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Generalized contractions in metric spaces endowed with a graph

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Abstract

A very interesting approach in the theory of fixed points in some general structures was recently given by Jachymski (Proc. Amer. Math. Soc. 136:1359-1373, 2008) and Gwóźdź-Lukawska and Jachymski (J. Math. Anal. Appl. 356:453-463, 2009) by using the context of metric spaces endowed with a graph. The purpose of this article is to present some new fixed point results for graphic contractions and for Ćirić-Reich-Rus G-contractions on complete metric spaces endowed with a graph. The particular case of almost contractions is also considered.
MSC: 47H10; 54H25

Keywords: metric space; connected graph; fixed point; graphic contraction; almost contraction; generalized contraction

1 Introduction

A very interesting approach in the theory of fixed points in some general structures was recently given by Jachymski [1] and Gwóźdź-Lukawska and Jachymski [2] by using the context of metric spaces endowed with a graph. More precisely, let (X, d) be a metric space and Δ be the diagonal of $X \times X$. Let G be a directed graph such that the set $V(G)$ of its vertices coincides with X and $\Delta \subseteq E(G)$, where $E(G)$ is the set of edges of the graph. Assume also that G has no parallel edges, and thus, one can identify G with the pair $(V(G), E(G))$.

By definition, an operator $f : X \to X$ is called a Banach G-contraction (see Definition 2.1 in Jachymski [1]) if and only if:

(a) for each $x, y \in X$ with $(x, y) \in E(G)$, we have $(f(x), f(y)) \in E(G)$;

(b) there exists $\alpha \in]0, 1[$ such that for each $x, y \in X$, the following implication holds:

$((x, y) \in E(G)$ implies $d(f(x), f(y)) \leq \alpha d(x, y))$.

If x and y are vertices of G, then a path in G from x to y of length $k \in \mathbb{N}$ is a finite sequence $(x_n)_{n \in \{0,1,2,\ldots,k\}}$ of vertices such that $x_0 = x$, $x_k = y$ and $(x_{i-1}, x_i) \in E(G)$ for $i \in \{1, 2, \ldots, k\}$. Notice that a graph G is connected if there is a path between any two vertices, and it is weakly connected if $\tilde{G}$ is connected, where $\tilde{G}$ denotes the undirected graph obtained from G by ignoring the direction of edges.

Denote by G^{-1} the graph obtained from G by reversing the direction of edges. Thus,

$$E(G^{-1}) = \{(x, y) \in X \times X : (y, x) \in E(G)\}. \tag{*}$$

Since it is more convenient to treat $\tilde{G}$ as a directed graph for which the set of its edges is symmetric, under this convention, we have that

$$E(\tilde{G}) = E(G) \cup E(G^{-1}). \tag{$**$}$$

If G is such that $E(G)$ is symmetric, then for $x \in V(G)$, the symbol $[x]_G$ denotes the equivalence class of the relation $\Re$ defined on $V(G)$ by the rule

$y \Re z$ if there is a path in G from y to z.

Recall that if $f : X \to X$ is an operator, then by

$$F_f := \{x \in X : x = f(x)\}$$

we denote the set of all fixed points of f. Denote also

$$X_f := \{x \in X : (x, f(x)) \in E(G)\}.$$

The operator $f : X \to X$ is called orbitally continuous if for all $x, a \in X$, for any sequence $(n(i))_{i \in \mathbb{N}}$ of positive integers,

$$\lim_{i \to \infty} f^{n(i)}(x) = a \quad \text{implies} \quad \lim_{i \to \infty} f\big(f^{n(i)}(x)\big) = f(a).$$

One of the main results in Jachymski [1] is Theorem 3.2. We present here (partially) this result. In this theorem, as well as in other results of this type, one uses the following property:

> for any sequence $(x_n)_{n \in \mathbb{N}} \subset X$, if $x_n \to x$ as $n \to +\infty$ and $(x_n, x_{n+1}) \in E(G)$ (or respectively $(x_{n+1}, x_n) \in E(G)$) for each $n \in \mathbb{N}$, then there exists a sub-sequence $(x_{k_n})_{n \in \mathbb{N}}$ of $(x_n)_{n \in \mathbb{N}}$ such that $(x_{k_n}, x) \in E(G)$ (or respectively $(x, x_{k_n}) \in E(G)$) for each $n \in \mathbb{N}$. $\tag{P}$

