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Convergence theorems of common fixed points for some semigroups of nonexpansive mappings in complete CAT(0) spaces

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Abstract

In this paper, we consider some iteration processes for one-parameter continuous semigroups of nonexpansive mappings in a nonempty compact convex subset C of a complete CAT(0) space X and prove that the proposed sequence converges to a common fixed point for these semigroups of nonexpansive mappings. Note that our results generalize Cho *et al.* result (Nonlinear Anal. 74:6050-6059, 2011) and related results.

Keywords: common fixed point; CAT(0) space; nonexpansive semigroup; implicit iteration process

1 Introduction

Fixed point theory in CAT(0) spaces was first studied by Kirk [1, 2]. He showed that every nonexpansive (single-valued) mapping defined on a bounded closed convex subset of a complete CAT(0) space always has a fixed point. Since then, the fixed point theory for single-valued and multivalued mappings in CAT(0) spaces has been rapidly developed, and many papers have appeared; for example, one can see [3–6] and related references.

Let (X, d) be a metric space. A geodesic path joining $x \in X$ to $y \in X$ (or, more briefly, a geodesic from x to y) is a map c from a closed interval $[0, \ell] \subseteq R$ to X such that $c(0) = x$, $c(\ell) = y$, and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, \ell]$. In particular, c is an isometry and $d(x, y) = \ell$. The image α of c is called a geodesic (or metric) segment joining x and y. When it is unique, this geodesic is denoted by $[x, y]$. The space (X, d) is said to be a geodesic space if every two points of X are joined by a geodesic, and X is said to be uniquely geodesic if there is exactly one geodesic joining x and y for each $x, y \in X$. A subset $Y \subseteq X$ is said to be convex if Y includes every geodesic segment joining any two of its points.

A geodesic triangle $\triangle(x_1, x_2, x_3)$ in a geodesic space (X, d) consists of three points x_1, x_2, and x_3 in X (the vertices of $\triangle$ and a geodesic segment between each pair of vertices (the edge of $\triangle$)). A comparison triangle for geodesic triangle $\triangle(x_1, x_2, x_3)$ in (X, d) is a triangle $\overline{\triangle}(x_1, x_2, x_3) := \triangle(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ in the Euclidean plane $\mathbb{E}_2$ such that $d_{\mathbb{E}_2}(\bar{x}_i, \bar{x}_j) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$.

A geodesic space is said to be a CAT(0) space if all geodesic triangles of appropriate size satisfy the following comparison axiom.

CAT(0): Let $\triangle$ be a geodesic triangle in X, and let $\overline{\triangle}$ be a comparison triangle for $\triangle$. Then $\triangle$ is said to satisfy the CAT(0) inequality if for all $x, y \in \triangle$ and all comparison points

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$\bar{x}, \bar{y} \in \overline{\Delta}$, $d(x, y) \leq d_{\mathbb{E}_2}(\bar{x}, \bar{y})$. It is well known that any complete, simply connected Riemannian manifold having nonpositive sectional curvature is a CAT(0) space. Other examples include pre-Hilbert spaces [7], R-trees [8], the complex Hilbert ball with a hyperbolic metric [9], and many others.

If x, y_1, y_2 are points in a CAT(0) space, and if y_0 is the midpoint of the segment $[y_1, y_2]$, then the CAT(0) inequality implies

$$d^2(x, y_0) \leq \frac{1}{2}d^2(x, y_1) + \frac{1}{2}d^2(x, y_2) - \frac{1}{4}d^2(y_1, y_2).$$

This is the (CN) inequality of Bruhat and Tits [10]. In fact, a geodesic space is a CAT(0) space if and only if it satisfies the (CN) inequality [7, p. 163].

In 2008, Dhompongsa and Panyanak [11] gave the following result, and the proof is similar to the proof of the remark in [12, p. 374].

Lemma 1.1 [11] *Let X be a CAT(0) space. Then*

$$d\big((1-t)x \oplus ty, z\big) \leq (1-t)d(x, z) + td(y, z)$$

for all $x, y, z \in X$ and $t \in [0, 1]$.

By the above lemma, we know that CAT(0) space is a convex metric space. Indeed, it is a metric space X with a convex structure if there exists a mapping $W : X \times X \times [0, 1] \to X$ such that

$$d\big(W(x, y, t), z\big) \leq td(x, z) + (1-t)d(y, z)$$

for all $x, y, z \in X$ and $t \in [0, 1]$ and this space X is called a convex metric space [13]. Furthermore, Takahashi [13] has proved that

$$d(x, y) = td\big(x, W(x, y, t)\big) + (1-t)d\big(y, W(x, y, t)\big)$$

for all $x, y, z \in X$ and $t \in [0, 1]$ when X is a convex metric space with a convex structure. So, we also get the following result, and it is proved in [11].

Lemma 1.2 [11] *Let X be a CAT(0) space and $x, y \in X$. For each $t \in [0, 1]$, there exists a unique point $z \in [x, y]$ such that $d(x, z) = td(x, y)$ and $d(y, z) = (1-t)d(x, y)$.*

For convenience, from now on we will use the notation $z = (1-t)x \oplus ty$. Therefore, we have

$$z = (1-t)x \oplus ty \quad \Leftrightarrow \quad z \in [x, y],$$

$$d(x, z) = td(x, y), \quad \text{and} \quad d(y, z) = (1-t)d(x, y).$$

Let C be a nonempty closed convex subset of a CAT(0) space X, and let T be a nonexpansive mapping on C, *i.e.*, $T : C \to C$ such that $d(Tx, Ty) \leq d(x, y)$ for all $x, y \in C$. We use $F(T)$ to denote the set of fixed points of T, *i.e.*, $F(T) := \{x \in C : Tx = x\}$. Let $\mathbb{N}$ be the set of

positive integers, $\mathbb{R}$ be the set of real numbers, and let $\mathbb{R}_+$ be the set of nonnegative real numbers.

A family $\{T(t) : t \geq 0\}$ of mappings is called a one-parameter continuous semigroup of nonexpansive mappings on a nonempty closed convex subset C of a CAT(0) space X if the following conditions hold:

$(SG)_1$ for each $t \in \mathbb{R}_+$, $T(t)$ is a nonexpansive mapping on C;
$(SG)_2$ $T(s + t) = T(t) \circ T(s)$ for all $t, s \in \mathbb{R}_+$;
$(SG)_3$ for each $x \in X$, the mapping $T(\cdot)x$ from $\mathbb{R}_+$ into C is continuous.

A family $\{T(t) : t \geq 0\}$ of mappings is called a one-parameter strongly continuous semigroup of nonexpansive mappings on a nonempty closed convex subset C of a CAT(0) space X if conditions $(SG)_i$, $i = 1, 2, 3$, and the following condition are satisfied:

$(SG)_4$ $T(0)x = x$ for all $x \in C$.

Note that if C is a nonempty compact subset of a Banach space and $\{T(t) : t \geq 0\}$ is a semigroup of nonexpansive mappings, then $\bigcap_{t \geq 0} F(T(t)) \neq \emptyset$ [14]; see also [15–17] and others. For the example of a one-parameter continuous semigroup of nonexpansive mappings, one can see [18].

