Shatanawi and Manro *Fixed Point Theory and Applications* 2012, **2012**:165
http://www.fixedpointtheoryandapplications.com/content/2012/1/165

Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH Open Access

Fixed point results for cyclic (ψ, ϕ, A, B)-contraction in partial metric spaces

Wasfi Shatanawi[1] and Saurabh Manro[2*]

*Correspondence: sauravmanro@hotmail.com
[2]School of Mathematics and Computer Applications, Thapar University, Patiala, Punjab, India
Full list of author information is available at the end of the article

Abstract

Very recently, Agarwal *et al.* (Fixed Point Theory Appl. 2012:40, 2012) initiated the study of fixed point theorems for mappings satisfying cyclical generalized contractive conditions in complete partial metric spaces. In the present paper, we study some fixed point theorems for a mapping satisfying a cyclical generalized contractive condition based on a pair of altering distance functions in complete partial metric spaces. Also, we introduce an example and an application to support the usability of our paper.

MSC: Primary 54H25; secondary 47H10

Keywords: partial metric spaces; fixed point; altering distance function; cyclic map

1 Introduction

The existence and uniqueness of fixed and common fixed point theorems of operators has been a subject of great interest since Banach [1] proved the Banach contraction principle in 1922. Many authors generalized the Banach contraction principle in various spaces such as quasi-metric spaces, generalized metric spaces, cone metric spaces and fuzzy metric spaces. Matthews [2] introduced the notion of partial metric spaces in such a way that each object does not necessarily have to have a zero distance from itself and proved a modified version of the Banach contraction principle. Afterwards, many authors proved many existing fixed point theorems in partial metric spaces (see [3–21] for examples).

We recall below the definition of partial metric space and some of its properties.

Definition 1 [2] A partial metric on a nonempty set X is a function $p : X \times X \to \mathbb{R}^+$ such that for all $x, y, z \in X$:

(p_1) $x = y \Longleftrightarrow p(x,x) = p(x,y) = p(y,y)$,
(p_2) $p(x,x) \le p(x,y)$,
(p_3) $p(x,y) = p(y,x)$,
(p_4) $p(x,y) \le p(x,z) + p(z,y) - p(z,z)$.

A partial metric space is a pair (X,p) such that X is a nonempty set and p is a partial metric on X. It is clear that, if $p(x,y) = 0$, then from (p_1) and (p_2), $x = y$. But if $x = y$, $p(x,y)$ may not be 0. The function $p(x,y) = \max\{x,y\}$ for all $x, y \in \mathbb{R}^+$ defines a partial metric on $\mathbb{R}^+$.

Each partial metric p on X generates a T_0 topology τ_p on X which has as a base the family of open p-balls $\{B_p(x,\varepsilon) : x \in X, \varepsilon > 0\}$, where $B_p(x,\varepsilon) = \{y \in X : p(x,y) < p(x,x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

If p is a partial metric on X, then the function $d_p : X \times X \to \mathbb{R}^+$ given by

$$d_p(x,y) = 2p(x,y) - p(x,x) - p(y,y)$$

is a metric on X.

Definition 2 Let (X,p) be a partial metric space. Then

(1) A sequence $\{x_n\}$ in a partial metric space (X,p) converges to a point $x \in X$ if and only if $p(x,x) = \lim_{n\to\infty} p(x,x_n)$.
(2) A sequence $\{x_n\}$ in a partial metric space (X,p) is called a Cauchy sequence iff $\lim_{n,m\to\infty} p(x_n,x_m)$ exists (and is finite).
(3) A partial metric space (X,p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p, to a point $x \in X$ such that $p(x,x) = \lim_{n,m\to\infty} p(x_n,x_m)$.
(4) A subset A of a partial metric space (X,p) is closed if whenever $\{x_n\}$ is a sequence in A such that $\{x_n\}$ converges to some $x \in X$, then $x \in A$.

Remark 1 The limit in a partial metric space is not unique.

Lemma 1 ([2, 17]) *Let (X,p) be a partial metric space.*

(a) *$\{x_n\}$ is a Cauchy sequence in (X,p) if and only if it is a Cauchy sequence in the metric space (X,d_p).*
(b) *A partial metric space (X,p) is complete if and only if the metric space (X,d_p) is complete. Furthermore, $\lim_{n\to\infty} d_p(x_n,x) = 0$ if and only if*

$$p(x,x) = \lim_{n\to\infty} p(x_n,x) = \lim_{n,m\to\infty} p(x_n,x_m).$$

Now, we define the cyclic map.

Definition 3 Let A and B be nonempty subsets of a metric space (X,d) and $T : A\cup B \to A\cup B$. Then T is called a cyclic map if $T(A) \subseteq B$ and $T(B) \subseteq A$.

In 2003, Kirk *et al.* [22] gave the following fixed point theorem for a cyclic map.

Theorem 1 [22] *Let A and B be nonempty closed subsets of a complete metric space (X,d). Suppose that $T : A\cup B \to A\cup B$ is a cyclic map such that*

$$d(Tx,Ty) \le kd(x,y) \quad \forall x \in A, \forall y \in B.$$

If $k \in [0,1)$, then T has a unique fixed point in $A\cap B$.