Theorem 1.3 (Jachymski [1]) *Let (X, d) be a complete metric space, and let G be a directed graph such that $V(G) = X$, $\Delta \subseteq E(G)$ and the triple (X, d, G) has the property* (P). *Let $f : X \to X$ be a G-contraction. Then the following statements hold*:

(1) *$F_f \neq \emptyset$ if and only if $X_f \neq \emptyset$;*

(2) *if $X_f \neq \emptyset$ and G is weakly connected, then f is a Picard operator, i.e., $F_f = \{x^*\}$ and $(f^n(x))_{n \in \mathbb{N}} \to x^*$ as $n \to \infty$ for all $x \in X$;*

(3) *for any $x \in X_f$, we have that $f|_{[x]_{\tilde{G}}}$ is a Picard operator;*

(4) *if $f \subseteq E(G)$, then f is a weakly Picard operator, i.e., $F_f \neq \emptyset$ and, for each $x \in X$, we have $(f^n(x))_{n \in \mathbb{N}} \to x^*(x) \in F_f$ as $n \to \infty$.*

The purpose of this paper is to extend the above result to the case of some generalized contractions. More precisely, the cases of Ćirić-Reich-Rus G-contractions, G-graphic contractions and of G-almost contractions in the sense of Berinde will be considered. As one can see from the recent literature (see [3–6]), these classes of operators are important extensions of the classical contraction mapping.

Our results also generalize and extend some fixed point theorems in partially ordered complete metric spaces given in Harjani and Sadarangani [7], Nicolae, O'Regan and Petruşel [8], Nieto and Rodríguez-López [9] and [10], Agarwal, El-Gebeily and O'Regan [16], Nieto, Pouso and Rodríguez-López [11], O'Regan and Petruşel [12], Petruşel and Rus [13], and Ran and Reurings [14].

2 Main results

Let G be a directed graph. Throughout this section, we will suppose that the graph G satisfies the following standard conditions: the set $V(G)$ of its vertices coincides with X, $E(G)$ is the set of edges of the graph, and Δ (the diagonal of $X \times X$) is a subset of $E(G)$. We also denote

$$X^f := \left\{ x \in X : \left(x, f(x) \right) \in E(G) \text{ or } \left(f(x), x \right) \in E(G) \right\}.$$

Definition 2.1 The mapping $f : X \to X$ is a G-graphic contraction if:

(1) f is edge preserving, *i.e.,* $(x, y) \in E(G)$ implies $(f(x), f(y)) \in E(G)$;

(2) there exists $a \in [0, 1[$ such that

$$d\left(f(x), f^2(x) \right) \leq a d\left(x, f(x) \right) \quad \text{for all } x \in X^f.$$

Lemma 2.1 *Let (X, d) be a metric space endowed with a graph G satisfying the above standard conditions. Let $f : X \to X$ be a G-graphic contraction. Then f is a G^{-1}-graphic contraction too.*

Proof Suppose that $f : X \to X$ is a G-graphic contraction. We have to prove that $f : X \to X$ is a G^{-1}-graphic contraction, *i.e.,*

(1) $(x, y) \in E(G^{-1})$ implies $(f(x), f(y)) \in E(G^{-1})$;

(indeed, $(x, y) \in E(G^{-1})$ implies $(y, x) \in E(G)$. Thus, $(f(y), f(x)) \in E(G)$ and hence $(f(x), f(y)) \in E(G^{-1})$.

(2) there exists $a \in [0, 1[$ such that $d(f(x), f^2(x)) \leq a d(x, f(x))$ for all $x \in X$ with $(x, f(x)) \in E(G^{-1})$ or $(f(x), x) \in E(G^{-1})$.

(indeed, let $x \in X$ such that $(x, f(x)) \in E(G^{-1})$ or $(f(x), x) \in E(G^{-1})$. Then $(f(x), x) \in E(G)$ or $(x, f(x)) \in E(G)$. In both cases, we get, by the G-graphic contraction condition, that there exists $a \in [0, 1[$ such that $d(f(x), f^2(x)) \leq a d(x, f(x))$. □

Notice now the fact that a G-graphic contraction is also a $\tilde{G}$-graphic contraction. This follows by $(*)$, $(**)$, the symmetry of the metric d and the above lemma.