Construction of common fixed points of a nonexpansive semigroup is an important subject in the theory of nonexpansive semigroup mappings and its applications. Fox example, one can refer to [19, 20]. In [21], Shioji and Takahashi introduced the implicit iteration

(A) $u_n = \alpha_n u + (1 - \alpha_n) \frac{1}{t_n} \int_0^{t_n} T(s) u_n \, ds$, $n \geq 0$,

where C is a nonempty closed convex subset of a real Hilbert space H, $u \in C$, $\{\alpha_n\}$ is a sequence in $(0, 1)$, $\{t_n\}$ is a sequence of positive real numbers divergent to ∞. Under suitable conditions, Shioji and Takahashi [21] proved strong convergence of $\{x_n\}$ to a member of $\bigcap_{t \geq 0} F(T(t))$. Note that their iterate x_n at step n is constructed through the average of a semigroup over the interval $(0, \infty)$.

In 2003, Suzuki [22] introduced the following implicit iteration process in a Hilbert space:

(B) $x_n = \alpha_n u + (1 - \alpha_n) T(t_n) x_n$, $n \geq 0$

for a nonexpansive semigroup, where C is a nonempty closed convex subset of a real Hilbert space H, $u \in C$, $\{\alpha_n\}$ is a sequence in $(0, 1)$, $\{t_n\}$ is a sequence of positive real numbers. Note that x_n is constructed directly from the $T(t_n)$. So, Zegeye and Shahzad [23] viewed Suzuki's iteration process (B) as an extension of the implicit process (A) to nonexpansive semigroups.

In 2005, Suzuki [24] considered an iterative process $\{x_n\}$ for a one-parameter continuous semigroup of nonexpansive mappings on C, where C is a nonempty compact convex subset of a Banach space E, defined by

$$
\begin{cases}
x_1 \in C \quad \text{chosen arbitrary,} \\
x_{n+1} := \lambda T(t_n) x_n + (1 - \lambda) x_n,
\end{cases}
\tag{1.1}
$$

where $\lambda \in (0, 1)$ and $\{t_n\} \subseteq [0, \infty)$. Then Suzuki [24] proved that $\{x_n\}$ converges strongly to a common fixed point of $\{T(t) : t \geq 0\}$ if

$$
\liminf_{n \to \infty} t_n < \limsup_{n \to \infty} t_n, \quad \text{and} \quad \lim_{n \to \infty} (t_{n+1} - t_n) = 0.
$$

In 2009, Dhompongsa *et al.* [25] gave the following important result for a strongly continuous semigroup of nonexpansive mappings.

Theorem 1.1 [25] *Let C be a nonempty bounded closed convex subset of a complete CAT(0) space X, and let $\{T(t) : t \geq 0\}$ be a strongly continuous semigroup of nonexpansive mappings on C. Let $\{\alpha_n\}$ and $\{t_n\}$ be sequences of real numbers satisfying $0 < \alpha_n < 1$, $t_n > 0$, and $\lim_{n\to\infty} t_n = \lim_{n\to\infty} \frac{\alpha_n}{t_n} = 0$. Let $x_0 \in C$, and let $\{x_n\}$ be a sequence in C with*

$$x_n = \alpha_n x_0 \oplus (1 - \alpha_n)T(t_n)x_n, \quad \forall n \in \mathbb{N}.$$

Then $\bigcap_{t\geq 0} F(T(t)) \neq \emptyset$ and $\{x_n\}$ converges to the element of $\bigcap_{t\geq 0} F(T(t))$ nearest to x_0.

In 2011, Cho *et al.* [26] gave the following result for a continuous semigroup of nonexpansive mappings on a nonempty compact convex subset C of a complete CAT(0) space X.

Theorem 1.2 [26] *Let C be a nonempty compact convex subset of a complete CAT(0) space X, and $\{T(t) : t \geq 0\}$ be a one-parameter continuous semigroup of nonexpansive mappings on C. Let $\{t_n\}$ be a sequence in $[0, \infty)$ satisfying*

$$\liminf_{n\to\infty} t_n < \limsup_{n\to\infty} t_n, \quad and \quad \lim_{n\to\infty}(t_{n+1} - t_n) = 0.$$

For any $\lambda \in (0,1)$, define a sequence $\{x_n\}$ in C by

$$\begin{cases} x_1 \in C \quad chosen\ arbitrary, \\ x_{n+1} := \lambda T(t_n)x_n \oplus (1 - \lambda)x_n. \end{cases} \tag{1.2}$$

Then $\{x_n\}$ converges to a common fixed point of the semigroup $\{T(t) : t \geq 0\}$.

Remark 1.1 By Theorems 1.1 and 1.2, we know that
 (a) if C is a nonempty bounded closed convex subset of a complete CAT(0) space X, and $\{T(t) : t \geq 0\}$ is a strongly continuous semigroup of nonexpansive mappings on C, then $\bigcap_{t\geq 0} F(T(t)) \neq \emptyset$;
 (b) if C is a compact convex subset of a complete CAT(0) space X, and $\{T(t) : t \geq 0\}$ is a one-parameter continuous semigroup of nonexpansive mappings on C, then $\bigcap_{t\geq 0} F(T(t)) \neq \emptyset$.

Motivated by the above works and related results, we study the following iteration processes for some families of one-parameter continuous semigroups of nonexpansive mappings on a nonempty compact convex subset C of a complete CAT(0) space X.

Let $\{T(t) : t \geq 0\}$ and $\{Q(q) : q \geq 0\}$ be one-parameter continuous semigroups of nonexpansive mappings on a nonempty compact convex subset C of a complete CAT(0) space X. We consider the following iteration processes (1.3), (1.4), and (1.5):

$$\begin{cases} x_1 \in C \quad chosen\ arbitrary, \\ y_n := (1 - \beta_n)x_n \oplus \beta_n T(t_n)x_n, \\ x_{n+1} := (1 - \alpha_n)x_n \oplus \alpha_n Q(q_n)y_n, \end{cases} \tag{1.3}$$

$$\begin{cases} x_1 \in C \quad \text{chosen arbitrary,} \\ y_n := (1 - \beta_n)x_n \oplus \beta_n T(t_n)x_n, \\ x_{n+1} := (1 - \alpha_n)y_n \oplus \alpha_n Q(q_n)y_n, \end{cases} \tag{1.4}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$, $\{t_n\}$ and $\{q_n\}$ are sequences in $[0,\infty)$. We prove that the proposed sequences converge to a common fixed point of these families of mappings. Note that our results generalize Theorem 1.1 (*i.e.*, Theorem 3.5 in [26]).

Besides, Thong [27] considered an implicit iteration for nonexpansive semigroups $\{T(t) : t \geq 0\}$ on a nonempty compact convex subset C of a real Banach space E as follows:

$$\begin{cases} x_1 \in C \quad \text{chosen arbitrary,} \\ x_n := \alpha_{n-1}x_n + (1 - \alpha_n)T(t_n)x_n, \end{cases} \tag{1.5}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$, and $\{t_n\}$ is a sequence in $[0,\infty)$.

In this paper, motivated by [27], we also consider the following implicit iteration process for some families of one-parameter continuous semigroups of nonexpansive mappings on a nonempty compact convex subset C of a complete CAT(0) space X:

$$\begin{cases} x_1 \in C \quad \text{chosen arbitrary,} \\ y_n := (1 - \beta_n)x_n \oplus \beta_n T(t_n)x_n, \\ x_{n+1} := (1 - \alpha_{n+1})Q(q_{n+1})x_{n+1} \oplus \alpha_{n+1}y_n, \end{cases} \tag{1.6}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$, $\{t_n\}$ and $\{q_n\}$ are sequences in $[0,\infty)$.

