bigl\{G_2(Sz,Sz,Tz), G_2(fz,fz,Sz), G_2(gz,gz,Tz), \\ &\quad \bigl[G_2(fz,fz,Tz) + G_2(gz,gz,Sz)\bigr]/2\bigr\} \\ &= k\max\bigl\{G_2(z,z,gz), G_2(z,z,z), G_2(gz,gz,gz), \\ &\quad \bigl[G_2(z,z,gz) + G_2(gz,gz,z)\bigr]/2\bigr\} \\ &= k\max\bigl\{G_2(z,z,gz), \bigl[G_2(z,z,gz) + G_2(gz,gz,z)\bigr]/2\bigr\} \\ &\leq \frac{k}{2}\bigl[G_1(z,z,gz) + G_1(gz,gz,z)\bigr], \end{aligned}$$

that is,

$$G_1(z,z,gz) \leq hG_1(gz,gz,z), \tag{2.11}$$

where $h = \frac{k}{2-k}$. Obviously, $0 \leq h < 1$. Using (2.2), we have

$$G_1(z,gz,gz) \leq hG_1(z,z,gz). \tag{2.12}$$

Combining the above two inequalities, we get

$$G_1(z,z,gz) \leq h^2 G_1(z,z,gz)$$

and so $z = gz$. Therefore, $fz = gz = Sz = Tz = z$. The proof is similar when f is continuous. Similarly, if (b) holds, then the result follows.

Now, suppose that the set of common fixed points of f, g, S and T is well ordered. We show that a common fixed point of f, g, S and T is unique. Let u be another common fixed point of f, g, S and T. Then, from (2.1), we have

$$\begin{aligned} G_1(z,z,u) &= G_1(fz,fz,gu) \\ &\leq k\max\bigl\{G_2(Sz,Sz,Tu),G_2(fz,fz,Sz),G_2(gu,gu,Tu), \\ &\quad \bigl[G_2(fz,fz,Tu)+G_2(gu,gu,Sz)\bigr]/2\bigr\} \\ &= k\max\bigl\{G_2(z,z,u),G_2(z,z,z),G_2(u,u,u), \\ &\quad \bigl[G_2(z,z,u)+G_2(u,u,z)\bigr]/2\bigr\} \\ &= \frac{k}{2}\bigl[G_2(z,z,u)+G_2(u,u,z)\bigr] \\ &\leq \frac{1}{2}G_1(z,z,u)+\frac{k}{2}G_1(u,u,z), \end{aligned}$$

that is,

$$G_1(z,z,u) \leq kG_1(z,u,u).$$

Similarly, using (2.2), we obtain

$$G_1(z,u,u) \leq kG_1(z,z,u).$$

Combining the above two inequalities, we get

$$G_1(z,z,u) \leq k^2G_1(z,z,u)$$

and hence $z = u$.

The converse follows immediately. This completes the proof. □

Example 2.2 Let $X = \{0,1,2,3\}$ be endowed with the usual ordering and G_1, G_2 be two G-metrics on X defined by Table 2. Then G_1 and G_2 are non-symmetric since $G_1(1,1,0) \neq$

Table 2 Two *G*-metrices

(x,y,z)	$G_1(x,y,z)$	$G_2(x,y,z)$
(0,0,0), (1,1,1), (2,2,2), (3,3,3),	0	0
(0,0,2), (0,2,0), (2,0,0), (0,2,2), (2,0,2), (2,2,0), (0,0,1), (0,1,0), (1,0,0), (0,0,3), (0,3,0), (3,0,0),	4	3
(0,1,1), (1,0,1), (1,1,0), (0,3,3), (3,0,3), (3,3,0), (1,1,2), (1,2,1), (2,1,1), (1,2,2), (2,1,2), (2,2,1), (1,1,3), (1,3,1), (3,1,1), (1,3,3), (3,1,3), (3,3,1), (2,2,3), (2,3,2), (3,2,2), (2,3,3), (3,2,3), (3,3,2),	8	6
(0,1,2), (0,1,3), (0,2,1), (0,2,3), (0,3,1), (0,3,2), (1,0,2), (1,0,3), (1,2,0), (1,2,3), (1,3,0), (1,3,2), (2,0,1), (2,0,3), (2,1,0), (2,1,3), (2,3,0), (2,3,1), (3,0,1), (3,0,2), (3,1,0), (3,1,2), (3,2,0), (3,2,1),	8	6