Definition 2.2 The operator $f : X \to X$ is called a Ćirić-Reich-Rus G-contraction if and only if:

(a) f is edge preserving, *i.e.* for each $x, y \in X$ with $(x, y) \in E(G)$ we have that $(f(x), f(y)) \in E(G)$;

(b) there exists $\alpha, \beta, \gamma \in \mathbb{R}_+$ with $\alpha + \beta + \gamma \in {]0, 1[}$ such that for each $x, y \in X$, we have $((x, y) \in E(G)$ implies $d(f(x), f(y)) \leq \alpha d(x, y) + \beta d(x, f(x)) + \gamma d(y, f(y)))$.

In particular, if $\beta = \gamma = 0$, then we get the concept of Banach G-contraction presented in the first section.

A similar result to the above one (see Lemma 2.1) is the following.

Lemma 2.2 *Let (X, d) be a metric space endowed with a graph G satisfying the standard conditions mentioned at the beginning of this section. Let $f : X \to X$ be a Ćirić-Reich-Rus G-contraction. Then f is a Ćirić-Reich-Rus G^{-1}-contraction too.*

Notice also that a Ćirić-Reich-Rus G-contraction is a Ćirić-Reich-Rus $\tilde{G}$-contraction. Now, we can prove the main results of this paper. We first present some auxiliary results.

Lemma 2.3 *Let (X, d) be a metric space endowed with a graph G satisfying the above standard conditions. Let $f : X \to X$ be a G-graphic contraction with the constant a. Then, given $x \in X^f$, there exists $r(x) \geq 0$ such that*

$$d\big(f^n(x), f^{n+1}(x)\big) \leq a^n r(x), \quad \text{for all } n \in \mathbb{N}.$$

Proof Let $x \in X^f$, *i.e.*, $(x, f(x)) \in E(G)$ or $(f(x), x) \in E(G)$. If $(x, f(x)) \in E(G)$, then by induction, we have that $(f^n(x), f^{n+1}(x)) \in E(G)$ for each $n \in \mathbb{N}$. Thus

$$d\big(f^n(x), f^{n+1}(x)\big) \leq ad\big(f^{n-1}(x), f^n(x)\big) \leq \cdots \leq a^n d\big(x, f(x)\big) := a^n r(x).$$

If $(f(x), x) \in E(G)$, again by induction, we have that $(f^{n+1}(x), f^n(x)) \in E(G)$ for each $n \in \mathbb{N}$. Thus

$$d\big(f^n(x), f^{n+1}(x)\big) \leq ad\big(f^{n-1}(x), f^n(x)\big) \leq \cdots \leq a^n d\big(x, f(x)\big) := a^n r(x). \qquad \square$$

Lemma 2.4 *Let (X, d) be a complete metric space endowed with a graph G satisfying the above standard conditions. Suppose that $f : X \to X$ is a G-graphic contraction. Then for each $x \in X^f$, there exists $x^*(x) \in X$ such that the sequence $(f^n(x))_{n \in \mathbb{N}}$ converges to $x^*(x)$ as $n \to +\infty$.*

Proof Let $x \in X^f$. By Lemma 2.3, we get that

$$d\big(f^n(x), f^{n+1}(x)\big) \leq a^n r(x), \quad \text{for all } n \in \mathbb{N},$$

where $r(x) := d(x, f(x))$. Hence $\sum_{n \geq 0} d(f^n(x), f^{n+1}(x)) < \infty$, proving by a standard argument that the sequence $(f^n(x))_{n \in \mathbb{N}}$ is Cauchy. Since the space X is complete with respect to d, we get that there exists $x^*(x) \in X$ such that the sequence $(f^n(x))_{n \in \mathbb{N}}$ converges to $x^*(x)$ as $n \to +\infty$. $\qquad \square$

The next result is also important.

Lemma 2.5 *Let (X, d) be a metric space endowed with a graph G satisfying the above standard conditions. Let $f : X \to X$ be a G-graphic contraction for which there exists $x_0 \in X$ such that $f(x_0) \in [x_0]_{\tilde{G}}$. Then the set $[x_0]_{\tilde{G}}$ is invariant with respect to f and $f|_{[x_0]_{\tilde{G}}}$ is a $\tilde{G}_{x_0}$-graphic contraction, where $\tilde{G}_{x_0}$ is the component of $\tilde{G}$ containing x_0.*