For a special case of the iteration process (1.6), we have the following types:

$$\begin{cases} x_1 \in C \quad \text{chosen arbitrary,} \\ x_{n+1} = (1 - \alpha_{n+1})Q(q_{n+1})x_{n+1} \oplus \alpha_{n+1}x_n, \end{cases} \tag{1.7}$$

$$\begin{cases} x_1 \in C \quad \text{chosen arbitrary,} \\ x_{n+1} := (1 - \beta_n)x_n \oplus \beta_n T(t_n)x_n. \end{cases} \tag{1.8}$$

We prove the proposed sequences converge to a common fixed point of three families of mappings. Our result for the iteration process (1.7) is similar to (1.5) on complete CAT(0) spaces. Our results for the iteration processes (1.6) and (1.8) generalize Theorem 1.2. Note that the iteration process (1.8) is also a special case of the iteration processes (1.3) and (1.4).

2 Preliminaries

In 2005, Suzuki [24] gave the following result, and it is an important tool in this paper.

Lemma 2.1 [24] *Let* $\{t_n\}$ *be a real sequence and* τ *be a real number satisfying* $\liminf_{n\to\infty} t_n \leq \tau \leq \limsup_{n\to\infty} t_n$. *Suppose that either of the following holds*:

  (i) $\limsup_{n\to\infty}(t_{n+1} - t_n) \leq 0$ *or*
  (ii) $\liminf_{n\to\infty}(t_{n+1} - t_n) \geq 0$.

Then τ is a cluster point of $\{t_n\}$. Moreover, for any $\varepsilon > 0$ and $k, m \in \mathbb{N}$, there exists $m_0 \geq m$ such that $|t_j - \tau| < \varepsilon$ for all $j \in \mathbb{N}$ with $m_0 \leq j \leq m_0 + k$.

Lemma 2.2 [11] *Let X be a CAT(0) space. Then*

$$d\big((1-t)x \oplus ty, z\big)^2 \leq (1-t)d(x,z)^2 + td(y,z)^2 - t(1-t)d(x,y)^2$$

for all $t \in [0,1]$ and $x, y, z \in X$.

Definition 2.1 Let $\{x_n\}$ be a bounded sequence in a CAT(0) space X, and let C be a subset of X. Now, we use the following notations:

   (i) $r(x, \{x_n\}) := \limsup_{n \to \infty} d(x, x_n)$.
  (ii) $r(\{x_n\}) := \inf_{x \in X} r(x, \{x_n\})$.
 (iii) $r_C(\{x_n\}) := \inf_{x \in C} r(x, \{x_n\})$.
 (iv) $A(\{x_n\}) := \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}$.
  (v) $A_C(\{x_n\}) := \{x \in C : r(x, \{x_n\}) = r_C(\{x_n\})\}$.

Note that $x \in X$ is called an asymptotic center of $\{x_n\}$ if $x \in A(\{x_n\})$. It is known that in a CAT(0) space, $A(\{x_n\})$ consists of exactly one point [28].

Definition 2.2 [6] Let (X, d) be a CAT(0) space. A sequence $\{x_n\}$ in X is said to be Δ-convergent to $x \in X$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. That is, $A(\{u_n\}) = \{x\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case, we write $\Delta\text{-}\lim_n x_n = x$ and call x the Δ-limit of $\{x_n\}$.

Lemma 2.3 [6] *Let (X, d) be a CAT(0) space. Then every bounded sequence in X has a Δ-convergent subsequence.*

Lemma 2.4 [29] *Let C be a nonempty closed convex subset of a CAT(0) space X. If $\{x_n\}$ is a bounded sequence in C, then the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is in C.*

Lemma 2.5 [6] *Let C be a nonempty closed convex subset of a complete CAT(0) space X, and let $T : C \to C$ be a nonexpansive mapping. Let $\{x_n\}$ be a bounded sequence in C with $\Delta\text{-}\lim_n x_n = x$ and $\lim_{n \to \infty} d(x_n, Tx_n) = 0$. Then $x \in C$ and $Tx = x$.*

Lemma 2.6 [30] *Let X be a CAT(0) space. Let $\{x_n\}$ and $\{y_n\}$ be two bounded sequences in X with $\lim_{n \to \infty} d(x_n, y_n) = 0$. If $\Delta\text{-}\lim_n x_n = x$, then $\Delta\text{-}\lim_n y_n = x$.*

Next, we give the following results and these results show that the intersection of the fixed point sets for a continuous semigroup of nonexpansive mappings is nonempty. Note that Theorem 2.1 is different from Theorem 1.1.

A: Common fixed point for a strongly nonexpansive semigroup

Lemma 2.7 *Let C be a nonempty bounded closed convex subset of a complete CAT(0) space X. Let $\{T(t) : t \in \mathbb{R}\}$ be a continuous semigroup of nonexpansive mappings on C. Let $\{t_n\}$ be a sequence in $\mathbb{R}_+$. Let $\{\alpha_n\}$ be a sequence in $(0,1)$, and let $\{x_n\}$ be defined as*

$$\begin{cases} x_0 \in C & \text{chosen arbitrary,} \\ x_n = \alpha_n x_{n-1} \oplus (1 - \alpha_n) T(t_n) x_n, & n \geq 1. \end{cases}$$

Assume that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} \frac{\alpha_n}{t_n} = 0$. *Then* $\bigcap_{t\geq0} F(T(t)) \neq \emptyset$ *if and only if* $\lim_{n\to\infty} d(T(0)x_n, x_n) = 0$.

Proof Suppose that $\lim_{n\to\infty} d(T(0)x_n, x_n) = 0$. Since C is bounded, there exist a subsequence $\{u_n\}$ of $\{x_n\}$ and $\bar{x} \in C$ such that $\Delta\text{-}\lim_{n\to\infty} u_n = \bar{x}$. By Lemma 2.5, $T(0)\bar{x} = \bar{x}$.

Take any $t > 0$ and let t be fixed. For $n > 0$ with $[\frac{t}{t_n}] > 1$, we have

$$d(x_n, T(t)\bar{x})$$

$$\leq d(x_n, T(0)x_n) + \sum_{k=0}^{[\frac{t}{t_n}]-1} d(T((k+1)t_n)x_n, T(kt_n)x_n)$$

$$+ d\left(T\left(\left[\frac{t}{t_n}\right]t_n\right)x_n, T\left(\left[\frac{t}{t_n}\right]t_n\right)\bar{x}\right) + d\left(T\left(\left[\frac{t}{t_n}\right]t_n\right)\bar{x}, T(t)\bar{x}\right)$$

$$\leq d(x_n, T(0)x_n) + \left[\frac{t}{t_n}\right]d(T(t_n)x_n, x_n) + d(x_n, \bar{x}) + d\left(T\left(t - \left[\frac{t}{t_n}\right]t_n\right)\bar{x}, T(0)\bar{x}\right)$$

$$\leq d(x_n, T(0)x_n) + \frac{t\alpha_n}{t_n}d(T(t_n)x_n, x_{n-1}) + d(x_n, \bar{x})$$

$$+ \max\{d(T(s)\bar{x}, T(0)\bar{x}) : 0 \leq s \leq t_n\}$$

$$\leq d(x_n, T(0)x_n) + \frac{t\alpha_n}{t_n}M + d(x_n, \bar{x}) + \max\{d(T(s)\bar{x}, T(0)\bar{x}) : 0 \leq s \leq t_n\}$$

for some $M > 0$ (note that C is bounded). Hence, we know that

$$\limsup_{n\to\infty} d(u_n, T(t)\bar{x}) \leq \limsup_{n\to\infty} d(u_n, \bar{x})$$

for each $t > 0$. Since $\Delta\text{-}\lim_{n\to\infty} u_n = \bar{x}$, $T(t)\bar{x} = \bar{x}$ for each $t > 0$. Therefore, $\bar{x} \in \bigcap_{t\geq0} F(T(t)) \neq \emptyset$.