Table 3 Self maps

x	$f(x)$	$g(x)$	$S(x)$	$T(x)$
0	0	0	0	0
1	0	0	2	2
2	0	2	2	3
3	0	0	3	3

Table 4 Dominated and dominating maps

$x \in X$	f is dominated	g is dominated	S is dominating	T is dominating
$x = 0$	$f(0) = 0$	$g(0) = 0$	$0 = S(0)$	$0 = T(0)$
$x = 1$	$f(1) = 0 < 1$	$g(1) = 0 < 1$	$1 < 2 = S(1)$	$1 < 2 = T(1)$
$x = 2$	$f(2) = 0 < 2$	$g(2) = 2$	$2 = S(2)$	$2 < 3 = T(2)$
$x = 3$	$f(3) = 0 < 3$	$g(3) = 0 < 3$	$3 = S(3)$	$3 = T(3)$

$G_1(1,0,0)$ and $G_2(1,1,0) \neq G_2(1,0,0)$ with $G_2(x,y,z) \leq G_1(x,y,z)$ for all $x,y,z \in X$. Let $f,g,S,T : X \to X$ be the mappings defined by Table 3. Clearly, $f(X) \subseteq T(X)$, $g(X) \subseteq S(X)$, f, g are dominated mappings and S, T are dominating mappings, see Table 4.

Now, we shall show that for all comparable $x,y \in X$, (2.1) and (2.2) are satisfied with $k = \frac{3}{4} \in [0,1)$. Note that for all $x,y \in \{0,1,3\}$, $G(fx,fx,gy) = G(fx,gy,gy) = 0$ and (2.1), (2.2) are satisfied obviously.

(1) When $x = 0$ and $y = 2$, then $fx = 0$, $gy = 2$, $Sx = 0$, $Ty = 3$ and so

$$\begin{aligned} G_1(fx,fx,gy) &= G_1(0,0,2) = 4 \\ &< \frac{3}{4}(6) = \frac{3}{4}G_2(2,2,3) = \frac{3}{4}G_2(gy,gy,Ty) \\ &\leq k\max\bigl\{G_2(Sx,Sx,Ty), G_2(fx,fx,Sx), G_2(gy,gy,Ty), \\ &\quad \bigl[G_2(fx,fx,Ty) + G_2(gy,gy,Sx)\bigr]/2\bigr\} \end{aligned}$$

and

$$\begin{aligned} G_1(fx,gy,gy) &= G_1(0,2,2) = 4 \\ &< \frac{3}{4}(6) = \frac{3}{4}G_2(2,3,3) = \frac{3}{4}G_2(gy,Ty,Ty) \\ &\leq k\max\bigl\{G_2(Sx,Ty,Ty), G_2(fx,Sx,Sx), G_2(gy,Ty,Ty), \\ &\quad \bigl[G_2(fx,Ty,Ty) + G_2(gy,Sx,Sx)\bigr]/2\bigr\}. \end{aligned}$$

(2) When $x = 1$ and $y = 2$, then $fx = 0$, $gy = 2$, $Sx = 2$, $Ty = 3$ and so

$$\begin{aligned} G_1(fx,fx,gy) &= G_1(0,0,2) = 4 \\ &< \frac{3}{4}(6) = \frac{3}{4}G_2(2,2,3) = \frac{3}{4}G_2(gy,gy,Ty) \\ &\leq k\max\bigl\{G_2(Sx,Sx,Ty), G_2(fx,fx,Sx), G_2(gy,gy,Ty), \\ &\quad \bigl[G_2(fx,fx,Ty) + G_2(gy,gy,Sx)\bigr]/2\bigr\} \end{aligned}$$

and

$$\begin{aligned} G_1(fx, gy, gy) &= G_1(0,2,2) = 4 \\ &< \frac{3}{4}(6) = \frac{3}{4}G_2(2,3,3) = \frac{3}{4}G_2(gy, Ty, Ty) \\ &\leq k\max\bigl\{G_2(Sx, Ty, Ty), G_2(fx, Sx, Sx), G_2(gy, Ty, Ty), \\ &\quad \bigl[G_2(fx, Ty, Ty) + G_2(gy, Sx, Sx)\bigr]/2\bigr\}. \end{aligned}$$

