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A three-step iterative scheme for solving nonlinear ϕ-strongly accretive operator equations in Banach spaces

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Abstract
In this paper, we study a three-step iterative scheme with error terms for solving nonlinear ϕ-strongly accretive operator equations in arbitrary real Banach spaces.

Keywords: three-step iterative scheme; ϕ-strongly accretive operator; ϕ-hemicontractive operator

1 Introduction

Let K be a nonempty subset of an arbitrary Banach space X and X^* be its dual space. The symbols $D(T)$, $R(T)$ and $F(T)$ stand for the domain, the range and the set of fixed points of T respectively (for a single-valued map $T : X \to X$, $x \in X$ is called a fixed point of T iff $T(x) = x$). We denote by J the normalized duality mapping from E to 2^{E^*} defined by

$$J(x) = \{f^* \in X^* : \langle x, f^* \rangle = \|x\|^2 = \|f^*\|^2\}.$$

Let $T : D(T) \subseteq X \to X$ be an operator. The following definitions can be found in [1–15] for example.

Definition 1 T is called *Lipshitzian* if there exists $L > 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\|,$$

for all $x, y \in K$. If $L = 1$, then T is called *nonexpansive*, and if $0 < L < 1$, T is called *contraction*.

Definition 2

(i) T is said to be strongly pseudocontractive if there exists a $t > 1$ such that for each $x, y \in D(T)$, there exists $j(x - y) \in J(x - y)$ satisfying

$$\operatorname{Re}\langle Tx - Ty, j(x - y)\rangle \leq \frac{1}{t}\|x - y\|^2.$$

(ii) T is said to be strictly hemicontractive if $F(T)$ is nonempty and if there exists a $t > 1$ such that for each $x \in D(T)$ and $q \in F(T)$, there exists $j(x-y) \in J(x-y)$ satisfying

$$\operatorname{Re}\langle Tx - q, j(x-q)\rangle \leq \frac{1}{t}\|x-q\|^2.$$

(iii) T is said to be ϕ-strongly pseudocontractive if there exists a strictly increasing function $\phi : [0,\infty) \to [0,\infty)$ with $\phi(0) = 0$ such that for each $x, y \in D(T)$, there exists $j(x-y) \in J(x-y)$ satisfying

$$\operatorname{Re}\langle Tx - Ty, j(x-y)\rangle \leq \|x-y\|^2 - \phi(\|x-y\|)\|x-y\|.$$

(iv) T is said to be ϕ-hemicontractive if $F(T)$ is nonempty and if there exists a strictly increasing function $\phi : [0,\infty) \to [0,\infty)$ with $\phi(0) = 0$ such that for each $x \in D(T)$ and $q \in F(T)$, there exists $j(x-y) \in J(x-y)$ satisfying

$$\operatorname{Re}\langle Tx - q, j(x-q)\rangle \leq \|x-q\|^2 - \phi(\|x-q\|)\|x-q\|.$$

Clearly, each strictly hemicontractive operator is ϕ-hemicontractive.

Definition 3

(i) T is called *accretive* if the inequality

$$\|x-y\| \leq \|x-y+s(Tx-Ty)\|$$

holds for every $x, y \in D(T)$ and for all $s > 0$.

(ii) T is called *strongly accretive* if, for all $x, y \in D(T)$, there exists a constant $k > 0$ and $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x-y)\rangle \geq k\|x-y\|^2.$$

(iii) T is called *ϕ-strongly accretive* if there exists $j(x-y) \in J(x-y)$ and a strictly increasing function $\phi : [0,\infty) \to [0,\infty)$ with $\phi(0) = 0$ such that for each $x, y \in X$,

$$\langle Tx - Ty, j(x-y)\rangle \geq \phi(\|x-y\|)\|x-y\|.$$

Remark 4 It has been shown in [11, 12] that the class of strongly accretive operators is a proper subclass of the class of ϕ-strongly accretive operators. If I denotes the identity operator, then T is called *strongly pseudocontractive* (respectively, *ϕ-strongly pseudocontractive*) if and only if $(I-T)$ is strongly accretive (respectively, ϕ-strongly accretive).