Karapınar and Erhan [23] introduced the following types of cyclic contractions:

Definition 4 [23] Let A and B be nonempty closed subsets of a metric space (X,d). A cyclic map $T: A\cup B\to A\cup B$ is said to be a Kannan type cyclic contraction if there exists $k\in(0,\frac{1}{2})$ such that

$$d(Tx,Ty)\le k\big(d(Tx,x)+d(Ty,y)\big)\quad \forall x\in A, \forall y\in B.$$

Definition 5 [23] Let A and B be nonempty closed subsets of a metric space (X,d). A cyclic map $T: A\cup B\to A\cup B$ is said to be a Reich type cyclic contraction if there exists $k\in(0,\frac{1}{3})$ such that

$$d(Tx,Ty)\le k\big(d(x,y)+d(Tx,x)+d(Ty,y)\big)\quad \forall x\in A, \forall y\in B.$$

Definition 6 [23] Let A and B be nonempty closed subsets of a metric space (X,d). A cyclic map $T: A\cup B\to A\cup B$ is said to be a Ćirić type cyclic contraction if there exists $k\in(0,\frac{1}{3})$ such that

$$d(Tx,Ty)\le k\max\big\{d(x,y),d(Tx,x),d(Ty,y)\big\}\quad \forall x\in A, \forall y\in B.$$

Moreover, Karapınar and Erhan [23] obtained the following results:

Theorem 2 [23] *Let A and B be nonempty closed subsets of a complete metric space (X,d), and let $T: A\cup B\to A\cup B$ be a Kannan type cyclic contraction. Then T has a unique fixed point in $A\cap B$.*

Theorem 3 [23] *Let A and B be nonempty closed subsets of a complete metric space (X,d), and let $T: A\cup B\to A\cup B$ be a Reich type cyclic contraction. Then T has a unique fixed point in $A\cap B$.*

Theorem 4 [23] *Let A and B be nonempty closed subsets of a complete metric space (X,d), and let $T: A\cup B\to A\cup B$ be a Ćirić type cyclic contraction. Then T has a unique fixed point in $A\cap B$.*

For more results on cyclic contraction mappings, see [24, 25].

Very recently, Agarwal *et al.* [26] initiated the study of fixed point theorems for mappings satisfying cyclical generalized contractive conditions in complete partial metric spaces.

Khan *et al.* [27] introduced the notion of altering distance function as follows.

Definition 7 (Altering distance function [27]) The function $\phi:[0,+\infty)\to[0,+\infty)$ is called an altering distance function if the following properties are satisfied:

(1) ϕ is continuous and nondecreasing.
(2) $\phi(t)=0$ if and only if $t=0$.

For some work on altering distance function, we refer the reader to [28–33].

The purpose of this paper is to study some fixed point theorems for a mapping satisfying a cyclical generalized contractive condition based on a pair of altering distance functions in partial metric spaces.

2 Main result

We start with the following definition.

Definition 8 Let (X,p) be a partial metric space and A, B be nonempty closed subsets of X. A mapping $T:X\to X$ is called a cyclic (ψ,ϕ,A,B)-contraction if

(1) ψ and ϕ are altering distance functions;

(2) $A\cup B$ has a cyclic representation w.r.t. T; that is, $T(A)\subseteq B$ and $T(B)\subseteq A$; and

(3)

$$\begin{aligned}\psi\bigl(p(Tx,Ty)\bigr)\le{}&\psi\left(\max\left\{p(x,y),p(x,Tx),p(y,Ty),\frac{1}{2}\bigl(p(x,Ty)+p(Tx,y)\bigr)\right\}\right)\\&-\phi\bigl(\max\bigl\{p(x,y),p(y,Ty)\bigr\}\bigr)\end{aligned}\tag{2.1}$$

for all $x\in A$ and $y\in B$.

From now on, by ψ and ϕ we mean altering distance functions unless otherwise stated.

In the rest of this paper, $\mathbf{N}$ stands for the set of nonnegative integer numbers.

Theorem 5 *Let A and B be nonempty closed subsets of a complete partial metric space (X,p). If $T:X\to X$ is a cyclic (ψ,ϕ,A,B)-contraction, then T has a unique fixed point $u\in A\cap B$.*

Proof Let $x_0\in A$. Since $TA\subseteq B$, we choose $x_1\in B$ such that $Tx_0=x_1$. Also, since $TB\subseteq A$, we choose $x_2\in A$ such that $Tx_1=x_2$. Continuing this process, we can construct sequences $\{x_n\}$ in X such that $x_{2n}\in A$, $x_{2n+1}\in B$, $x_{2n+1}=Tx_{2n}$ and $x_{2n+2}=Tx_{2n+1}$. If $x_{2n_0+1}=x_{2n_0+2}$ for some $n\in\mathbf{N}$, then $x_{2n_0+1}=Tx_{2n_0+1}$. Thus, x_{2n_0+1} is a fixed point of T in $A\cap B$. Thus, we may assume that $x_{2n+1}\neq x_{2n+2}$ for all $n\in\mathbf{N}$.