Proof Let $x \in [x_0]_{\tilde{G}}$. Then there exist $(x_0, x_1, \ldots, x_{k-1}, x_k = x)$ which form a path between x_0 and x in $\tilde{G}$, *i.e.*, $(x_i, x_{i+1}) \in E(\tilde{G})$ for $i \in \{0, \ldots, k-1\}$. Since f is a $\tilde{G}$-graphic contraction (thus edge preserving), we get that $(f(x_i), f(x_{i+1})) \in E(\tilde{G})$ for $i \in \{0, \ldots, k-1\}$, *i.e.*, we have

a path from $f(x_0)$ to $f(x)$. Hence $f(x) \in [f(x_0)]_{\tilde{G}} = [x_0]_{\tilde{G}}$ since $f(x_0) \in [x_0]_{\tilde{G}}$. Thus, the set $[x_0]_{\tilde{G}}$ is invariant with respect to f.

Let $(x, y) \in E(\tilde{G}_{x_0})$. Then there is a path from x_0 to y passing through x, i.e., $(x_0, x_1, \ldots, x_{k-1} = x, x_k = y)$ and $(x_i, x_{i+1}) \in E(\tilde{G})$ for $i \in \{0, \ldots, k-1\}$. As before, by the edge preserving condition, we get that $(f(x_i), f(x_{i+1})) \in E(\tilde{G})$ for $i \in \{0, \ldots, k-1\}$. Let $(y_0, y_1, \ldots, y_{l-1}, y_l)$ be a path between x_0 and f_{x_0}. Thus

$$\left(x_0 = y_0, y_1, \ldots, y_l = f(x_0), f(x_1), \ldots, f(x_{k-1}) = f(x), f(x_k) = f(y)\right)$$

is a path in $\tilde{G}$ from x_0 to $f(y)$ such that $(f(x), f(y)) \in E(\tilde{G}_{x_0})$. Moreover, f is a $\tilde{G}_{x_0}$-graphic contraction since $E(\tilde{G}_{x_0}) \subset E(\tilde{G})$, and f is a $\tilde{G}$-graphic contraction. □

For our next result, we need the following definition.

Definition 2.3 Let (X, d) be a metric space endowed with a graph G satisfying the above standard conditions. The operator $f : X \to X$ is called orbitally G-continuous if, for all $x, a \in X$ and for any sequence $(n(i))_{i \in \mathbb{N}}$ of positive integers,

$$\lim_{i \to \infty} f^{n(i)}(x) = a, \qquad \left(f^{n(i)}(x), f^{n(i)+1}(x)\right) \in E(G), \quad i \in \mathbb{N} \quad \Rightarrow \quad \lim_{i \to \infty} f\left(f^{n(i)}(x)\right) = f(a).$$

We can now give the main result for G-graphic contractions. We will make use of the property (P) introduced in Section 1.

Theorem 2.1 *Let (X, d) be a complete metric space and G be a directed graph satisfying the above standard conditions such that the triple (X, d, G) satisfies the property* (P). *Let $f : X \to X$ be a G-graphic contraction which is G-orbitally continuous. Then the following statements hold*:

(1) *$F_f \neq \emptyset$ if and only if $X^f \neq \emptyset$*;
(2) *if $X^f \neq \emptyset$ and G is weakly connected, then f is a weakly Picard operator*;
(3) *for any $x \in X^f$, we have that $f|_{[x]_{\tilde{G}}}$ is a weakly Picard operator.*

Proof We start by proving statement (3). Let $x \in X^f$. Applying Lemma 2.3, we have that there exists $r(x) \geq 0$ such that

$$d\left(f^n(x), f^{n+1}(x)\right) \leq a^n r(x), \quad \text{for all } n \in \mathbb{N}.$$

This yields, as in the proof of Lemma 2.4, that there exists $x^*(x) \in X$ such that $\lim_{n \to \infty} f^n(x) = x^*(x)$. Because $x \in X^f$, condition (1) in Definition 2.1 implies that $f^n(x) \in X^f$ for every $n \in \mathbb{N}$. Suppose now, for example, that $(x, f(x)) \in E(G)$. (A similar deduction can be made if $(f(x), x) \in E(G)$.) Using the property (P), we can find a subsequence $(f^{k_n}(x))_{n \in \mathbb{N}}$ of $(f^n(x))_{n \in \mathbb{N}}$ such that $(f^{k_n}(x), x^*(x)) \in E(G)$ for each $n \in \mathbb{N}$. Then the points $x, f(x), \ldots, f^{k_1}(x), x^*(x)$ form a path in G, and thus $x^*(x) \in [x]_{\tilde{G}}$. Since f is G-orbitally continuous, we obtain that $x^*(x)$ is a fixed point for $f|_{[x]_{\tilde{G}}}$.