Chidume [1] showed that the Mann iterative method can be used to approximate fixed points of Lipschitz strongly pseudocontractive operators in L_p (or l_p) spaces for $p \in [2,\infty)$. Chidume and Osilike [4] proved that each strongly pseudocontractive operator with a fixed point is strictly hemicontractive, but the converse does not hold in general. They also proved that the class of strongly pseudocontractive operators is a proper subclass of the class of ϕ-strongly pseudocontractive operators and pointed out that the class of ϕ-strongly pseudocontractive operators with a fixed point is a proper subclass of the class of ϕ-hemicontractive operators. These classes of nonlinear operators have been studied by various researchers (see, for example, [7–25]). Liu *et al.* [26] proved that, under certain conditions, a three-step iteration scheme with error terms converges strongly to the unique fixed point of ϕ-hemicontractive mappings.

In this paper, we study a three-step iterative scheme with error terms for nonlinear ϕ-strongly accretive operator equations in arbitrary real Banach spaces.

2 Preliminaries

We need the following results.

Lemma 5 [27] *Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be three sequences of nonnegative real numbers with $\sum_{n=1}^{\infty} b_n < \infty$ and $\sum_{n=1}^{\infty} c_n < \infty$. If*

$$a_{n+1} \leq (1 + b_n)a_n + c_n, \quad n \geq 1,$$

then the limit $\lim_{n\to\infty} a_n$ exists.

Lemma 6 [28] *Let $x, y \in X$. Then $\|x\| \leq \|x + ry\|$ for every $r > 0$ if and only if there is $f \in J(x)$ such that $\operatorname{Re}\langle y, f\rangle \geq 0$.*

Lemma 7 [9] *Suppose that X is an arbitrary Banach space and $A : E \to E$ is a continuous ϕ-strongly accretive operator. Then the equation $Ax = f$ has a unique solution for any $f \in E$.*

3 Strong convergence of a three-step iterative scheme to a solution of the system of nonlinear operator equations

For the rest of this section, L denotes the Lipschitz constant of $T_1, T_2, T_3 : X \to X$, $L_* = (1 + L)$ and $R(T_1)$, $R(T_2)$ and $R(T_3)$ denote the ranges of T_1, T_2 and T_3 respectively. We now prove our main results.

Theorem 8 *Let X be an arbitrary real Banach space and $T_1, T_2, T_3 : X \to X$ Lipschitz ϕ-strongly accretive operators. Let $f \in R(T_1) \cap R(T_2) \cap R(T_3)$ and generate $\{x_n\}$ from an arbitrary $x_0 \in X$ by*

$$\begin{aligned} x_{n+1} &= a_n x_n + b_n\big(f + (I - T_1)y_n\big) + c_n v_n, \\ y_n &= a'_n x_n + b'_n\big(f + (I - T_2)z_n\big) + c'_n u_n, \\ z_n &= a''_n x_n + b''_n\big(f + (I - T_3)x_n\big) + c''_n w_n, \quad n \geq 0, \end{aligned} \tag{3.1}$$

where $\{v_n\}_{n=0}^{\infty}$, $\{u_n\}_{n=0}^{\infty}$ and $\{w_n\}_{n=0}^{\infty}$ are bounded sequences in X and $\{a_n\}$, $\{c_n\}$, $\{a'_n\}$, $\{b'_n\}$, $\{c'_n\}$, $\{a''_n\}$, $\{b''_n\}$, $\{c''_n\}$ are sequences in $[0,1]$ and $\{b_n\}$ in $(0,1)$ satisfying the following conditions: (i) $a_n + b_n + c_n = 1 = a'_n + b'_n + c'_n = a''_n + b''_n + c''_n$, (ii) $\sum_{n=0}^{\infty} b_n = \infty$, (iii) $\sum_{n=0}^{\infty} b_n^2 < \infty$, $\sum_{n=0}^{\infty} b'_n < \infty$, (iv) $\sum_{n=0}^{\infty} c_n < \infty$, $\sum_{n=0}^{\infty} c'_n < \infty$ and $\sum_{n=0}^{\infty} c''_n < \infty$. Then the sequence $\{x_n\}$ converges strongly to the solution of the system $T_i x = f$; $i = 1, 2, 3$.