Given $n\in\mathbf{N}$. If n is even, then $n=2t$ for some $t\in\mathbf{N}$. By (2.1), we have

$$\begin{aligned}&\psi\bigl(p(x_{n+1},x_{n+2})\bigr)\\&\quad=\psi\bigl(p(x_{2t+1},x_{2t+2})\bigr)\\&\quad=\psi\bigl(p(Tx_{2t},Tx_{2t+1})\bigr)\\&\quad\le\psi\left(\max\left\{p(x_{2t},x_{2t+1}),p(Tx_{2t},x_{2t}),p(Tx_{2t+1},x_{2t+1}),\right.\right.\\&\qquad\left.\left.\frac{1}{2}\bigl(p(x_{2t},Tx_{2t+1})+p(Tx_{2t},x_{2t+1})\bigr)\right\}\right)\\&\qquad-\phi\bigl(\max\bigl\{p(x_{2t},x_{2t+1}),p(Tx_{2t+1},x_{2t+1})\bigr\}\bigr)\\&\quad=\psi\left(\max\left\{p(x_{2t},x_{2t+1}),p(x_{2t+2},x_{2t+1}),\right.\right.\\&\qquad\left.\left.\frac{1}{2}\bigl(p(x_{2t},x_{2t+2})+p(x_{2t+1},x_{2t+1})\bigr)\right\}\right)\\&\qquad-\phi\bigl(\max\bigl\{p(x_{2t},x_{2t+1}),p(x_{2t+2},x_{2t+1})\bigr\}\bigr).\end{aligned}$$

By (p_4), we have

$$\begin{aligned}
&\psi\big(p(x_{n+1},x_{n+2})\big)\\
&\quad= \psi\big(p(x_{2t+1},x_{2t+2})\big)\\
&\quad\le \psi\left(\max\left\{p(x_{2t},x_{2t+1}),p(x_{2t+2},x_{2t+1}),\frac{1}{2}\big(p(x_{2t},x_{2t+1})+p(x_{2t+1},x_{2t+2})\big)\right\}\right)\\
&\qquad- \phi\big(\max\big\{p(x_{2t},x_{2t+1}),p(x_{2t+2},x_{2t+1})\big\}\big)\\
&\quad\le \psi\big(\max\big\{p(x_{2t},x_{2t+1}),p(x_{2t+2},x_{2t+1})\big\}\big) - \phi\big(\max\big\{p(x_{2t},x_{2t+1}),p(x_{2t+2},x_{2t+1})\big\}\big)\\
&\quad\le \psi\big(\max\big\{p(x_{2t},x_{2t+1}),p(x_{2t+2},x_{2t+1})\big\}\big).
\end{aligned}$$

If

$$\max\big\{p(x_{2t},x_{2t+1}),p(x_{2t+2},x_{2t+1})\big\} = p(x_{2t+2},x_{2t+1}),$$

then

$$\psi\big(p(x_{2t+1},x_{2t+2})\big) \le \psi\big(p(x_{2t+2},x_{2t+1})\big) - \phi\big(p(x_{2t+2},x_{2t+1})\big).$$

Therefore, $\phi(p(x_{2t+1},x_{2t+2})) = 0$, and hence $p(x_{2t+1},x_{2t+2}) = 0$. By (p_1) and (p_2), we have $x_{2t+1} = x_{2t+2}$, which is a contradiction. Therefore,

$$\max\big\{p(x_{2t},x_{2t+1}),p(x_{2t+2},x_{2t+1})\big\} = p(x_{2t},x_{2t+1}).$$

Hence,

$$p(x_{n+1},x_{n+2}) = p(x_{2t+2},x_{2t+1}) \le p(x_{2t},x_{2t+1}) = p(x_n,x_{n+1}) \tag{2.2}$$

and

$$\psi\big(p(x_{n+1},x_{n+2})\big) \le \psi\big(p(x_n,x_{n+1})\big) - \phi\big(p(x_n,x_{n+1})\big). \tag{2.3}$$

If n is odd, then $n = 2t + 1$ for some $t \in \mathbf{N}$. By (2.1), we have

$$\begin{aligned}
&\psi\big(p(x_{n+1},x_{n+2})\big)\\
&\quad= \psi\big(p(x_{2t+2},x_{2t+3})\big)\\
&\quad= \psi\big(p(x_{2t+3},x_{2t+2})\big)\\
&\quad= \psi\big(p(Tx_{2t+2},Tx_{2t+1})\big)\\
&\quad\le \psi\left(\max\left\{p(x_{2t+2},x_{2t+1}),p(Tx_{2t+2},x_{2t+2}),p(Tx_{2t+1},x_{2t+1}),\right.\right.\\
&\qquad\left.\left.\frac{1}{2}\big(p(x_{2t+2},Tx_{2t+1})+p(Tx_{2t+2},x_{2t+1})\big)\right\}\right)\\
&\qquad- \phi\big(\max\big\{p(x_{2t+2},x_{2t+1}),p(Tx_{2t+1},x_{2t+1})\big\}\big)
\end{aligned}$$

$$= \psi\left(\max\left\{p(x_{2t+3},x_{2t+2}),p(x_{2t+2},x_{2t+1}),\frac{1}{2}\big(p(x_{2t+2},x_{2t+2})+p(x_{2t+3},x_{2t+1})\big)\right\}\right) - \phi\big(\max\{p(x_{2t+2},x_{2t+1}),p(x_{2t+2},x_{2t+1})\}\big).$$