Conversely, suppose that $\bigcap_{t\geq0} F(T(t)) \neq \emptyset$. Now, take any $w \in \bigcap_{t\geq0} F(T(t))$, and let w be fixed. Then we have

$$d(x_n, w)^2$$

$$\leq \alpha_n d(x_{n-1}, w)^2 + (1-\alpha_n)d(T(t_n)x_n, w)^2 - \alpha_n(1-\alpha_n)d(x_{n-1}, T(t_n)x_n)$$

$$\leq \alpha_n d(x_{n-1}, w)^2 + (1-\alpha_n)d(x_n, w)^2 - \alpha_n(1-\alpha_n)d(x_{n-1}, T(t_n)x_n)$$

$$\leq \alpha_n d(x_{n-1}, w)^2 + (1-\alpha_n)d(x_n, w)^2.$$

And this implies that $d(x_n, w) \leq d(x_{n-1}, w)$ and $\lim_{n\to\infty} d(x_n, w)$ exists. Hence,

$$d(x_{n-1}, T(t_n)x_n) \leq d(x_{n-1}, w)^2 - d(x_n, w)^2 + \alpha_n d(x_{n-1}, T(t_n)x_n).$$

By assumption, $\lim_{n\to\infty} d(x_{n-1}, T(t_n)x_n) = 0$. Next, we get

$$d(x_n, x_{n-1}) = d(\alpha_n x_{n-1} \oplus (1-\alpha_n)T(t_n)x_n, x_{n-1}) = (1-\alpha_n)d(T(t_n)x_n, x_{n-1}).$$

This implies that $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$ and $\lim_{n\to\infty} d(x_n, T(t_n)x_n) = 0$. Furthermore, it follows that

$$
\begin{aligned}
\limsup_{n\to\infty} & d\big(T(0)x_n, x_n\big) \\
&\leq \limsup_{n\to\infty} \big(d\big(T(0)x_n, T(0 + t_n)x_n\big) + d\big(T(t_n)x_n, x_n\big)\big) \\
&\leq \limsup_{n\to\infty} \big(d\big(T(0)x_n, T(0) \circ T(t_n)x_n\big) + d\big(T(t_n)x_n, x_n\big)\big) \\
&\leq \limsup_{n\to\infty} 2d\big(x_n, T(t_n)x_n\big) = 0.
\end{aligned}
$$

Therefore, the proof is completed. □

By Lemma 2.7, we get the following theorem, and it is different from Theorem 1.1.

Theorem 2.1 *Let C be a nonempty bounded closed convex subset of a complete CAT(0) space X. Let $\{T(t) : t \in \mathbb{R}\}$ be a strongly continuous semigroup of nonexpansive mappings on C. Let $\{t_n\}$ be a sequence in $\mathbb{R}_+$, and $\{\alpha_n\}$ be a sequence in $(0,1)$ with $\lim_{n\to\infty} t_n = \lim_{n\to\infty} \frac{\alpha_n}{t_n} = 0$. Let $\{x_n\}$ be defined as*

$$
\begin{cases}
x_0 \in C & \text{chosen arbitrary,} \\
x_n = \alpha_n x_{n-1} \oplus (1 - \alpha_n) T(t_n)x_n, & n \geq 1.
\end{cases}
$$

Then $\bigcap_{t\geq 0} F(T(t)) \neq \emptyset$. Furthermore, if C is a compact set, then $\lim_{n\to\infty} d(x_n, \bar{x}) = 0$ for some $\bar{x} \in \Omega$.

Proof Since $\{T(t) : t \in \mathbb{R}\}$ is a strongly continuous semigroup of nonexpansive mappings on C, it is easy to see that $\lim_{n\to\infty} d(T(0)x_n, x_n) = 0$. By Lemma 2.7, $\bigcap_{t\geq 0} F(T(t)) \neq \emptyset$, and the proof is completed. □

B: Common fixed point for a nonexpansive semigroup

Lemma 2.8 *Let C be a nonempty bounded closed convex subset of a complete CAT(0) space X, and $\{T(t) : t \geq 0\}$ be a one-parameter continuous semigroup of nonexpansive mappings on C. Let $\{t_n\}$ be a sequence in $[0, \infty)$ satisfying*

$$
\liminf_{n\to\infty} t_n < \limsup_{n\to\infty} t_n, \quad \text{and} \quad \lim_{n\to\infty} (t_{n+1} - t_n) = 0.
$$

For any $\lambda \in (0,1)$, define a sequence $\{x_n\}$ in C by

$$
\begin{cases}
x_1 \in C & \text{chosen arbitrary,} \\
x_{n+1} := \lambda T(t_n)x_n \oplus (1 - \lambda)x_n.
\end{cases}
$$

Then $\bigcap_{t\geq 0} F(T(t)) \neq \emptyset$ if and only if $\lim_{n\to\infty} d(x_n, T(0)x_n) = 0$.

Proof Suppose that $\lim_{n\to\infty} d(x_n, T(0)x_n) = 0$. Following the same argument as in the proof of Theorem 4 in [24], we get $\bigcap_{t\geq 0} F(T(t)) \neq \emptyset$. Conversely, if $\bigcap_{t\geq 0} F(T(t)) \neq \emptyset$, it is easy to see that $\lim_{n\to\infty} d(x_n, T(0)x_n) = 0$. □

By Lemma 2.8, we get the following theorem. Notice also that it is a consequence of Theorem 1.2.

Theorem 2.2 *Let C be a nonempty compact convex subset of a complete CAT(0) space X, and $\{T(t) : t \geq 0\}$ be a one-parameter continuous semigroup of nonexpansive mappings on C. Let $\{t_n\}$ be a sequence in $[0, \infty)$ satisfying*

$$\liminf_{n \to \infty} t_n < \limsup_{n \to \infty} t_n, \quad and \quad \lim_{n \to \infty}(t_{n+1} - t_n) = 0.$$

For any $\lambda \in (0, 1)$, define a sequence $\{x_n\}$ in C by

$$\begin{cases} x_1 \in C & chosen\ arbitrary, \\ x_{n+1} := \lambda T(t_n)x_n \oplus (1 - \lambda)x_n. \end{cases}$$

Then $\bigcap_{t \geq 0} F(T(t)) \neq \emptyset$.