Proof By Lemma 7, the system $T_i x = f$; $i = 1, 2, 3$ has the unique solution $x^* \in X$. Following the techniques of [5, 8–12, 26, 29], define $S_i : X \to X$ by $S_i x = f + (I - T_i)x$; $i = 1, 2, 3$; then each S_i is demicontinuous and x^* is the unique fixed point of S_i; $i = 1, 2, 3$, and for all $x, y \in X$, we have

$$\begin{aligned}
&\big\langle (I - S_i)x - (I - S_i)y, j(x - y)\big\rangle \\
&\quad \geq \phi_i\big(\|x - y\|\big)\|x - y\| \\
&\quad \geq \frac{\phi_i(\|x - y\|)}{(1 + \phi_i(\|x - y\|) + \|x - y\|)}\|x - y\|^2 \\
&\quad = \theta_i(x, y)\|x - y\|^2,
\end{aligned}$$

where $\theta_i(x, y) = \frac{\phi_i(\|x-y\|)}{(1+\phi_i(\|x-y\|)+\|x-y\|)} \in [0, 1)$ for all $x, y \in X$; $i = 1, 2, 3$. Let $x^* \in \bigcap_{i=1}^{3} F(S_i)$ be the fixed point set of S_i, and let $\theta(x, y) = \inf \min_i\{\theta_i(x, y)\} \in [0, 1]$. Thus

$$\big\langle (I - S_i)x - (I - S_i)y, j(x - y)\big\rangle \geq \theta(x, y)\|x - y\|^2; \quad i = 1, 2, 3. \tag{3.2}$$

It follows from Lemma 6 and inequality (3.2) that

$$\|x - y\| \leq \big\|x - y + \lambda\big[(I - S_i)x - \theta(x, y)x - \big((I - S_i)y - \theta(x, y)y\big)\big]\big\|, \tag{3.3}$$

for all $x, y \in X$ and for all $\lambda > 0$; $i = 1, 2, 3$.

Set $\alpha_n = b_n + c_n$, $\beta_n = b'_n + c'_n$ and $\gamma_n = b''_n + c''_n$, then (3.1) becomes

$$\begin{aligned}
x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n S_1 y_n + c_n(v_n - S_1 y_n), \\
y_n &= (1 - \beta_n)x_n + \beta_n S_2 z_n + c'_n(u_n - S_2 z_n), \\
z_n &= (1 - \gamma_n)x_n + \gamma_n S_3 x_n + c''_n(w_n - S_3 x_n), \quad n \geq 0.
\end{aligned} \tag{3.4}$$

We have

$$\begin{aligned}
x_n = {}& (1 + \alpha_n)x_{n+1} + \alpha_n\big[(I - S_1)x_{n+1} - \theta\big(x_{n+1}, x^*\big)x_{n+1}\big] \\
& - \big(1 - \theta\big(x_{n+1}, x^*\big)\big)\alpha_n x_n + \big(2 - \theta\big(x_{n+1}, x^*\big)\big)\alpha_n^2(x_n - S_1 y_n) \\
& + \alpha_n(S_1 x_{n+1} - S_1 y_n) - \big[1 + \big(2 - \theta\big(x_{n+1}, x^*\big)\big)\alpha_n\big]c_n(v_n - S_1 y_n).
\end{aligned}$$

Furthermore,

$$x^* = (1 + \alpha_n)x^* + \alpha_n\big[(I - S_1)x^* - \theta\big(x_{n+1}, x^*\big)x^*\big] - \big(1 - \theta\big(x_{n+1}, x^*\big)\big)\alpha_n x^*,$$

so that

$$\begin{aligned}
x_n - x^* = {}& (1 + \alpha_n)\big(x_{n+1} - x^*\big) + \alpha_n\big[(I - S_1)x_{n+1} - \theta\big(x_{n+1}, x^*\big)x_{n+1} \\
& - \big((I - S_1)x^* - \theta\big(x_{n+1}, x^*\big)x^*\big)\big] \\
& - \big(1 - \theta\big(x_{n+1}, x^*\big)\big)\alpha_n\big(x_n - x^*\big) + \big(2 - \theta\big(x_{n+1}, x^*\big)\big)\alpha_n^2(x_n - S_1 y_n) \\
& + \alpha_n(S_1 x_{n+1} - S_1 y_n) - \big[1 + \big(2 - \theta\big(x_{n+1}, x^*\big)\big)\alpha_n\big]c_n(v_n - S_1 y_n).
\end{aligned}$$