By (p_4), we have

$$\begin{aligned}
&\psi\big(p(x_{n+1},x_{n+2})\big)\\
&\quad= \psi\big(p(x_{2t+3},x_{2t+2})\big)\\
&\quad\le \psi\left(\max\left\{p(x_{2t+3},x_{2t+2}),p(x_{2t+2},x_{2t+1}),\frac{1}{2}\big(p(x_{2t+3},x_{2t+2})+p(x_{2t+2},x_{2t+1})\big)\right\}\right)\\
&\qquad - \phi\big(\max\{p(x_{2t+2},x_{2t+1}),p(Tx_{2t+1},x_{2t+1})\}\big)\\
&\quad\le \psi\big(\max\{p(x_{2t+3},x_{2t+2}),p(x_{2t+2},x_{2t+1})\}\big) - \phi\big(\max\{p(x_{2t+2},x_{2t+1}),p(Tx_{2t+1},x_{2t+1})\}\big)\\
&\quad\le \psi\big(\max\{p(x_{2t+3},x_{2t+2}),p(x_{2t+2},x_{2t+1})\}\big).
\end{aligned}$$

If

$$\max\{p(x_{2t+3},x_{2t+2}),p(x_{2t+2},x_{2t+1})\} = p(x_{2t+3},x_{2t+2}),$$

then

$$\phi\big(p(x_{2t+3},x_{2t+2})\big) \le \psi\big(p(x_{2t+3},x_{2t+2})\big) - \phi\big(p(x_{2t+2},x_{2t+1})\big).$$

Therefore, $\phi(p(x_{2t+2},x_{2t+1})) = 0$, and hence $p(x_{2t+3},x_{2t+2}) = 0$. By (p_1) and (p_2), we have $x_{2t+2} = x_{2t+1}$, which is a contradiction. Therefore,

$$\max\{p(x_{2t+3},x_{2t+2}),p(x_{2t+2},x_{2t+1})\} = p(x_{2t+2},x_{2t+1}).$$

Hence,

$$p(x_{n+2},x_{n+1}) = p(x_{2t+3},x_{2t+2}) \le p(x_{2t+2},x_{2t+1}) = p(x_{n+1},x_n), \tag{2.4}$$

$$\psi\big(p(x_{n+2},x_{n+1})\big) \le \psi\big(p(x_n,x_{n+1})\big) - \phi\big(p(x_n,x_{n+1})\big). \tag{2.5}$$

From (2.2) and (2.4), we have $\{p(x_{n+1},x_n) : n \in \mathbf{N}\}$ is a nonincreasing sequence and hence there exists $r \ge 0$ such that

$$\lim_{n\to+\infty} p(x_n,x_{n+1}) = r.$$

Also, from (2.3) and (2.5), we have

$$\psi\big(p(x_{n+2},x_{n+1})\big) \le \psi\big(p(x_n,x_{n+1})\big) - \phi\big(p(x_n,x_{n+1})\big) \quad \forall n \in \mathbf{N}. \tag{2.6}$$

Letting $n \to +\infty$ in (2.6) and using the fact that ψ and ϕ are continuous, we get that

$$\psi(r) \le \psi(r) - \phi(r).$$

Therefore, $\phi(r) = 0$ and hence $r = 0$. Thus

$$\lim_{n\to+\infty} p(x_n, x_{n+1}) = 0. \tag{2.7}$$

By (p_2), we get that

$$\lim_{n\to+\infty} p(x_n, x_n) = 0. \tag{2.8}$$

Since $d_p(x,y) \leq 2p(x,y)$ for all $x, y \in X$, we get that

$$\lim_{n\to+\infty} d_p(x_n, x_{n+1}) = 0. \tag{2.9}$$

Next, we show that $\{x_n\}$ is a Cauchy sequence in the metric space $(A \cup B, d_p)$. It is sufficient to show that $\{x_{2n}\}$ is a Cauchy sequence in $(A \cup B, d_p)$. Suppose the contrary; that is, $\{x_{2n}\}$ is not a Cauchy sequence in $(A \cup B, d_p)$. Then there exists $\epsilon > 0$ for which we can find two subsequences $\{x_{2m(i)}\}$ and $\{x_{2n(i)}\}$ of $\{x_{2n}\}$ such that $n(i)$ is the smallest index for which

$$n(i) > m(i) > i, \quad d_p(x_{2m(i)}, x_{2n(i)}) \geq \epsilon. \tag{2.10}$$

This means that

$$d_p(x_{2m(i)}, x_{2n(i)-2}) < \epsilon. \tag{2.11}$$

From (2.10), (2.11) and the triangular inequality, we get that

$$\begin{aligned}
\epsilon &\leq d_p(x_{2m(i)}, x_{2n(i)}) \\
&\leq d_p(x_{2m(i)}, x_{2n(i)-2}) + d_p(x_{2n(i)-2}, x_{2n(i)-1}) \\
&\quad + d_p(x_{2n(i)-1}, x_{2n(i)}) \\
&< \epsilon + d_p(x_{2n(i)-2}, x_{2n(i)-1}) + d_p(x_{2n(i)-1}, x_{2n(i)}).
\end{aligned}$$