Proof Following the same argument as in the proof of Theorem 3.5 in [26], we can prove that $\lim_{n \to \infty} d(x_n, T(0)x_n) = 0$. By Lemma 2.8, $\bigcap_{t \geq 0} F(T(t)) \neq \emptyset$. □

3 Main results

Theorem 3.1 *Let C be a nonempty compact convex subset of a complete CAT(0) space X. Let $\{T(t) : t \in \mathbb{R}_+\}$ and $\{Q(q) : q \in \mathbb{R}_+\}$ be continuous semigroups of nonexpansive mappings on C. Let $\{t_n\}$ and $\{q_n\}$ be sequences in $\mathbb{R}_+$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[0, 1]$. Suppose that*

$$\Omega := \left(\bigcap_{t \geq 0} F(T(t)) \right) \cap \left(\bigcap_{q \geq 0} F(Q(q)) \right) \neq \emptyset.$$

Let $\{x_n\}$ be defined as

$$\begin{cases} x_1 \in C & chosen\ arbitrary, \\ y_n := (1 - \beta_n)x_n \oplus \beta_n T(t_n)x_n, \\ x_{n+1} := (1 - \alpha_n)x_n \oplus \alpha_n Q(q_n)y_n. \end{cases}$$

Assume that

   (i) $0 < a \leq \alpha_n \leq b < 1$, $\liminf_{n \to \infty} \beta_n(1 - \beta_n) > 0$;
   (ii) $\max\{\liminf_{n \to \infty} t_n, \liminf_{n \to \infty} q_n\} < \min\{\limsup_{n \to \infty} t_n, \limsup_{n \to \infty} q_n\}$;
   (iii) *either* $\limsup_{n \to \infty}(t_{n+1} - t_n) \leq 0$ *or* $\liminf_{n \to \infty}(t_{n+1} - t_n) \geq 0$ *holds*;
   (iv) *either* $\limsup_{n \to \infty}(q_{n+1} - q_n) \leq 0$ *or* $\liminf_{n \to \infty}(q_{n+1} - q_n) \geq 0$ *holds*.
Then $\lim_{n \to \infty} x_n = \bar{x}$ for some $\bar{x} \in \Omega$.

Proof Take any $w \in \Omega$, and let w be fixed. Then for each $n \in \mathbb{N}$,

$$d(y_n, w)^2$$
$$= d\big((1 - \beta_n)x_n \oplus \beta_n T(t_n)x_n, w\big)^2$$

$$\leq (1 - \beta_n)d(x_n, w)^2 + \beta_n d\big(T(t_n)x_n, w\big)^2 - \beta_n(1 - \beta_n)d\big(x_n, T(t_n)x_n\big)^2$$

$$\leq (1 - \beta_n)d(x_n, w)^2 + \beta_n d(x_n, w)^2 - \beta_n(1 - \beta_n)d\big(x_n, T(t_n)x_n\big)^2$$

$$\leq d(x_n, w)^2, \tag{3.1}$$

and

$$d(x_{n+1}, w)^2$$

$$= d\big((1 - \alpha_n)x_n \oplus \alpha_n Q(q_n)y_n, w\big)^2$$

$$\leq (1 - \alpha_n)d(x_n, w)^2 + \alpha_n d\big(Q(q_n)y_n, w\big)^2 - \alpha_n(1 - \alpha_n)d\big(x_n, Q(q_n)y_n\big)^2$$

$$\leq (1 - \alpha_n)d(x_n, w)^2 + \alpha_n d\big(Q(q_n)y_n, w\big)^2$$

$$\leq (1 - \alpha_n)d(x_n, w)^2 + \alpha_n d(y_n, w)^2$$

$$\leq d(x_n, w)^2. \tag{3.2}$$

Hence, $\lim_{n \to \infty} d(x_n, w)$ exists, and $\{x_n\}$ and $\{y_n\}$ are bounded sequences. By (3.2), we get

$$d(x_n, w)^2 + \frac{d(x_{n+1}, w)^2 - d(x_n, w)^2}{a}$$

$$\leq d(x_n, w)^2 + \frac{d(x_{n+1}, w)^2 - d(x_n, w)^2}{\alpha_n}$$

$$= \frac{d(x_{n+1}, w)^2 - (1 - \alpha_n)d(x_n, w)^2}{\alpha_n}$$

$$\leq d\big(Q(q_n)y_n, w\big)^2$$

$$\leq d(y_n, w)^2$$

$$\leq d(x_n, w)^2. \tag{3.3}$$

And this implies that

$$\lim_{n \to \infty} d(x_n, w) = \lim_{n \to \infty} d(y_n, w) = \lim_{n \to \infty} d\big(Q(q_n)y_n, w\big). \tag{3.4}$$

By (3.1) and (3.4), we have

$$\lim_{n \to \infty} \beta_n(1 - \beta_n)d\big(x_n, T(t_n)x_n\big)^2 = 0. \tag{3.5}$$

By assumption and (3.5),

$$\lim_{n \to \infty} d\big(x_n, T(t_n)x_n\big) = 0. \tag{3.6}$$

By (3.2), we also have

$$\alpha_n(1 - \alpha_n)d\big(x_n, Q(q_n)y_n\big)^2 \leq d(x_n, w)^2 - d(x_{n+1}, w)^2,$$

and this inequality and assumption imply that

$$\lim_{n \to \infty} d\big(x_n, Q(q_n)y_n\big) = 0. \tag{3.7}$$

By (3.6), we have

$$
\begin{aligned}
\limsup_{n\to\infty} d(x_n, y_n) &= \limsup_{n\to\infty} d\big((1-\beta_n)x_n \oplus \beta_n T(t_n)x_n, x_n\big) \\
&\le \limsup_{n\to\infty} d\big(x_n, T(t_n)x_n\big) = 0.
\end{aligned}
\tag{3.8}
$$

By (3.7), it is easy to see that

$$
\lim_{n\to\infty} d(x_{n+1}, x_n) = 0.
\tag{3.9}
$$

By (3.7), (3.8), and (3.9), it is easy to see that

$$
\lim_{n\to\infty} d\big(y_n, Q(q_n)y_n\big) = \lim_{n\to\infty} d\big(x_{n+1}, Q(q_n)y_n\big) = 0.
\tag{3.10}
$$

Furthermore, by (3.6), it follows that

$$
\begin{aligned}
\limsup_{n\to\infty} d\big(T(0)x_n, x_n\big) \\
&\le \limsup_{n\to\infty} \big(d\big(T(0)x_n, T(0+t_n)x_n\big) + d\big(T(t_n)x_n, x_n\big)\big) \\
&\le \limsup_{n\to\infty} \big(d\big(T(0)x_n, T(0) \circ T(t_n)x_n\big) + d\big(T(t_n)x_n, x_n\big)\big) \\
&\le \limsup_{n\to\infty} 2d\big(x_n, T(t_n)x_n\big) = 0.
\end{aligned}
\tag{3.11}
$$

By (3.10), it follows that

$$
\limsup_{n\to\infty} d\big(Q(0)y_n, y_n\big) = 0.
\tag{3.12}
$$

Next, fix $\tau, \delta \in \mathbb{R}$ with

$$
\max\Big\{ \liminf_{n\to\infty} t_n, \liminf_{n\to\infty} q_n \Big\} < \tau < \tau + \delta < \min\Big\{ \limsup_{n\to\infty} t_n, \limsup_{n\to\infty} q_n \Big\}.
$$

Following the same argument as in the proof of Theorem 4 in [24], we choose a subsequence $\{n_i\}$ of $\{n\}$ such that

$$
\begin{aligned}
\lim_{n\to\infty} \frac{d(T(t_{n_i})x_{n_i}, x_{n_i})}{t_{n_i} - \tau} &= \lim_{n\to\infty} \frac{d(Q(q_{n_i})y_{n_i}, y_{n_i})}{q_{n_i} - \tau} = 0, \\
\lim_{i\to\infty} t_{n_i} &= \lim_{i\to\infty} q_{n_i} = \tau.
\end{aligned}
\tag{3.13}
$$