Hence,

$$\begin{aligned}
\|x_n - x^*\| &\geq (1+\alpha_n)\bigg\| x_{n+1} - x^* + \frac{\alpha_n}{(1+\alpha_n)}\big[(I-S_1)x_{n+1} - \theta\big(x_{n+1}, x^*\big)x_{n+1} \\
&\quad - \big((I-S_1)x^* - \theta\big(x_{n+1}, x^*\big)x^*\big)\big]\bigg\| \\
&\quad - \big(1-\theta\big(x_{n+1}, x^*\big)\big)\alpha_n\|x_n - x^*\| - \big(2-\theta\big(x_{n+1}, x^*\big)\big)\alpha_n^2\|x_n - S_1y_n\| \\
&\quad - \alpha_n\|S_1x_{n+1} - S_1y_n\| - \big[1+\big(2-\theta\big(x_{n+1}, x^*\big)\big)\alpha_n\big]c_n\|v_n - S_1y_n\| \\
&\geq (1+\alpha_n)\|x_{n+1} - x^*\| - \big(1-\theta\big(x_{n+1}, x^*\big)\big)\alpha_n\|x_n - x^*\| \\
&\quad - \big(2-\theta\big(x_{n+1}, x^*\big)\big)\alpha_n^2\|x_n - S_1y_n\| - \alpha_n\|S_1x_{n+1} - S_1y_n\| \\
&\quad - \big[1+\big(2-\theta\big(x_{n+1}, x^*\big)\big)\alpha_n\big]c_n\|v_n - S_1y_n\|.
\end{aligned}$$

Hence,

$$\begin{aligned}
\|x_{n+1} - x^*\| &\leq \frac{[1+(1-\theta(x_{n+1}, x^*))\alpha_n]}{(1+\alpha_n)}\|x_n - x^*\| + 2\alpha_n^2\|x_n - S_1y_n\| \\
&\quad + \alpha_n\|S_1x_{n+1} - S_1y_n\| + \big[1+\big(2-\theta\big(x_{n+1}, x^*\big)\big)\alpha_n\big]c_n\|v_n - S_1y_n\| \\
&\leq \big[1+\big(1-\theta\big(x_{n+1}, x^*\big)\big)\alpha_n\big]\big[1-\alpha_n+\alpha_n^2\big]\|x_n - x^*\| \\
&\quad + 2\alpha_n^2\|x_n - S_1y_n\| + \alpha_n\|S_1x_{n+1} - S_1y_n\| + 3c_n\|v_n - S_1y_n\| \\
&\leq \big[1-\theta\big(x_{n+1}, x^*\big)\alpha_n + \alpha_n^2\big]\|x_n - x^*\| + 2\alpha_n^2\|x_n - S_1y_n\| \\
&\quad + \alpha_n\|S_1x_{n+1} - S_1y_n\| + 3c_n\|v_n - S_1y_n\|. \qquad (3.5)
\end{aligned}$$

Furthermore, we have the following estimates:

$$\begin{aligned}
\|z_n - x^*\| &= \big\|(1-\gamma_n)\big(x_n - x^*\big) + \gamma_n\big(S_3x_n - x^*\big) + c_n''(w_n - S_3x_n)\big\| \\
&\leq (1-\gamma_n)\|x_n - x^*\| + \gamma_n\|S_3x_n - x^*\| + c_n''\|w_n - S_3x_n\| \\
&\leq (1-\gamma_n)\|x_n - x^*\| + L_*\gamma_n\|x_n - x^*\| \\
&\quad + c_n''\big(\|w_n - x^*\| + \|S_3x_n - x^*\|\big) \\
&\leq \big(1+(L_*-1)\gamma_n + L_*c_n''\big)\|x_n - x^*\| + c_n''\|w_n - x^*\| \\
&\leq (3L_*-1)\|x_n - x^*\| + c_n''\|w_n - x^*\|, \qquad (3.6) \\
\|y_n - x^*\| &= \big\|(1-\beta_n)\big(x_n - x^*\big) + \beta_n\big(S_2z_n - x^*\big) + c_n'(u_n - S_2z_n)\big\| \\
&\leq (1-\beta_n)\|x_n - x^*\| + \beta_n\|S_2z_n - x^*\| + c_n'\|u_n - S_2z_n\| \\
&\leq (1-\beta_n)\|x_n - x^*\| + L_*\beta_n\|z_n - x^*\| \\
&\quad + c_n'\big(\|u_n - x^*\| + L_*\|z_n - x^*\|\big) \\
&\leq \big(1-\beta_n + L_*(3L_*-1)\beta_n + L_*(3L_*-1)c_n'\big)\|x_n - x^*\| \\
&\quad + \big(L_*\beta_nc_n'' + L_*c_n'c_n''\big)\|w_n - x^*\| + c_n'\|u_n - x^*\| \\
&\leq \big[3L_*(3L_*-1)-1\big]\|x_n - x^*\| + 3L_*c_n''\|w_n - x^*\| + c_n'\|u_n - x^*\|, \qquad (3.7)
\end{aligned}$$