(3) When $x = 2$ and $y = 2$, then $fx = 0$, $gy = 2$, $Sx = 2$, $Ty = 3$ and so

$$\begin{aligned} G_1(fx, fx, gy) &= G_1(0,0,2) = 4 \\ &< \frac{3}{4}(6) = \frac{3}{4}G_2(2,2,3) = \frac{3}{4}G_2(Sx, Sx, Ty) \\ &\leq k\max\bigl\{G_2(Sx, Sx, Ty), G_2(fx, fx, Sx), G_2(gy, gy, Ty), \\ &\quad \bigl[G_2(fx, fx, Ty) + G_2(gy, gy, Sx)\bigr]/2\bigr\} \end{aligned}$$

and

$$\begin{aligned} G_1(fx, gy, gy) &= G_1(0,2,2) = 4 \\ &< \frac{3}{4}(6) = \frac{3}{4}G_2(2,3,3) = \frac{3}{4}G_2(Sx, Ty, Ty) \\ &\leq k\max\bigl\{G_2(Sx, Ty, Ty), G_2(fx, Sx, Sx), G_2(gy, Ty, Ty), \\ &\quad \bigl[G_2(fx, Ty, Ty) + G_2(gy, Sx, Sx)\bigr]/2\bigr\}. \end{aligned}$$

(4) Finally, when $x = 3$ and $y = 2$, then $fx = 0$, $gy = 2$, $Sx = 3$, $Ty = 3$ and so

$$\begin{aligned} G_1(fx, fx, gy) &= G_1(0,0,2) = 4 \\ &< \frac{3}{4}(6) = \frac{3}{4}G_2(2,2,3) = \frac{3}{4}G_2(gy, gy, Ty) \\ &\leq k\max\bigl\{G_2(Sx, Sx, Ty), G_2(fx, fx, Sx), G_2(gy, gy, Ty), \\ &\quad \bigl[G_2(fx, fx, Ty) + G_2(gy, gy, Sx)\bigr]/2\bigr\} \end{aligned}$$

and

$$\begin{aligned} G_1(fx, gy, gy) &= G_1(0,2,2) = 4 \\ &< \frac{3}{4}(6) = \frac{3}{4}G_2(2,3,3) = \frac{3}{4}G_2(gy, Ty, Ty) \\ &\leq k\max\bigl\{G_2(Sx, Ty, Ty), G_2(fx, Sx, Sx), G_2(gy, Ty, Ty), \\ &\quad \bigl[G_2(fx, Ty, Ty) + G_2(gy, Sx, Sx)\bigr]/2\bigr\}. \end{aligned}$$

Thus, for all cases, the contractions (2.1) and (2.2) are satisfied. Hence all of the conditions of Theorem 2.1 are satisfied. Moreover, 0 is the unique common fixed point of f, g, S and g.

If we consider the same set equipped with two metrics given by $d_1(x,y) = |x-y|$ and $d_2(x,y) = \frac{1}{2}|x-y|$ for all $x, y \in X$, then for $x = 1$ and $y = 2$, we have

$$\begin{aligned} d_1(fx,gy) &= d_1(0,2) = 2 \nleq 2k \\ &\leq k \max\{d_2(2,3), d_2(0,2), d_2(2,3), [d_2(0,3) + d_2(2,2)]/2\} \\ &= k \max\{d_2(Sx,Ty), d_2(fx,Sx), d_2(gy,Ty), [d_2(fx,Ty) + d_2(gy,Sx)]/2\} \end{aligned}$$

for any $k \in [0,1)$. So corresponding results in ordinary metric spaces cannot be applied in this case.

Theorem 2.1 can be viewed as an extension of Theorem 2.1 of [8] to the case of two ordered G-metric spaces.