For completeness, we give the following proof. By (3.6) and (3.10), there exists $m_1 \in \mathbb{N}$ such that

$$
d\big(x_n, T(t_n)x_n\big) < \frac{1}{3^2}, \quad \text{and} \quad d\big(y_n, Q(q_n)y_n\big) < \frac{1}{3^2}
$$

for all $n \ge m_1$. By Lemma 2.1, we note that $\tau + \frac{\delta}{2}$ is a cluster point of $\{t_n\}$ and $\{q_n\}$. Hence, there exists $n_1 > m_1$ such that

$$
\tau + \frac{\delta}{3} < t_{n_1}, q_{n_1} < \tau + \delta.
$$

By (3.6) and (3.10) again, there exists $m_2 > n_1$ such that

$$d\big(x_n, T(t_n)x_n\big) < \frac{1}{4^2}, \quad \text{and} \quad d\big(y_n, Q(q_n)y_n\big) < \frac{1}{4^2}$$

for all $n \geq m_2$. By Lemma 2.1 again, we note that $\tau + \frac{\delta}{3}$ is a cluster point of $\{t_n\}$ and $\{q_n\}$. Hence, there exists $n_2 > m_2$ such that

$$\tau + \frac{\delta}{4} < t_{n_2}, q_{n_2} < \tau + \frac{\delta}{2}.$$

Continuing this argument, we can define a subsequence $\{n_i\}$ of $\{n\}$ satisfying

$$\max\big\{d\big(x_{n_i}, T(t_{n_i})x_{n_i}\big), d\big(y_{n_i}, Q(q_{n_i})y_{n_i}\big)\big\} < \frac{1}{(i+2)^2}, \quad \text{and}$$

$$\tau + \frac{\delta}{i+2} < t_{n_i}, q_{n_i} < \tau + \frac{\delta}{i}$$

for all $i \in \mathbb{N}$. Then it is obvious that $\tau < \min\{t_{n_i}, q_{n_i}\}$ for all $i \in \mathbb{N}$,

$$\lim_{i \to \infty} t_{n_i} = \lim_{i \to \infty} q_{n_i} = \tau,$$

and

$$\lim_{i \to \infty} d\big(T(t_{n_i})x_{n_i}, x_{n_i}\big) = \lim_{i \to \infty} d\big(Q(q_{n_i})y_{n_i}, y_{n_i}\big) = 0.$$

We also have

$$0 \leq \limsup_{i \to \infty} \frac{d(T(t_{n_i})x_{n_i}, x_{n_i})}{t_{n_i} - \tau} \leq \limsup_{i \to \infty} \frac{1/(i+2)^2}{\delta/(i+2)}$$

$$= \limsup_{i \to \infty} \frac{1}{\delta(i+2)} = 0.$$

Similarly,

$$\lim_{i \to \infty} \frac{d(Q(q_{n_i})y_{n_i}, y_{n_i})}{q_{n_i} - \tau} = 0.$$

Since $\{x_n\}$ is a bounded sequence, there exist a subsequence $\{x_{n_{i_k}}\}$ of $\{x_{n_i}\}$ and $\bar{x} \in C$ such that $\Delta\text{-}\lim_{k \to \infty} x_{n_{i_k}} = \bar{x}$. Let $u_k = x_{n_{i_k}}$, $v_k = y_{n_{i_k}}$, $a_k = t_{n_{i_k}}$, and $b_k = s_{n_{i_k}}$. Then we get

$$\limsup_{k \to \infty} d\big(u_k, T(\tau)\bar{x}\big)$$

$$\leq \limsup_{k \to \infty}\big(d\big(u_k, T(a_k)u_k\big) + d\big(T(a_k)u_k, T(a_k)\bar{x}\big) + d\big(T(a_k)\bar{x}, T(\tau)\bar{x}\big)\big)$$

$$\leq \limsup_{k \to \infty} d\big(u_k, T(a_k)u_k\big) + \limsup_{k \to \infty} d\big(u_k, \bar{x}\big) + \limsup_{k \to \infty} d\big(T(a_k - \tau)\bar{x}, T(0)\bar{x}\big)$$

$$\leq \limsup_{k \to \infty} d(u_k, \bar{x}),$$

and

$$\limsup_{k\to\infty} d\big(v_k, Q(\tau)\bar{x}\big)$$

$$\leq \limsup_{k\to\infty}\big(d\big(v_k, Q(b_k)v_k\big) + d\big(Q(b_k)v_k, Q(b_k)\bar{x}\big) + d\big(Q(b_k)\bar{x}, Q(\tau)\bar{x}\big)\big)$$

$$\leq \limsup_{k\to\infty} d\big(v_k, Q(b_k)v_k\big) + \limsup_{k\to\infty} d\big(v_k, \bar{x}\big) + \limsup_{k\to\infty} d\big(Q(b_k - \tau)\bar{x}, Q(0)\bar{x}\big)$$

$$\leq \limsup_{k\to\infty} d\big(v_k, \bar{x}\big).$$

Since $\Delta\text{-}\lim_{k\to\infty} u_k = \bar{x}$, we know that $T(\tau)\bar{x} = \bar{x}$. By (3.8), $\Delta\text{-}\lim_{k\to\infty} u_k = \bar{x}$, and Lemma 2.6, we know that $\Delta\text{-}\lim_{k\to\infty} v_k = \bar{x}$, and this implies that $Q(\tau)\bar{x} = \bar{x}$. Also, we have

$$T(0)\bar{x} = T(0) \circ T(\tau)\bar{x} = T(\tau + 0)\bar{x} = T(\tau)\bar{x} = \bar{x}.$$

Similarly, $Q(0)\bar{x} = \bar{x}$.

Take any $t > 0$, and let t be fixed. Then for $k \in \mathbb{N}$ with $a_k - \tau < t$, we have

$$d\big(u_k, T(t)\bar{x}\big)$$

$$= d\big(u_k, T(t) \circ T(\tau)^{[t/(a_k-\tau)]}\bar{x}\big)$$

$$= d\big(u_k, T\big([t/(a_k - \tau)]\tau + t\big)\bar{x}\big)$$

$$\leq d\big(u_k, T(0)u_k\big) + \sum_{m=0}^{[t/(a_k-\tau)]-1} d\big(T\big((m+1)a_k\big)u_k, T(ma_k)u_k\big)$$

$$\quad + d\big(T\big([t/(a_k - \tau)]a_k\big)u_k, T\big([t/(a_k - \tau)]a_k\big)\bar{x}\big)$$

$$\quad + d\big(T\big([t/(a_k - \tau)]a_k\big)\bar{x}, T\big([t/(a_k - \tau)]\tau + t\big)\bar{x}\big)$$

$$\leq d\big(u_k, T(0)u_k\big) + [t/(a_k - \tau)] \cdot d\big(T(a_k)u_k, u_k\big) + d(u_k, \bar{x})$$

$$\quad + d\big(T\big(t - [t/(a_k - \tau)](a_k - \tau)\big)\bar{x}, T(0)\bar{x}\big)$$

$$\leq d\big(u_k, T(0)u_k\big) + \frac{t \cdot d(T(a_k)u_k, u_k)}{a_k - \tau} + d(u_k, \bar{x})$$

$$\quad + \max\big\{d\big(T(s)\bar{x}, T(0)\bar{x}\big) : 0 \leq s \leq a_k - \tau\big\}.$$

By (3.11) and (3.13), we get

$$\limsup_{k\to\infty} d\big(u_k, T(t)\bar{x}\big) \leq \limsup_{k\to\infty} d(u_k, \bar{x}).$$

Since $\Delta\text{-}\lim_{k\to\infty} u_k = \bar{x}$, we get $T(t)\bar{x} = \bar{x}$ for each $t \geq 0$. Therefore, $\bar{x}$ is a common fixed point of $\{T(t) : t \geq 0\}$. Furthermore, we know that $\bar{x} \in \Omega$ by following the same argument.