$$\begin{aligned}\|x_n - S_1y_n\| &\le \|x_n - x^*\| + L_*\|y_n - x^*\| \\ &\le \big[1 + L_*\big[3L_*(3L_* - 1) - 1\big]\big]\|x_n - x^*\| \\ &\quad + 3L_*^2 c_n''\|w_n - x^*\| + L_* c_n'\|u_n - x^*\|, \end{aligned} \tag{3.8}$$

$$\begin{aligned}\|S_1x_{n+1} - S_1y_n\| &\le L_*\|x_{n+1} - y_n\| \\ &= L_*\big\|(1-\alpha_n)(x_n - y_n) + \alpha_n(S_1y_n - y_n) + c_n(v_n - S_1y_n)\big\| \\ &\le L_*\big[(1-\alpha_n)\|x_n - y_n\| + \alpha_n\|S_1y_n - y_n\| + c_n\|v_n - S_1y_n\|\big] \\ &\le L_*\big[\|x_n - y_n\| + \alpha_n\|S_1y_n - y_n\| + c_n\|v_n - S_1y_n\|\big]. \end{aligned} \tag{3.9}$$

Using (3.4) and (3.6),

$$\begin{aligned}\|x_n - y_n\| &= \big\|\beta_n(x_n - S_2z_n) - c_n'(u_n - S_2z_n)\big\| \\ &\le \beta_n\|x_n - S_2z_n\| + c_n'\|u_n - S_2z_n\| \\ &\le \big[\big[1 + L_*(3L_* - 1)\big]\beta_n + L_*(3L_* - 1)c_n'\big]\|x_n - x^*\| \\ &\quad + L_*\big(\beta_n + c_n'\big)c_n''\|w_n - x^*\| + c_n'\|u_n - x^*\| \\ &\le \big[\big[1 + L_*(3L_* - 1)\big]\beta_n + L_*(3L_* - 1)c_n'\big]\|x_n - x^*\| \\ &\quad + 3L_*c_n''\|w_n - x^*\| + c_n'\|u_n - x^*\|. \end{aligned} \tag{3.10}$$

Using (3.7),

$$\begin{aligned}\|S_1y_n - y_n\| &\le \|S_1y_n - x^*\| + \|y_n - x^*\| \\ &\le (1 + L_*)\|y_n - x^*\| \\ &\le (1 + L_*)[3L_*(3L_* - 1) - 1]\|x_n - x^*\| \\ &\quad + 3L_*(1 + L_*)c_n''\|w_n - x^*\| + (1 + L_*)c_n'\|u_n - x^*\|. \end{aligned} \tag{3.11}$$

Again, using (3.7),

$$\begin{aligned}\|v_n - S_1y_n\| &\le \|v_n - x^*\| + L_*\|y_n - x^*\| \\ &\le L_*\big[3L_*(3L_* - 1) - 1\big]\|x_n - x^*\| + \|v_n - x^*\| \\ &\quad + 3L_*^2c_n''\|w_n - x^*\| + L_*c_n'\|u_n - x^*\|. \end{aligned} \tag{3.12}$$