Since the class of weakly compatible mappings includes R-weakly commuting mappings, Theorem 2.1 generalizes the comparable results in [8].

Corollary 2.3 *Let $(X, \preceq)$ be a partially ordered set and G_1, G_2 be two G-metrics on X such that $G_2(x,y,z) \leq G_1(x,y,z)$ for all $x, y, z \in X$ with a complete metric G_1 on X. Suppose that f, g, S and T are self-mappings on X satisfying the following properties:*

$$\begin{aligned} G_1(fx,fx,gy) \leq{}& a_1 G_2(Sx,Sx,Ty) + a_2 G_2(Sx,Sx,fx) + a_3 G_2(Ty,Ty,gy) \\ &+ a_4[G_2(Sx,Sx,gy) + G_2(Ty,Ty,fx)] \end{aligned} \tag{2.13}$$

and

$$\begin{aligned} G_1(fx,gy,gy) \leq{}& a_1 G_2(Sx,Ty,Ty) + a_2 G_2(Sx,fx,fx) + a_3 G_2(Ty,gy,gy) \\ &+ a_4[G_2(Sx,gy,gy) + G_2(Ty,fx,fx)] \end{aligned} \tag{2.14}$$

for all comparable $x, y \in X$, where $a_1 + a_2 + a_3 + 2a_4 < 1$. Suppose that $f(X) \subseteq T(X)$, $g(X) \subseteq S(X)$ and f, g are dominated mappings and S, T are dominating mappings. If, for any nonincreasing sequence $\{x_n\}$ with $y_n \preceq x_n$ for all $n \in \mathbb{N}$, $y_n \to u$ implies that $u \preceq x_n$ and either

(a) f, S are compatible, f or S is continuous and g, T are weakly compatible

or

(b) g, T are compatible, g or T is continuous and f, S are weakly compatible,

then f, g, S and T have a common fixed point in X. Moreover, the set of common fixed points of f, g, S and T is well-ordered if and only if f, g, S and T have one and only one common fixed point in X.

Example 2.4 Let $X = [0,1]$ be endowed with the usual ordering and G_1, G_2 be two G-metrics on X given in [13]:

$$\begin{aligned} G_1(a,b,c) &= |a-b| + |b-c| + |c-a|, \\ G_2(a,b,c) &= \frac{1}{2}[|a-b| + |b-c| + |c-a|]. \end{aligned}$$

Define the mappings $f, g, S, T : X \to X$ as

$$fx = \frac{x}{12}, \qquad gx = \begin{cases} \frac{x}{4} & \text{if } x \in [0, \frac{1}{2}), \\ \frac{x}{6} & \text{if } x \in [\frac{1}{2}, 1), \end{cases} \qquad S(x) = \frac{3x}{2}, \qquad T(x) = \frac{5x}{2}$$

for all $x \in X$. Clearly, f, g are dominated mappings and S, T are dominating mappings with $f(X) \subseteq T(X)$ and $g(X) \subseteq S(X)$. Also, f, S are compatible, f is continuous and g, T are weakly compatible. Now, for all comparable $x, y \in X$, we check the following cases:

(1) If $x, y \in [0, \frac{1}{2})$, then we have

$$\begin{aligned} G_1(fx, fx, gy) &= \frac{1}{12}|x - 3y| \le \frac{1}{12}(x + 3y) \\ &\le \frac{3}{10}\left(\frac{17}{12}x\right) + \frac{3}{10}\left(\frac{9}{4}y\right) \\ &= a_2 G_2(fx, fx, Sx) + a_3 G_2(gy, gy, Ty) \\ &\le a_1 G_2(Sx, Sx, Ty) + a_2 G_2(fx, fx, Sx) + a_3 G_2(gy, gy, Ty) \\ &\quad + a_4\bigl[G_2(fx, fx, Ty) + G_2(gy, gy, Sx)\bigr]. \end{aligned}$$