For (1), notice that from (3) it follows that $F_f \neq \emptyset$ if $X^f \neq \emptyset$. Suppose $F_f \neq \emptyset$. Since we work under the assumption that $\Delta \subseteq E(G)$, we immediately get that $X^f \neq \emptyset$. Hence, (1) also holds.

To prove (2), let $x \in X^f$. Because G is weakly connected, we have that $X = [x]_{\tilde{G}}$ and we only need to apply (3). □

Remark 2.1 In the above result, we can remove the condition that the triple (X, d, G) satisfies the property (P) and f is G-orbitally continuous if we assume that the mapping f is orbitally continuous.

Remark 2.2 If we relax the definition of the G-graphic contraction, by replacing the condition

(1) f is edge preserving, *i.e.*, $(x, y) \in E(G)$ implies $(f(x), f(y)) \in E(G)$

with

(1′) $(x, f(x)) \in E(G)$ implies $(f(x), f^2(x)) \in E(G)$,

then part of the above results take place. More precisely, we do not know if a similar result to Lemma 2.5 takes place.

The following example shows that our result is a non-trivial extension of the above mentioned papers.

Example 2.1 Let $X = [0, 1]$ be endowed with the usual distance. Consider

$$E(G) = \big\{(0, 0)\big\} \cup \big\{(0, x) : x \geq 1/2\big\} \cup \big\{(x, y) : x, y \in (0, 1]\big\},$$

and $f : X \to X$,

$$f(x) = \begin{cases} x/2, & \text{if } x \in (0, 1), \\ 3/4, & \text{if } x = 0, \\ 1, & \text{if } x = 1. \end{cases}$$

Then G is weakly connected, X^f is nonempty and f is a G-graphic contraction which fails to be a G-contraction. Moreover, $Fix(f) = \{1\}$.

Proof It can be easily checked that G is weakly connected and f is edge preserving. To see that f is not a G-contraction, notice that $|f(0) - f(1/2)| = 1/2 = |0 - 1/2|$. A simple case-by-case calculation shows that f is a G-graphic contraction. Moreover, $Fix(f) = \{1\}$. □

We will consider next the case of almost contractions; see Berinde [3]. See also Păcurar [15] for a detailed and comprehensive study of almost contractions.

Definition 2.4 The mapping $f : X \to X$ is a G-almost contraction if:
   (1) f is edge preserving, *i.e.*, $(x, y) \in E(G)$ implies $(f(x), f(y)) \in E(G)$;
   (2) there exists $\alpha \in\,]0, 1[$ and $L \geq 0$ such that for each $x, y \in X$ the following implication holds:

$$(x, y) \in E(G) \quad \text{implies} \quad d\big(f(x), f(y)\big) \leq ad(x, y) + Ld\big(y, f(x)\big).$$

Notice that if f is a G-almost contraction, then f is a G-graphic contraction with the constant $a \in [0, 1[$. Thus all the above results take place.

We will consider now the case of Ćirić-Reich-Rus G-contractions.

Lemma 2.6 *Let (X, d) be a metric space endowed with a graph G satisfying the above standard conditions. Let $f : X \to X$ be a Ćirić-Reich-Rus G-contraction with constants α, β, γ. Then, given $x \in X^f$, there exists $r(x) \geq 0$ such that*

$$d\big(f^n(x), f^{n+1}(x)\big) \leq a^n r(x), \quad \text{for all } n \in \mathbb{N},$$

where $a := \frac{\alpha + \beta}{1 - \gamma}$.

Proof Let $x \in X^f$, i.e., $(x, f(x)) \in E(G)$ or $(f(x), x) \in E(G)$. If $(x, f(x)) \in E(G)$, then by induction, we have that $(f^n(x), f^{n+1}(x)) \in E(G)$ for each $n \in \mathbb{N}$. Hence

$$d\big(f^n(x), f^{n+1}(x)\big) \leq \alpha d\big(f^{n-1}(x), f^n(x)\big) + \beta d\big(f^{n-1}(x), f^n(x)\big) + \gamma d\big(f^n(x), f^{n+1}(x)\big).$$