On letting $i \to +\infty$ in the above inequalities and using (2.9), we have

$$\lim_{i\to+\infty} d_p(x_{2m(i)}, x_{2n(i)}) = \epsilon. \tag{2.12}$$

Again, from (2.10) and the triangular inequality, we get that

$$\begin{aligned}
\epsilon &\leq d_p(x_{2m(i)}, x_{2n(i)}) \\
&\leq d_p(x_{2n(i)}, x_{2n(i)-1}) + d_p(x_{2n(i)-1}, x_{2m(i)}) \\
&\leq d_p(x_{2n(i)}, x_{2n(i)-1}) + d_p(x_{2n(i)-1}, x_{2m(i)+1}) + d_p(x_{2m(i)+1}, x_{2m(i)}) \\
&\leq d_p(x_{2n(i)}, x_{2n(i)-1}) + d_p(x_{2n(i)-1}, x_{2m(i)}) + 2d_p(x_{2m(i)+1}, x_{2m(i)}) \\
&\leq 2d_p(x_{2n(i)}, x_{2n(i)-1}) + d_p(x_{2n(i)}, x_{2m(i)}) + 2d_p(x_{2m(i)+1}, x_{2m(i)}).
\end{aligned}$$

Letting $i \to +\infty$ in the above inequalities and using (2.9) and (2.12), we get that

$$\begin{aligned}\lim_{i\to+\infty} d_p(x_{2m(i)}, x_{2n(i)}) &= \lim_{i\to+\infty} d_p(x_{2m(i)+1}, x_{2n(i)-1}) \\ &= \lim_{i\to+\infty} d_p(x_{2m(i)+1}, x_{2n(i)}) \\ &= \lim_{i\to+\infty} d_p(x_{2m(i)}, x_{2n(i)-1}) \\ &= \epsilon.\end{aligned}$$

Since

$$d_p(x,y) = 2p(x,y) - p(x,x) - p(y,y)$$

for all $x, y \in X$, then

$$\begin{aligned}\lim_{i\to+\infty} p(x_{2m(i)}, x_{2n(i)}) &= \lim_{i\to+\infty} p(x_{2m(i)+1}, x_{2n(i)-1}) \\ &= \lim_{i\to+\infty} p(x_{2m(i)+1}, x_{2n(i)}) \\ &= \lim_{i\to+\infty} p(x_{2m(i)}, x_{2n(i)-1}) \\ &= \frac{\epsilon}{2}.\end{aligned}$$

By (2.1), we have

$$\begin{aligned}\psi\big(p(x_{2m(i)+1}, x_{2n(i)})\big) &= \psi\big(p(Tx_{2m(i)}, Tx_{2n(i)-1})\big) \\ &\le \psi\Bigg(\max\Bigg\{p(x_{2m(i)}, x_{2n(i)-1}), p(x_{2m(i)}, Tx_{2m(i)}), p(x_{2n(i)-1}, Tx_{2n(i)-1}), \\ &\quad \frac{1}{2}\big(p(x_{2m(i)}, Tx_{2n(i)-1}) + p(x_{2n(i)-1}, Tx_{2m(i)})\big)\Bigg\}\Bigg) \\ &\quad - \phi\big(\max\big\{p(x_{2m(i)}, x_{2n(i)-1}), p(x_{2n(i)-1}, Tx_{2n(i)-1})\big\}\big) \\ &= \psi\Bigg(\max\Bigg\{p(x_{2m(i)}, x_{2n(i)-1}), p(x_{2m(i)}, x_{2m(i)+1}), p(x_{2n(i)-1}, x_{2n(i)}), \\ &\quad \frac{1}{2}\big(p(x_{2m(i)}, x_{2n(i)}) + p(x_{2n(i)-1}, x_{2m(i)+1})\big)\Bigg\}\Bigg) \\ &\quad - \phi\big(\max\big\{p(x_{2m(i)}, x_{2n(i)-1}), p(x_{2n(i)-1}, x_{2n(i)})\big\}\big).\end{aligned}$$

Letting $i \to +\infty$ and using the continuity of ϕ and ψ, we get that

$$\psi\left(\frac{\epsilon}{2}\right) \le \psi\left(\frac{\epsilon}{2}\right) - \phi\left(\frac{\epsilon}{2}\right).$$

Therefore, we get that $\phi(\frac{\epsilon}{2}) = 0$. Hence, $\epsilon = 0$ is a contradiction. Thus $\{x_n\}$ is a Cauchy sequence in $(A \cup B, d_p)$. Since (X, p) is complete and $A \cup B$ is a closed subspace of (X, p), then we have $(A \cup B, p)$ is complete. From Lemma 1, the sequence $\{x_n\}$ converges in the metric space $(A \cup B, d_p)$, say $\lim_{n\to\infty} d_p(x_n, u) = 0$. Again from Lemma 1, we have

$$p(u,u) = \lim_{n\to\infty} p(x_n, u) = \lim_{n,m\to\infty} p(x_n, x_m). \tag{2.13}$$

Moreover, since $\{x_n\}$ is a Cauchy sequence in the metric space $(A \cup B, d_p)$, we have

$$\lim_{n,m\to\infty} d_p(x_n, x_m) = 0. \tag{2.14}$$

From the definition of d_p we have

$$d_p(x_n, x_m) = 2p(x_n, x_m) - p(x_n, x_n) - p(x_m, x_m).$$

Letting $n, m \to +\infty$ in the above equality and using (2.8) and (2.14), we get

$$\lim_{n,m\to\infty} p(x_n, x_m) = 0.$$

Thus by (2.13), we have

$$\lim_{n\to+\infty} p(x_n, u) = p(u, u) = 0. \tag{2.15}$$

Since $p(x_{2n}, u) \to 0 = p(u, u)$, $\{x_{2n}\}$ is a sequence in A, and A is closed in (X, p), we have $u \in A$. Similarly, we have $u \in B$, that is $u \in A \cap B$. Again, from the definition of p, we have

$$\begin{aligned} p(x_n, Tu) &\le p(x_n, u) + p(u, Tu) - p(u, u) \\ &\le p(x_n, u) + p(u, x_n) + p(x_n, Tu) - p(x_n, x_n) - p(u, u). \end{aligned}$$

Letting $n \to +\infty$ in the above inequalities and using (2.9) and (2.15), we get that

$$\lim_{n\to+\infty} p(x_n, Tu) = p(u, Tu).$$

Now, we claim that $Tu = u$.