In fact, since C is a compact set, we may assume that $\lim_{k\to\infty} d(x_{n_{i_k}}, \bar{x}) = 0$. So, $\bar{x}$ is a cluster point of $\{x_n\}$ and $\liminf_{n\to\infty} d(x_n, \bar{x}) = 0$. Since $\lim_{n\to\infty} d(x_n, w)$ exists for each $w \in \Omega$, $\lim_{n\to\infty} d(x_n, \bar{x}) = 0$. Therefore, the proof is completed. □

Remark 3.1 Theorem 3.1 generalizes Theorem 1.2.

Theorem 3.2 *Let C be a nonempty compact convex subset of a complete CAT(0) space X. Let $\{T(t) : t \in \mathbb{R}_+\}$ and $\{Q(q) : q \in \mathbb{R}_+\}$ be continuous semigroups of nonexpansive mappings on C. Let $\{t_n\}$ and $\{q_n\}$ be sequences in $\mathbb{R}_+$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[0, 1]$. Suppose that*

$$\Omega := \left(\bigcap_{t \geq 0} F(T(t)) \right) \cap \left(\bigcap_{q \geq 0} F(Q(q)) \right) \neq \emptyset.$$

Let $\{x_n\}$ be defined as

$$\begin{cases} x_1 \in C \quad \text{chosen arbitrary}, \\ y_n := (1 - \beta_n)x_n \oplus \beta_n T(t_n)x_n, \\ x_{n+1} := (1 - \alpha_n)y_n \oplus \alpha_n Q(q_n)y_n. \end{cases}$$

Assume that

(i) $\liminf_{n \to \infty} \alpha_n(1 - \alpha_n) > 0$, $\liminf_{n \to \infty} \beta_n(1 - \beta_n) > 0$;

(ii) $\max\{\liminf_{n \to \infty} t_n, \liminf_{n \to \infty} q_n\} < \min\{\limsup_{n \to \infty} t_n, \limsup_{n \to \infty} q_n\}$;

(iii) *either* $\limsup_{n \to \infty}(t_{n+1} - t_n) \leq 0$ *or* $\liminf_{n \to \infty}(t_{n+1} - t_n) \geq 0$ *holds*;

(iv) *either* $\limsup_{n \to \infty}(q_{n+1} - q_n) \leq 0$ *or* $\liminf_{n \to \infty}(q_{n+1} - q_n) \geq 0$ *holds*.

Then $\lim_{n \to \infty} x_n = \bar{x}$ for some $\bar{x} \in \Omega$.

Proof Take any $w \in \Omega$, and let w be fixed. Then for each $n \in \mathbb{N}$,

$$\begin{aligned} d(y_n, w)^2 \\ &= d\big((1 - \beta_n)x_n \oplus \beta_n T(t_n)x_n, w\big)^2 \\ &= (1 - \beta_n)d(x_n, w)^2 + \beta_n d\big(T(t_n)x_n, w\big)^2 - \beta_n(1 - \beta_n)d\big(x_n, T(t_n)x_n\big)^2 \\ &\leq (1 - \beta_n)d(x_n, w)^2 + \beta_n d(x_n, w)^2 - \beta_n(1 - \beta_n)d\big(x_n, T(t_n)x_n\big)^2 \\ &\leq d(x_n, w)^2, \end{aligned} \tag{3.14}$$

and

$$\begin{aligned} d(x_{n+1}, w)^2 \\ &= d\big((1 - \alpha_n)y_n \oplus \alpha_n Q(q_n)y_n, w\big)^2 \\ &= (1 - \alpha_n)d(y_n, w)^2 + \alpha_n d\big(Q(q_n)y_n, w\big)^2 - \alpha_n(1 - \alpha_n)d\big(y_n, Q(q_n)y_n\big)^2 \\ &\leq d(y_n, w)^2 - \alpha_n(1 - \alpha_n)d\big(y_n, Q(q_n)y_n\big)^2 \\ &\leq d(x_n, w)^2 - \alpha_n(1 - \alpha_n)d\big(y_n, Q(q_n)y_n\big)^2. \end{aligned} \tag{3.15}$$

Hence, $\lim_{n \to \infty} d(x_n, w)$ exists, $\lim_{n \to \infty} d(x_n, w) = \lim_{n \to \infty} d(y_n, w)$, and $\{x_n\}$ and $\{y_n\}$ are bounded sequences. By (3.14) and $\lim_{n \to \infty} d(x_n, w) = \lim_{n \to \infty} d(y_n, w)$, we have

$$\lim_{n \to \infty} \beta_n(1 - \beta_n)d\big(x_n, T(t_n)x_n\big)^2 = 0. \tag{3.16}$$

By assumption and (3.16),

$$\lim_{n \to \infty} d\big(x_n, T(t_n)x_n\big) = 0. \tag{3.17}$$

By (3.15) and $\lim_{n\to\infty} d(x_n, w)$ exists,

$$\lim_{n\to\infty} d(y_n, Q(q_n)y_n) = 0. \tag{3.18}$$

By (3.17), we have

$$
\begin{aligned}
\limsup_{n\to\infty} d(x_n, y_n) &= \limsup_{n\to\infty} d(x_n, (1 - \beta_n)x_n \oplus \beta_n T(t_n)x_n) \\
&\leq \limsup_{n\to\infty} d(x_n, T(t_n)x_n) = 0.
\end{aligned} \tag{3.19}
$$

Next, following the same argument as in the proof of Theorem 3.1, we get the proof of Theorem 3.2. □

Remark 3.2 Theorem 1.2 is also a special case of Theorem 3.2.

Theorem 3.3 *Let C be a nonempty compact convex subset of a complete CAT(0) space X. Let $\{T(t) : t \in \mathbb{R}\}$ and $\{Q(q) : q \in \mathbb{R}\}$ be continuous semigroups of nonexpansive mappings on C. Let $\{t_n\}$ and $\{q_n\}$ be sequences in $\mathbb{R}_+$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[0, 1]$ with $0 < \alpha_n < 1$, $\liminf_{n\to\infty} \beta_n(1 - \beta_n) > 0$. Suppose that $\Omega := (\bigcap_{t\geq 0} F(T(t))) \cap (\bigcap_{q\geq 0} F(Q(q))) \neq \emptyset$. Let $\{x_n\}$ be defined as*

$$
\begin{cases}
x_1 \in C \quad \text{chosen arbitrary,} \\
y_n = (1 - \beta_n)x_n \oplus \beta_n T(t_n)x_n, \\
x_{n+1} = (1 - \alpha_{n+1})Q(q_{n+1})x_{n+1} \oplus \alpha_{n+1}y_n.
\end{cases}
$$

Assume that

(i) $\liminf_{n\to\infty} t_n < \limsup_{n\to\infty} t_n$;
(ii) *either* $\limsup_{n\to\infty}(t_{n+1} - t_n) \leq 0$ *or* $\liminf_{n\to\infty}(t_{n+1} - t_n) \geq 0$ *holds*;
(iii) $\lim_{n\to\infty} q_n = \lim_{n\to\infty} \frac{\alpha_n}{q_n} = 0$.