(2) If $x \in [0, \frac{1}{2})$ and $y \in [\frac{1}{2}, 1]$, then we have

$$\begin{aligned} G_1(fx, fx, gy) &= \frac{1}{12}|x - 2y| \le \frac{1}{12}(x + 2y) \\ &\le \frac{3}{10}\left(\frac{17}{12}x\right) + \frac{3}{10}\left(\frac{14}{6}y\right) \\ &= a_2 G_2(fx, fx, Sx) + a_3 G_2(gy, gy, Ty) \\ &\le a_1 G_2(Sx, Sx, Ty) + a_2 G_2(fx, fx, Sx) + a_3 G_2(gy, gy, Ty) \\ &\quad + a_4\bigl[G_2(fx, fx, Ty) + G_2(gy, gy, Sx)\bigr]. \end{aligned}$$

(3) If $y \in [0, \frac{1}{2})$ and $x \in [\frac{1}{2}, 1]$, then we have

$$\begin{aligned} G_1(fx, fx, gy) &= \frac{1}{12}|x - 3y| \le \frac{1}{12}(x + 3y) \\ &\le \frac{3}{10}\left(\frac{17}{12}x\right) + \frac{3}{10}\left(\frac{9}{4}y\right) \\ &= a_2 G_2(fx, fx, Sx) + a_3 G_2(gy, gy, Ty) \\ &\le a_1 G_2(Sx, Sx, Ty) + a_2 G_2(fx, fx, Sx) + a_3 G_2(gy, gy, Ty) \\ &\quad + a_4\bigl[G_2(fx, fx, Ty) + G_2(gy, gy, Sx)\bigr]. \end{aligned}$$

(4) If $x, y \in [\frac{1}{2}, 1]$, then we obtain

$$\begin{aligned} G_1(fx, fx, gy) &= \frac{1}{12}|x - 2y| \le \frac{1}{12}(x + 2y) \\ &\le \frac{3}{10}\left(\frac{17}{12}x\right) + \frac{3}{10}\left(\frac{14}{6}y\right) \end{aligned}$$

$$
\begin{aligned}
&= a_2 G_2(fx, fx, Sx) + a_3 G_2(gy, gy, Ty) \\
&\le a_1 G_2(Sx, Sx, Ty) + a_2 G_2(fx, fx, Sx) + a_3 G_2(gy, gy, Ty) \\
&\quad + a_4\big[G_2(fx, fx, Ty) + G_2(gy, gy, Sx)\big].
\end{aligned}
$$

Thus (2.13) is satisfied with $a_1 = a_4 = \frac{1}{10}$ and $a_2 = a_3 = \frac{3}{10}$, where $a_1 + a_2 + a_3 + 2a_4 < 1$. Similarly, (2.14) is satisfied. Thus all the conditions of Corollary 2.3 are satisfied. Moreover, 0 is the unique common fixed point of f and g.

3 Application

Let $X = L^2(\Omega)$, the set of comparable functions on Ω whose square is integrable on Ω where $\Omega = [0,1]$, be a bounded set in $\mathbb{R}$. We endow X with the partial ordered $\preceq$ given by: $x, y \in X$, $x \preceq y \Leftrightarrow x(t) \le y(t)$, for all $t \in \Omega$. We consider the integral equations

$$
\begin{aligned}
x(t) &= \int_\Omega q_1\big(t, s, x(s)\big)\, ds - c(t), \\
y(t) &= \int_\Omega q_2\big(t, s, y(s)\big)\, ds - c(t),
\end{aligned}
\tag{3.1}
$$

where $q_1, q_2 : \Omega \times \Omega \times \mathbb{R} \to \mathbb{R}$ and $c : \Omega \to \mathbb{R}^+$, to be given continuous mappings. Recently, Abbas *et al.* [35] obtained a common solution of integral equations (3.1) as an application of their results in the setup of ordered generalized metric spaces. Here we study a sufficient condition for the existence of a common solution of integral equations in the framework of two generalized metric spaces. Define $G_1, G_2 : X \times X \times X \to \mathbb{R}^+$ by

$$
\begin{aligned}
G_1(x, y, z) &= \sup_{t\in\Omega}\big|x(t) - y(t)\big| + \sup_{t\in\Omega}\big|y(t) - z(t)\big| + \sup_{t\in\Omega}\big|z(t) - x(t)\big|, \\
G_2(x, y, z) &= \frac{1}{2}\Big[\sup_{t\in\Omega}\big|x(t) - y(t)\big| + \sup_{t\in\Omega}\big|y(t) - z(t)\big| + \sup_{t\in\Omega}\big|z(t) - x(t)\big|\Big].
\end{aligned}
$$