Since $x_{2n} \in A$ and $u \in B$, by (2.1) we have

$$\begin{aligned} \psi\big(p(x_{2n+1}, Tu)\big) &= \psi\big(p(Tx_{2n}, Tu)\big) \\ &\le \psi\bigg(\max\bigg\{p(x_{2n}, u), p(Tx_{2n}, x_{2n}), p(Tu, u), \\ &\qquad \frac{1}{2}\big(p(x_{2n}, Tu) + p(u, Tx_{2n})\big)\bigg\}\bigg) \\ &\qquad - \phi\big(\max\big\{p(x_{2n}, u), p(Tu, u)\big\}\big) \\ &= \psi\bigg(\max\bigg\{p(x_{2n}, u), p(x_{2n}, x_{2n+1}), p(Tu, u), \\ &\qquad \frac{1}{2}\big(p(x_{2n}, Tu) + p(u, x_{2n+1})\big)\bigg\}\bigg) \\ &\qquad - \phi\big(\max\big\{p(x_{2n}, u), p(u, Tu)\big\}\big). \end{aligned}$$

Letting $n \to +\infty$, we get that

$$\psi\big(p(u, Tu)\big) \le \psi\big(p(u, Tu)\big) - \phi\big(p(u, Tu)\big).$$

Therefore, $\phi(p(u, Tu)) = 0$. Since ϕ is an altering distance function, $p(u, Tu) = 0$, that is, $u = Tu$.

Therefore, u is a fixed point of T. To prove the uniqueness of the fixed point, we let v be any other fixed point of T in $A \cap B$. It is an easy matter to prove that $p(v,v) = 0$. Now, we prove that $u = v$. Since $u \in A \cap B \subseteq A$ and $v \in A \cap B \subseteq B$, we have

$$\begin{aligned} \psi\big(p(u,v)\big) &= \psi\big(p(Tu,Tv)\big) \\ &\leq \psi\big(\max\{p(u,v),p(u,u),p(v,v)\}\big) - \phi\big(\max\{p(u,v),p(v,v)\}\big) \\ &= \psi\big(p(u,v)\big) - \phi\big(p(u,v)\big). \end{aligned}$$

Thus $\phi(p(u,v)) = 0$ and hence $p(u,v) = 0$. Therefore, $u = v$. □

Taking $\psi = I_{[0,+\infty)}$ (the identity function) in Theorem 5, we have the following result.

Corollary 1 *Let A and B be nonempty closed subsets of a complete partial metric space (X,p). Let $T : X \to X$ be a mapping such that $A \cup B$ has a cyclic representation w.r.t. T. Suppose there exists an altering distance function ϕ such that*

$$\begin{aligned} p(Tx,Ty) \leq{} & \max\left\{p(x,y),p(x,Tx),p(y,Ty),\frac{1}{2}\big(p(x,Ty)+p(Tx,y)\big)\right\} \\ & - \phi\big(\max\{p(x,y),p(y,Ty)\}\big) \end{aligned}$$

for all $x \in A$ and $y \in B$. Then T has a unique fixed point $u \in A \cap B$.

Corollary 2 *Let A and B be nonempty closed subsets of a complete partial metric space (X,p). Let $T : X \to X$ be a mapping such that $A \cup B$ has a cyclic representation w.r.t. T. Suppose there exists an altering distance function ϕ such that*

$$p(Tx,Ty) \leq \max\{p(x,y),p(x,Tx),p(y,Ty)\} - \phi\big(\max\{p(x,y),p(x,Tx),p(y,Ty)\}\big)$$

for all $x \in A$ and $y \in B$. Then T has a unique fixed point $u \in A \cap B$.

Now, we introduce an example to support the usability of our results.

Example 1 Let $X = [0,1]$. Define the partial metric p on X by

$$p(x,y) = \begin{cases} 0, & \text{if } x = y; \\ \max\{x,y\}, & \text{if } x \neq y. \end{cases}$$

Also, define the mapping $T : X \to X$ by $T(x) = \frac{x^2}{1+x}$ and the functions $\psi,\phi : [0,+\infty) \to [0,+\infty)$ by $\psi(t) = 2t$ and $\phi(t) = \frac{t}{1+2t}$. Take $A = [0,\frac{1}{2}]$ and $B = [0,1]$. Then

(1) (X,p) is a complete partial metric space.
(2) $A \cup B$ has a cyclic representation w.r.t. T.
(3) For all $x \in A$ and $y \in B$, we have

$$\begin{aligned} \psi\big(p(Tx,Ty)\big) \leq{} & \psi\left(\max\left\{p(x,y),p(x,Tx),p(y,Ty),\frac{1}{2}\big(p(x,Ty)+p(Tx,y)\big)\right\}\right) \\ & - \phi\big(\max\{p(x,y),p(y,Ty)\}\big). \end{aligned}$$