Substituting (3.10)-(3.12) in (3.9), we obtain

$$\begin{aligned}\|S_1x_{n+1} - S_1y_n\| &\le L_*\big[1 + L_*(3L_* - 1)\big]\beta_n + L_*(3L_* - 1)c_n' \\ &\quad + \big[3L_*(3L_* - 1) - 1\big]\big[(1 + L_*)\alpha_n + L_*c_n\big]\|x_n - x^*\| \\ &\quad + 3L_*\big[L_*c_n'' + \big[(1 + L_*)\alpha_n + L_*c_n\big]c_n''\big]\|w_n - x^*\| \\ &\quad + L_*\big[c_n' + \big[(1 + L_*)\alpha_n + L_*c_n\big]c_n'\big]\|u_n - x^*\| \\ &\quad + L_*c_n\|v_n - x^*\|. \end{aligned} \tag{3.13}$$

Substituting (3.8), (3.12) and (3.13) in (3.5), we obtain

$$\begin{aligned}\|x_{n+1}-x^*\| &\le \bigl[1+\bigl[3+L_*(3+L_*)3L_*(3L_*-1)-1\bigr]\bigr]\alpha_n^2\\&\quad+L_*\bigl[3L_*(3L_*-1)-1\bigr]\alpha_n\beta_n+L_*^2(3L_*-1)\alpha_n c_n'\\&\quad+L_*\bigl[3L_*(3L_*-1)-1\bigr]\alpha_n c_n+3L_*\bigl[3L_*(3L_*-1)-1\bigr]c_n\|x_n-x^*\|\\&\quad-\theta\bigl(x_{n+1},x^*\bigr)\alpha_n\|x_n-x^*\|+\bigl[3L_*(1+3L_*)\alpha_n^2c_n''+3L_*^2\alpha_n c_n''+3L_*^2\alpha_n c_n c_n''\\&\quad+9L_*^2c_nc_n''\bigr]\|w_n-x^*\|+\bigl[L_*(3+L_*)\alpha_n^2c_n'+L_*\alpha_n c_n'\\&\quad+L_*^2\alpha_n c_n c_n'+3L_*c_nc_n'\bigr]\|u_n-x^*\|+(2L_*+3)c_n\|v_n-x^*\|.\end{aligned}\tag{3.14}$$

Since $\{v_n\}$, $\{u_n\}$ and $\{w_n\}$ are bounded, we set

$$M=\sup_{n\ge0}\|v_n-x^*\|+\sup_{n\ge0}\|u_n-x^*\|+\sup_{n\ge0}\|w_n-x^*\|<\infty.$$

Then it follows from (3.14) that

$$\begin{aligned}\|x_{n+1}-x^*\| &\le \bigl[1+\bigl[3+L_*(3+L_*)\bigl[3L_*(3L_*-1)-1\bigr]\bigr]\alpha_n^2\\&\quad+L_*\bigl[3L_*(3L_*-1)-1\bigr]\alpha_n\beta_n+L_*^2(3L_*-1)\alpha_n c_n'\\&\quad+L_*\bigl[3L_*(3L_*-1)-1\bigr]\alpha_n c_n+3L_*\bigl[3L_*(3L_*-1)-1\bigr]c_n\bigr]\|x_n-x^*\|\\&\quad-\theta\bigl(x_{n+1},x^*\bigr)\alpha_n\|x_n-x^*\|+\bigl[3L_*(1+3L_*)\alpha_n^2c_n''+3L_*^2\alpha_n c_n''+3L_*^2\alpha_n c_n c_n''\\&\quad+9L_*^2c_nc_n''\bigr]M+\bigl[L_*(3+L_*)\alpha_n^2c_n'+L_*\alpha_n c_n'\\&\quad+L_*^2\alpha_n c_n c_n'+3L_*c_nc_n'\bigr]M+(2L_*+3)c_nM\\&=(1+\delta_n)\|x_n-x^*\|-\theta\bigl(x_{n+1},x^*\bigr)\alpha_n\|x_n-x^*\|+\sigma_n\\&\le(1+\delta_n)\|x_n-x^*\|+\sigma_n,\end{aligned}\tag{3.15}$$