Then $\lim_{n\to\infty} x_n = \bar{x}$ for some $\bar{x} \in \Omega$.

Proof Take any $w \in \Omega$ and let w be fixed. Then for each $n \in \mathbb{N}$,

$$
\begin{aligned}
d(y_n, w)^2 &= d((1 - \beta_n)x_n \oplus \beta_n T(t_n)x_n, w)^2 \\
&\leq (1 - \beta_n)d(x_n, w)^2 + \beta_n d(T(t_n)x_n, w)^2 - \beta_n(1 - \beta_n)d(x_n, T(t_n)x_n)^2 \\
&\leq (1 - \beta_n)d(x_n, w)^2 + \beta_n d(x_n, w)^2 - \beta_n(1 - \beta_n)d(x_n, T(t_n)x_n)^2 \\
&\leq d(x_n, w)^2,
\end{aligned} \tag{3.20}
$$

and

$$
\begin{aligned}
d(x_{n+1}, w)^2 &= d((1 - \alpha_{n+1})Q(q_{n+1})x_{n+1} \oplus \alpha_{n+1}y_n, w)^2
\end{aligned}
$$

$$\leq (1 - \alpha_{n+1})d\big(Q(q_{n+1})x_{n+1}, w\big)^2 + \alpha_{n+1}d(y_n, w)^2 - \alpha_{n+1}(1 - \alpha_{n+1})d\big(Q(q_{n+1})x_{n+1}, y_n\big)^2$$

$$\leq (1 - \alpha_{n+1})d(x_{n+1}, w)^2 + \alpha_{n+1}d(y_n, w)^2. \tag{3.21}$$

By (3.20) and (3.21),

$$d(x_{n+1}, w) \leq d(y_n, w) \leq d(x_n, w). \tag{3.22}$$

Hence, $\lim_{n\to\infty} d(x_n, w)$ exists, $\lim_{n\to\infty} d(x_n, w) = \lim_{n\to\infty} d(y_n, w)$, $\{x_n\}$ and $\{y_n\}$ are bounded sequences. By (3.20) and assumptions,

$$\lim_{n\to\infty} d\big(x_n, T(t_n)x_n\big) = 0. \tag{3.23}$$

By (3.21) and assumptions,

$$\lim_{n\to\infty} d\big(Q(q_{n+1})x_{n+1}, y_n\big) = 0. \tag{3.24}$$

By (3.23) and (3.24), we have

$$\lim_{n\to\infty} d(x_n, y_n) = 0 \quad \text{and} \quad \lim_{n\to\infty} d\big(Q(q_{n+1})x_{n+1}, x_{n+1}\big) = 0. \tag{3.25}$$

Furthermore, by (3.23) and (3.25), it follows that

$$\lim_{n\to\infty} d\big(T(0)x_n, x_n\big) = 0 \quad \text{and} \quad \lim_{n\to\infty} d\big(Q(0)x_n, x_n\big) = 0. \tag{3.26}$$

Following the same argument as in the proof of Theorem 3.1, there exist a subsequence $\{u_n\}$ of $\{x_n\}$ and $\bar{x} \in C$ such that $\lim_{n\to\infty} d(u_n, \bar{x}) = 0$, and $\bar{x}$ is a common fixed point of $\{T(t) : t \geq 0\}$. Hence, $\liminf_{n\to\infty} d(x_n, \bar{x}) = 0$.

For $q = 0$, by (3.26), we get $\lim_{n\to\infty} d(Q(0)u_n, u_n) = 0$, and it is easy to see that $Q(0)\bar{x} = \bar{x}$. Next, for each $q > 0$,

$$d\big(x_n, Q(q)\bar{x}\big)$$

$$\leq d\big(x_n, Q(0)x_n\big) + \sum_{k=0}^{[\frac{q}{q_n}]-1} d\big(Q((k+1)q_n)x_n, Q(kq_n)x_n\big)$$

$$+ d\left(Q\left(\left[\frac{q}{q_n}\right]q_n\right)x_n, Q\left(\left[\frac{q}{q_n}\right]q_n\right)\bar{x}\right) + d\left(Q\left(\left[\frac{q}{q_n}\right]q_n\right)\bar{x}, Q(q)\bar{x}\right)$$

$$\leq d\big(x_n, Q(0)x_n\big) + \left[\frac{q}{q_n}\right]d\big(Q(q_n)x_n, x_n\big) + d(x_n, \bar{x}) + d\left(Q\left(q - \left[\frac{q}{q_n}\right]q_n\right)\bar{x}, Q(0)\bar{x}\right)$$

$$\leq d\big(x_n, Q(0)x_n\big) + \frac{q\alpha_n}{q_n}d\big(Q(q_n)x_n, y_n\big) + d(x_n, \bar{x}) + \max\big\{d\big(Q(s)\bar{x}, Q(0)\bar{x}\big) : 0 \leq s \leq q_n\big\}$$

$$\leq d\big(x_n, Q(0)x_n\big) + \frac{q\alpha_n}{q_n}M + d(x_n, \bar{x}) + \max\big\{d\big(Q(s)\bar{x}, Q(0)\bar{x}\big) : 0 \leq s \leq q_n\big\}$$

for some $M > 0$.

By assumptions and (3.26), we know that

$$\limsup_{n\to\infty} d\big(u_n, Q(q)\bar{x}\big) = 0$$

for each $q > 0$, and this implies that $\bar{x}$ is also a common fixed point of $\{Q(q) : q \geq 0\}$. Now, $\bar{x} \in \Omega$ and this implies that $\lim_{n \to \infty} d(x_n, \bar{x})$ exists. Since $\liminf_{n \to \infty} d(x_n, \bar{x}) = 0$, we know that $\lim_{n \to \infty} d(x_n, \bar{x}) = 0$, and the proof is completed. □

Remark 3.3 Theorem 3.3 is also a generalization of Theorem 1.2.

The following result is similar to Theorem 2.3 in [27].

Corollary 3.1 *Let C be a nonempty compact convex subset of a complete $CAT(0)$ space X. Let $\{Q(q) : q \in \mathbb{R}\}$ be a continuous semigroup of nonexpansive mappings on C. Let $\{q_n\}$ be a sequence in $\mathbb{R}_+$. Let $\{\alpha_n\}$ be a sequence in $[0,1]$ with $0 < \alpha_n < 1$. Let $\{x_n\}$ be defined as*

$$\begin{cases} x_1 \in C \quad \text{chosen arbitrary,} \\ x_{n+1} = (1 - \alpha_{n+1})Q(q_{n+1})x_{n+1} \oplus \alpha_{n+1}x_n. \end{cases}$$

Assume that $\lim_{n \to \infty} q_n = \lim_{n \to \infty} \frac{\alpha_n}{q_n} = 0$. Then $\lim_{n \to \infty} x_n = \bar{x}$ for some $\bar{x} \in \Omega$.

Proof For each $t \geq 0$, let $T(t) : C \to C$ be defined by $T(t)x := x$ for each $x \in C$. Clearly, $\{T(t) : t \geq 0\}$ is a continuous semigroup of nonexpansive mappings on C. Since C is a compact set, $\Omega := \bigcap_{q \geq 0} F(Q(q)) \neq \emptyset$. By Theorem 3.3, we get the conclusion. □

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