Proof Note that $TA = [0, \frac{1}{6}] \subseteq B$ and $TB = [0, \frac{1}{2}] \subseteq A$. Thus $A \cup B$ has a cyclic representation of T. To prove (3), given $x \in A$ and $y \in B$, without loss of generality, we may assume that $x \leq y$. So,

$$\begin{aligned}
&\psi\left(p(Tx,Ty)\right) = \psi\left(p\left(\frac{x^2}{1+x}, \frac{y^2}{1+y}\right)\right) = \psi\left(\frac{y^2}{1+y}\right) = \frac{2y^2}{1+y},\\
&\psi\left(\max\left\{p(x,y), p(x,Tx), p(y,Ty), \frac{1}{2}\left(P(x,Ty) + p(Tx,y)\right)\right\}\right)\\
&\quad = \psi\left(\max\left\{y, p\left(x, \frac{x^2}{1+x}\right), p\left(y, \frac{y^2}{1+y}\right), \frac{1}{2}\left(p\left(x, \frac{y^2}{1+y}\right) + p\left(\frac{x^2}{1+x}, y\right)\right)\right\}\right)\\
&\quad \leq \psi(y) = 2y,
\end{aligned}$$

and

$$\phi\left(\max\left\{p(x,y), p(y,Ty)\right\}\right) = \phi\left(\max\left\{y, p\left(y, \frac{y^2}{1+y}\right)\right\}\right) = \phi(y) = \frac{y}{1+2y}.$$

Since

$$\frac{2y^2}{1+y} \leq 2y - \frac{y}{1+2y},$$

we have

$$\begin{aligned}
\psi\left(p(Tx,Ty)\right) \leq{}& \psi\left(\max\left\{p(x,y), p(x,Tx), p(y,Ty), \frac{1}{2}\left(p(x,Ty) + p(Tx,y)\right)\right\}\right)\\
&- \phi\left(\max\left\{p(x,y), p(y,Ty)\right\}\right).
\end{aligned}$$ □

Note that Example 1 satisfies all the hypotheses of Theorem 5.

3 Application

Denote by Λ the set of functions $\mu : [0, +\infty) \to [0, +\infty)$ satisfying the following hypotheses:

(h1) μ is a Lebesgue-integrable mapping on each compact of $[0, +\infty)$.

(h2) For every $\epsilon > 0$, we have

$$\int_0^\epsilon \mu(t)\,dt > 0.$$

Theorem 6 *Let A and B be nonempty closed subsets of a complete partial metric space (X, p). Let $T : X \to X$ be a mapping such that $A \cup B$ has a cyclic representation w.r.t. T. Suppose that for $x \in A$ and $y \in B$, we have*

$$\begin{aligned}
\int_0^{p(Tx,Ty)} \mu_1(t)\,dt \leq{}& \int_0^{\max\{p(x,y),p(x,Tx),p(y,Ty),\frac{1}{2}(p(x,Ty)+p(Tx,y))\}} \mu_1(t)\,dt\\
&- \int_0^{\max\{p(x,y),p(y,Ty)\}} \mu_2(t)\,dt,
\end{aligned}$$

where $\mu_1, \mu_2 \in \Lambda$. Then T has a unique fixed point $u \in A \cap B$.

Proof Follows from Theorem 5 by defining $\psi, \phi : [0, +\infty) \to [0, +\infty)$ *via* $\psi(t) = \int_0^t \mu_1(s)\,ds$ and $\phi(t) = \int_0^t \mu_2(s)\,ds$ and noting that ψ, ϕ are altering distance functions. □

Remark 2 Theorem 2.1 of [23] is a special case of Corollary 2.

Remark 3 Theorem 2.3 of [23] is a special case of Corollary 2.

Remark 4 Theorem 2.4 of [23] is a special case of Corollary 2.

Remark 5 Theorem 1.1 of [22] is a special case of Corollary 2.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the manuscript.

Author details
[1]Department of Mathematics, Hashemite University, P.O. Box 150459, Zarqa, 13115, Jordan. [2]School of Mathematics and Computer Applications, Thapar University, Patiala, Punjab, India.

Acknowledgements
The authors thank the Editor and the referees for their useful comments and suggestions.