where

$$\begin{aligned}\delta_n&=\bigl[3+L_*(3+L_*)\bigl[3L_*(3L_*-1)-1\bigr]\bigr]\alpha_n^2\\&\quad+L_*\bigl[3L_*(3L_*-1)-1\bigr]\alpha_n\beta_n+L_*^2(3L_*-1)\alpha_n c_n'\\&\quad+L_*\bigl[3L_*(3L_*-1)-1\bigr]\alpha_n c_n+3L_*\bigl[3L_*(3L_*-1)-1\bigr]c_n,\\\sigma_n&=M\bigl[3L_*(1+3L_*)\alpha_n^2c_n''+3L_*^2\alpha_n c_n''+3L_*^2\alpha_n c_n c_n''+9L_*^2c_nc_n''\\&\quad L_*(3+L_*)\alpha_n^2c_n'+L_*\alpha_n c_n'+L_*^2\alpha_n c_n c_n'+3L_*c_nc_n'\\&\quad+(2L_*+3)c_n\bigr].\end{aligned}$$

Since $b_n\in(0,1)$, the conditions (iii) and (iv) imply that $\sum_{n=0}^{\infty}\delta_n<\infty$ and $\sum_{n=0}^{\infty}\sigma_n<\infty$. It then follows from Lemma 5 that $\lim_{n\to\infty}\|x_n-x^*\|$ exists. Let $\lim_{n\to\infty}\|x_n-x^*\|=\delta\ge0$. We now prove that $\delta=0$. Assume that $\delta>0$. Then there exists a positive integer N_0 such that $\|x_n-x^*\|\ge\frac{\delta}{2}$ for all $n\ge N_0$. Since

$$\theta\bigl(x_{n+1},x^*\bigr)\|x_n-x^*\|=\frac{\phi(\|x_{n+1}-x^*\|)}{1+\phi(\|x_{n+1}-x^*\|)+\|x_{n+1}-x^*\|}\|x_n-x^*\|\ge\frac{\phi(\frac{\delta}{2})\delta}{2(1+\phi(D)+D)},$$

for all $n \geq N_0$, it follows from (3.15) that

$$\|x_{n+1} - x^*\| \leq \|x_n - x^*\| - \frac{\phi(\frac{\delta}{2})\delta}{2(1 + \phi(D) + D)}\alpha_n + \lambda_n \quad \text{for all } n \geq N_0.$$

Hence,

$$\frac{\phi(\frac{\delta}{2})\delta}{2(1 + \phi(D) + D)}\alpha_n \leq \|x_n - x^*\| - \|x_{n+1} - x^*\| + \lambda_n \quad \text{for all } n \geq N_0.$$

This implies that

$$\frac{\phi(\frac{\delta}{2})\delta}{2(1 + \phi(D) + D)} \sum_{j=N_0}^{n} \alpha_j \leq \|x_{N_0} - x^*\| + \sum_{j=N_0}^{n} \lambda_j.$$

Since $b_n \leq \alpha_n$,

$$\frac{\phi(\frac{\delta}{2})\delta}{2(1 + \phi(D) + D)} \sum_{j=N_0}^{n} b_j \leq \|x_{N_0} - x^*\| + \sum_{j=N_0}^{n} \lambda_j$$

yields $\sum_{n=0}^{\infty} b_n < \infty$, contradicting the fact that $\sum_{n=0}^{\infty} b_n = \infty$. Hence, $\lim_{n\to\infty} \|x_n - x^*\| = 0$. □

Corollary 9 *Let X be an arbitrary real Banach space and $T_1, T_2, T_3 : X \to X$ be three Lipschitz ϕ-strongly accretive operators, where ϕ is in addition continuous. Suppose $\liminf_{r\to\infty} \phi(r) > 0$ or $\|T_i x\| \to \infty$ as $\|x\| \to \infty$; $i = 1, 2, 3$. Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{a'_n\}$, $\{b'_n\}$, $\{c'_n\}$, $\{a''_n\}$, $\{b''_n\}$, $\{c''_n\}$, $\{w_n\}$, $\{u_n\}$, $\{v_n\}$, $\{y_n\}$ and $\{x_n\}$ be as in Theorem 8. Then, for any given $f \in X$, the sequence $\{x_n\}$ converges strongly to the solution of the system $T_i x = f$; $i = 1, 2, 3$.*

Proof The existence of a unique solution to the system $T_i x = f$; $i = 1, 2, 3$ follows from [9] and the result follows from Theorem 8. □