Thus, we get that

$$d\big(f^n(x), f^{n+1}(x)\big) \leq \frac{\alpha + \beta}{1 - \gamma} d\big(f^{n-1}(x), f^n(x)\big) \leq \cdots \leq a^n d\big(x, f(x)\big) := a^n r(x).$$

If $(f(x), x) \in E(G)$, again by induction, we have that $(f^{n+1}(x), f^n(x)) \in E(G)$ for each $n \in \mathbb{N}$. Thus we get the same relation as before. This completes the proof. □

Lemma 2.7 *Let (X, d) be a complete metric space endowed with a graph G satisfying the above standard conditions. Suppose that $f : X \to X$ is a Ćirić-Reich-Rus G-contraction with constants α, β, γ. Then for each $x \in X^f$, there exists $x^*(x) \in X$ such that the sequence $(f^n(x))_{n \in \mathbb{N}}$ converges to $x^*(x)$ as $n \to +\infty$.*

Proof Let $x \in X^f$. By Lemma 2.6, we get that

$$d\big(f^n(x), f^{n+1}(x)\big) \leq a^n r(x), \quad \text{for all } n \in \mathbb{N},$$

where $r(x) := d(x, f(x))$. Hence $\sum_{n \geq 0} d(f^n(x), f^{n+1}(x)) < \infty$, proving by a standard argument that the sequence $(f^n(x))_{n \in \mathbb{N}}$ is Cauchy. Since the space X is complete with respect to d, we get that there exists $x^*(x) \in X$ such that the sequence $(f^n(x))_{n \in \mathbb{N}}$ converges to $x^*(x)$ as $n \to +\infty$. □

We can now give the main result for Ćirić-Reich-Rus G-contractions. We will make use again of the property (P) introduced in Section 1.

Theorem 2.2 *Let (X, d) be a complete metric space and G be a directed graph satisfying the above standard conditions such that the triple (X, d, G) satisfies the property (P). Let $f : X \to X$ be Ćirić-Reich-Rus G-contraction with constants α, β, γ such that it is G-orbitally continuous. Then the following statements hold:*

(1) *$F_f \neq \emptyset$ if and only if $X^f \neq \emptyset$;*

(2) *if $X^f \neq \emptyset$ and G is weakly connected, then f is a weakly Picard operator;*

(3) *for any $x \in X^f$, we have that $f|_{[x]_{\tilde{G}}}$ is a weakly Picard operator.*

Proof We start by proving statement (3). Let $x \in X^f$. Applying Lemma 2.6, we have that there exists $r(x) \geq 0$ such that

$$d\big(f^n(x), f^{n+1}(x)\big) \leq a^n r(x), \quad \text{for all } n \in \mathbb{N}.$$

This yields, as in the proof of Lemma 2.7, that there exists $x^*(x) \in X$ such that $\lim_{n \to \infty} f^n(x) = x^*(x)$. Because $x \in X^f$, condition (1) in Definition 2.2 implies that $f^n(x) \in X^f$ for every $n \in \mathbb{N}$. Suppose now, for example, that $(x, f(x)) \in E(G)$. (A similar deduction can be made if $(f(x), x) \in E(G)$.) Using the property (P), we can find a subsequence $(f^{k_n}(x))_{n \in \mathbb{N}}$ of $(f^n(x))_{n \in \mathbb{N}}$ such that $(f^{k_n}(x), x^*(x)) \in E(G)$ for each $n \in \mathbb{N}$. Then the points $x, f(x), \ldots, f^{k_1}(x), x^*(x)$ form a path in G, and thus $x^*(x) \in [x]_{\tilde{G}}$. Since f is G-orbitally continuous, we obtain that $x^*(x)$ is a fixed point for $f|_{[x]_{\tilde{G}}}$.

For (1), notice that from (3) it follows that $F_f \neq \emptyset$ if $X^f \neq \emptyset$. Suppose $F_f \neq \emptyset$. Since we work under the assumption that $\Delta \subseteq E(G)$, we immediately get that $X^f \neq \emptyset$. Hence, (1) also holds.

To prove (2), let $x \in X^f$. Because G is weakly connected, we have that $X = [x]_{\tilde{G}}$, and we only need to apply (3). □

Remark 2.3 If in the above result we put $\beta = \gamma = 0$, then we get the results given in Jachymski [1].

Remark 2.4 It is an open problem if in some appropriate assumptions, one can prove that a Ćirić-Reich-Rus G-contraction is a Picard operator.

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