Obviously, $G_2(x, y, z) \le G_1(x, y, z)$ for all $x, y, z \in X$. Suppose that the following hypotheses hold:

(i) For each $s, t \in \Omega$,

$$
\int_\Omega q_1\big(t, s, u(s)\big)\, ds \le u(s)
$$

and

$$
\int_\Omega q_2\big(t, s, u(s)\big)\, ds \le u(s)
$$

hold.

(ii) There exists $r : \Omega \to \Omega$ such that

$$
\int_\Omega \big|q_1\big(t, s, u(t)\big) - q_2\big(t, s, v(t)\big)\big|\, dt \le r(t)\big|u(t) - v(t)\big|
$$

for each $s, t \in \Omega$ with $\sup_{t\in\Omega} r(t) \le k$ where $k \in [0,1)$.

Then the integral equations (3.1) have a common solution in $L^2(\Omega)$.

Proof Define $fx(t) = \int_\Omega q_1(t,s,x(t))\,dt - c(t)$ and $gx(t) = \int_\Omega q_2(t,s,x(t))\,dt - c(t)$. As $fx(t) \leq x(t)$ and $gx(t) \leq x(t)$, so f and g are dominated maps. Now, for all comparable $x, y \in X$,

$$\begin{aligned}
G_1(fx,fx,gy) &= 2\sup_{t\in\Omega}\bigl|fx(t) - gy(t)\bigr| \\
&= 2\sup_{t\in\Omega}\left|\int_\Omega q_1\bigl(t,s,x(t)\bigr)\,dt - \int_\Omega q_2\bigl(t,s,y(t)\bigr)\,dt\right| \\
&\leq 2\sup_{t\in\Omega}\int_\Omega \bigl|q_1\bigl(t,s,x(t)\bigr) - q_2\bigl(t,s,y(t)\bigr)\bigr|\,dt \\
&\leq 2\sup_{t\in\Omega} r(t)\bigl|x(t) - y(t)\bigr| \\
&\leq 2k\sup_{t\in\Omega}\bigl|x(t) - y(t)\bigr| \\
&= kG_2(x,y,y) \\
&\leq k\max\bigl\{G_2(x,x,y), G_2(fx,fx,x), G_2(gy,gy,y), \\
&\quad \bigl[G_2(fx,fx,y) + G_2(gy,gy,x)\bigr]/2\bigr\}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
G_1(fx,gy,gy) \leq k\max\bigl\{&G_2(x,y,y), G_2(fx,x,x), G_2(gy,y,y), \\
&\bigl[G_2(fx,y,y) + G_2(gy,x,x)\bigr]/2\bigr\}
\end{aligned}$$

is satisfied. Now we can apply Theorem 2.1 by taking S and T as identity maps to obtain the common solutions of integral equations (3.1) in $L^2(\Omega)$. □

Remarks

(1) If we take $f = g$ in Theorem 2.1, then it generalizes Corollary 2.3 in [8] to a more general class of commuting mappings in the setup of two ordered G-metric spaces.

(2) If we take $S = T$ in Theorem 2.1, then Corollary 2.4 in [8] is a special case of Theorem 2.1.

(3) If $S = T = I_X$ (: the identity mapping on X) in Theorem 2.1, then we obtain Corollary 2.5 in [8] in a more general setup.

(4) Corollary 2.6 of [8] becomes a special case of Theorem 2.1 if we take $f = g$ and $S = T = I_X$.

(5) A G-metric naturally induces a metric d_G given by $d_G(x,y) = G(x,y,y) + G(x,x,y)$. If the G-metric is not symmetric, then the inequalities (2.1), (2.2), (2.13) and (2.14) do not reduce to any metric inequality with the metric d_G. Hence our results do not reduce to fixed point problems in the corresponding metric space $(X, \preceq, d_G)$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

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