Theorem 10 *Let X be a real Banach space and K be a nonempty closed convex subset of X. Let $T_1, T_2, T_3 : K \to K$ be three Lipschitz ϕ-strong pseudocontractions with a nonempty fixed point set. Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{a'_n\}$, $\{b'_n\}$, $\{c'_n\}$, $\{a''_n\}$, $\{b''_n\}$, $\{c''_n\}$, $\{w_n\}$, $\{u_n\}$ and $\{v_n\}$ be as in Theorem 8. Let $\{x_n\}$ be the sequence generated iteratively from an arbitrary $x_0 \in K$ by*

$$\begin{aligned} x_{n+1} &= a_n x_n + b_n T_1 y_n + c_n v_n, \\ y_n &= a'_n x_n + b'_n T_2 z_n + c'_n u_n, \\ z_n &= a''_n x_n + b''_n T_3 x_n + c''_n w_n, \quad n \geq 0. \end{aligned}$$

Then $\{x_n\}$ converges strongly to the common fixed point of T_1, T_2, T_3.

Proof As in the proof of Theorem 8, set $\alpha_n = b_n + c_n$, $\beta_n = b'_n + c'_n$, $\gamma_n = b''_n + c''_n$ to obtain

$$\begin{aligned} x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n T_1 y_n + c_n(v_n - T_1 y_n), \\ y_n &= (1 - \beta_n)x_n + \beta_n T_2 z_n + c_n(u_n - T_2 z_n), \end{aligned}$$

$$z_n = (1-\gamma_n)x_n + \gamma_n T_3 x_n + c_n(w_n - T_3 x_n), \quad n \geq 0.$$

Since each T_i; $i = 1,2,3$ is a ϕ-strong pseudocontraction, $(I - T_i)$ is ϕ-strongly accretive so that for all $x, y \in X$, there exist $j(x-y) \in J(x-y)$ and a strictly increasing function $\phi : (0,\infty) \to (0,\infty)$ with $\phi(0) = 0$ such that

$$\langle (I-T_i)x - (I-T_i)y, j(x-y)\rangle \geq \phi(\|x-y\|)\|x-y\| \geq \theta(x,y)\|x-y\|^2; \quad i = 1,2,3.$$

The rest of the argument now follows as in the proof of Theorem 8. □

Remark 11 The example in [4] shows that the class of ϕ-strongly pseudocontractive operators with nonempty fixed point sets is a proper subclass of the class of ϕ-hemicontractive operators. It is easy to see that Theorem 8 easily extends to the class of ϕ-hemicontractive operators.

Remark 12

(i) If we set $b''_n = 0 = c''_n$ for all $n \geq 0$ in our results, we obtain the corresponding results for the Ishikawa iteration scheme with error terms in the sense of Xu [15].

(ii) If we set $b''_n = 0 = c''_n = b'_n = 0 = c'_n$ for all $n \geq 0$ in our results, we obtain the corresponding results for the Mann iteration scheme with error terms in the sense of Xu [15].

Remark 13 Let $\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences satisfying the following conditions:

(i) $0 \leq \alpha_n, \beta_n \leq 1$, $n \geq 0$,

(ii) $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = 0$,

(iii) $\sum_{n=0}^{\infty} \alpha_n = \infty$,

(iv) $\sum_{n=0}^{\infty} \beta_n < \infty$, and

(v) $\sum_{n=0}^{\infty} \alpha_n^2 < \infty$.

If we set $a'_n = (1-\beta_n)$, $b'_n = \beta_n$, $c'_n = 0$, $a_n = (1-\alpha_n)$, $b_n = \alpha_n$, $c_n = 0$, $b''_n = 0 = c''_n$ for all $n \geq 0$ in Theorems 8 and 10 respectively, we obtain the corresponding convergence theorems for the original Ishikawa [18] and Mann [30] iteration schemes.

Remark 14

(i) Gurudwan and Sharma [29] studied a strong convergence of multi-step iterative scheme to a common solution for a finite family of ϕ-strongly accretive operator equations in a reflexive Banach space with weakly continuous duality mapping. Some remarks on their work can be seen in [31].

(ii) All the above results can be extended to a finite family of ϕ-strongly accretive operators.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All the authors studied and approved the manuscript.

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