Received: 2 July 2012 Accepted: 31 August 2012 Published: 28 September 2012

References
1. Banach, S: Sur les operations dans les ensembles et leur application aux equation sitegrales. Fundam. Math. **3**, 133-181 (1922)
2. Matthews, SG: Partial metric topology. Proc. 8th Summer Conference on General Topology and Applications. Ann. N.Y. Acad. Sci. **728**, 183-197 (1994)
3. Abdeljawad, T, Karapinar, E, Taş, K: Existence and uniqueness of a common fixed point on partial metric spaces. Appl. Math. Lett. **24**, 1900-1904 (2011)
4. Abdeljawad, T, Karapinar, E, Taş, K: A generalized contraction principle with control functions on partial metric spaces. Comput. Math. Appl. **6**, 716-719 (2012)
5. Abdeljawad, T: Fixed points for generalized weakly contractive mappings in partial metric spaces. Math. Comput. Model. **54**, 2923-2927 (2011)
6. Altun, I, Erduran, A: Fixed point theorems for monotone mappings on partial metric spaces. Fixed Point Theory Appl. **2011**, Article ID 508730 (2011)
7. Altun, I, Sola, F, Simsek, H: Generalized contractions on partial metric spaces. Topol. Appl. **157**, 2778-2785 (2010)
8. Aydi, H: Some fixed point results in ordered partial metric spaces. J. Nonlinear Sci. Appl **4**, 1-12 (2011)
9. Aydi, H: Some coupled fixed point results on partial metric spaces. Int. J. Math. Math. Sci. **2011**, Article ID 647091 (2011)
10. Aydi, H: Fixed point theorems for generalized weakly contractive condition in ordered partial metric spaces. J. Nonlinear Anal. Optim. **2**, 33-48 (2011)
11. Aydi, H, Karapinar, E, Shatanawi, W: Coupled fixed point results for (ψ, φ)-weakly contractive condition in ordered partial metric spaces. Comput. Math. Appl. **62**, 4449-4460 (2011)
12. Golubović, Z, Kadelburg, Z, Radenović, S: Coupled coincidence points of mappings in ordered partial metric spaces. Abstr. Appl. Anal. **2012**, Article ID 192581 (2012)
13. Heckmann, R: Approximation of metric spaces by partial metric spaces. Appl. Categ. Struct. **7**, 71-83 (1999)
14. Karapinar, E: Generalizations of Caristi Kirk's theorem on partial metric spaces. Fixed Point Theory Appl. (2011, in press)
15. Karapinar, E, Erhan, IM: Fixed point theorems for operators on partial metric spaces. Appl. Math. Lett. (2011). doi:10.1016/j.aml.2011.05.013
16. Nashine, HK, Kadelburg, Z, Radenović, S: Common fixed point theorems for weakly isotone increasing mappings in ordered partial metric spaces. Math. Comput. Model. (2012, in press)
17. Oltra, S, Valero, O: Banach's fixed point theorem for partial metric spaces. Rend. Ist. Mat. Univ. Trieste **36**, 17-26 (2004)
18. Romaguera, S: A Kirk type characterization of completeness for partial metric spaces. Fixed Point Theory Appl. **2010**, Article ID 493298 (2010)
19. Samet, B, Rajović, M, Lazović, R, Stoiljković, R: Common fixed point results for nonlinear contractions in ordered partial metric spaces. Fixed Point Theory Appl. **2011**, 71 (2011)
20. Shatanawi, W, Nashine, HK: A generalization of Banach's contraction principle for nonlinear contraction in a partial metric space. J. Nonlinear Sci. Appl **5**, 37-43 (2012)
21. Valero, O: On Banach fixed point theorems for partial metric spaces. Appl. Gen. Topol. **6**, 229-240 (2005)

22. Kirk, WA, Srinavasan, PS, Veeramani, P: Fixed points for mapping satisfying cyclical contractive conditions. Fixed Point Theory **4**, 79-89 (2003)
23. Karapinar, E, Erhan, IM: Best proximity point on different type contractions. Appl. Math. Inf. Sci. **5**, 342-353 (2011)
24. Karapinar, E, Erhan, IM, Ulus, AY: Fixed point theorem for cyclic maps on partial metric spaces. Appl. Math. Inf. Sci. **6**, 239-244 (2012)
25. Karapinar, E, Erhan, IM: Cyclic contractions and fixed point theorems. Filomat **26**, 777-782 (2012)
26. Agarwal, RP, Alghamdi, MA, Shahzad, N: Fixed point theory for cyclic generalized contractions in partial metric spaces. Fixed Point Theory Appl. **2012**, 40 (2012)
27. Khan, MS, Swaleh, M, Sessa, S: Fixed point theorems by altering distances between the points. Bull. Aust. Math. Soc. **30**, 1-9 (1984)
28. Cho, YJ, Rhoades, BE, Saadati, R, Samet, B, Shatanawi, W: Nonlinear coupled fixed point theorems in ordered generalized metric spaces with integral type. Fixed Point Theory Appl. **2012**, 8 (2012). doi:10.1186/1687-1812-2012-8
29. Aydi, H, Postolache, M, Shatanawi, W: Coupled fixed point results for (ψ,ϕ)-weakly contractive mappings in ordered G-metric spaces. Comput. Math. Appl. **63**, 298-309 (2012)
30. Lakzian, H, Samet, B: Fixed points for (ψ,ϕ)-weakly contractive mappings in generalized metric spaces. Appl. Math. Lett. **25**, 902-906 (2012)
31. Shatanawi, W, Al-Rawashdeh, A: Common fixed points of almost generalized (ψ,ϕ)-contractive mappings in ordered metric spaces. Fixed Point Theory Appl. (accepted)
32. Shatanawi, W, Mustafa, Z, Tahat, N: Some coincidence point theorems for nonlinear contraction in ordered metric spaces. Fixed Point Theory Appl. **2011**, 68 (2011)
33. Shatanawi, W, Samet, B: On (ψ,ϕ)-weakly contractive condition in partially ordered metric spaces. Comput. Math. Appl. **62**, 3204-3214 (2011)

doi:10.1186/1687-1812-2012-165
Cite this article as: Shatanawi and Manro: **Fixed point results for cyclic (ψ,ϕ,A,B)-contraction in partial metric spaces.** *Fixed Point Theory and Applications* 2012 **2012**:165.