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Fixed point theory for the cyclic weaker Meir-Keeler function in complete metric spaces

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Abstract

In this article, we introduce the notions of cyclic weaker $\phi \circ \phi$-contractions and cyclic weaker (ϕ, ϕ)-contractions in complete metric spaces and prove two theorems which assure the existence and uniqueness of a fixed point for these two types of contractions. Our results generalize or improve many recent fixed point theorems in the literature.

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1 Introduction and preliminaries

Throughout this article, by $\mathbb{R}^+$, $\mathbb{R}$ we denote the sets of all nonnegative real numbers and all real numbers, respectively, while $\mathbb{N}$ is the set of all natural numbers. Let (X, d) be a metric space, D be a subset of X and $f: D \to X$ be a map. We say f is contractive if there exists $\alpha \in [0,1)$ such that for all $x, y \in D$,

$$d(fx, fy) \leq \alpha \cdot d(x, y).$$

The well-known Banach's fixed point theorem asserts that if $D = X$, f is contractive and (X, d) is complete, then f has a unique fixed point in X. It is well known that the Banach contraction principle [1] is a very useful and classical tool in nonlinear analysis. In 1969, Boyd and Wong [2] introduced the notion of Φ-contraction. A mapping $f: X \to X$ on a metric space is called Φ-contraction if there exists an upper semi-continuous function $\Phi: [0, \infty) \to [0, \infty)$ such that

$$d(fx, fy) \leq \Phi(d(x, y)) \quad \text{for all} \quad x, y \in X.$$

Generalization of the above Banach contraction principle has been a heavily investigated branch research. (see, e.g., [3,4]). In 2003, Kirk et al. [5] introduced the following notion of cyclic representation.

Definition 1 [5]*Let X be a nonempty set, $m \in \mathbb{N}$ and $f: X \to X$ an operator. Then $X = \cup_{i=1}^{m} A_i$is called a cyclic representation of X with respect to f if*

(1) A_i, $i = 1, 2,..., m$ are nonempty subsets of X;
(2) $f(A_1) \subset A_2, f(A_2) \subset A_3,..., f(A_{m-1}) \subset A_m, f(A_m) \subset A_1$.

Kirk et al. [5] also proved the below theorem.

Theorem 1 [5]*Let* (X, d) *be a complete metric space,* $m \in \mathbb{N}$, $A_1, A_2, \ldots, A_m$, *closed nonempty subsets of* X *and* $X = \cup_{i=1}^{m} A_i$. *Suppose that f satisfies the following condition.*

$$d(fx, fy) \le \psi(d(x, y)), \quad \text{for all} \quad x \in A_i, \quad y \in A_{i+1}, \quad i \in \{1, 2, \ldots, m\},$$

where ψ: $[0, \infty) \to [0, \infty)$ *is upper semi-continuous from the right and* $0 \le \psi(t) < t$ *for* $t > 0$. *Then, f has a fixed point* $z \in \cap_{i=1}^{n} A_i$.

Recently, the fixed theorems for an operator f: $X \to X$ that defined on a metric space X with a cyclic representation of X with respect to f had appeared in the literature. (see, e.g., [6-10]). In 2010, Păcurar and Rus [7] introduced the following notion of cyclic weaker ϕ-contraction.

Definition 2 [7]*Let* (X, d) *be a metric space,* $m \in \mathbb{N}$, $A_1, A_2, \ldots, A_m$ *closed nonempty subsets of* X *and* $X = \cup_{i=1}^{m} A_i$. *An operator* f: $X \to X$ *is called a cyclic weaker* ϕ*-contraction if*

(1) $X = \cup_{i=1}^{m} A_i$ *is a cyclic representation of* X *with respect to* f;
(2) there exists a continuous, non-decreasing function ϕ: $[0, \infty) \to [0, \infty)$ *with* $\phi(t) > 0$ *for* $t \in (0, \infty)$ *and* $\phi(0) = 0$ *such that*

$$d(fx, fy) \le d(x, y) - \varphi(d(x, y)),$$

for any $x \in A_i$, $y \in A_{i+1}$, $i = 1, 2, \ldots, m$ *where* $A_{m+1} = A_1$.

And, Păcurar and Rus [7] proved the below theorem.

Theorem 2 [7]*Let* (X, d) *be a complete metric space,* $m \in \mathbb{N}$, $A_1, A_2, \ldots, A_m$ *closed nonempty subsets of* X *and* $X = \cup_{i=1}^{m} A_i$. *Suppose that f is a cyclic weaker* ϕ*-contraction. Then, f has a fixed point* $z \in \cap_{i=1}^{n} A_i$.

In this article, we also recall the notion of Meir-Keeler function (see [11]). A function ϕ: $[0, \infty) \to [0, \infty)$ is said to be a Meir-Keeler function if for each $\eta > 0$, there exists $\delta > 0$ such that for $t \in [0, \infty)$ with $\eta \le t < \eta + \delta$, we have $\phi(t) < \eta$. We now introduce the notion of weaker Meir-Keeler function ϕ: $[0, \infty) \to [0, \infty)$, as follows:

Definition 3 *We call* ϕ: $[0, \infty) \to [0, \infty)$ *a weaker Meir-Keeler function if for each* $\eta > 0$, *there exists* $\delta > 0$ *such that for* $t \in [0, \infty)$ *with* $\eta \le t < \eta + \delta$, *there exists* $n_0 \in \mathbb{N}$ *such that* $\phi^{n_0}(t) < \eta$.

2 Fixed point theory for the cyclic weaker $\phi \circ \phi$-contractions

The main purpose of this section is to present a generalization of Theorem 1. In the section, we let ϕ: $[0, \infty) \to [0, \infty)$ be a weaker Meir-Keeler function satisfying the following conditions:

(ϕ_1) $\phi(t) > 0$ for $t > 0$ and $\phi(0) = 0$;
(ϕ_2) for all $t \in (0, \infty)$, $\{\phi^n(t)\}_{n \in \mathbb{N}}$ is decreasing;
(ϕ_3) for $t_n \in [0, \infty)$, we have that
  (a) if $\lim_{n\to\infty} t_n = \gamma > 0$, then $\lim_{n\to\infty} \phi(t_n) < \gamma$, and
  (b) if $\lim_{n\to\infty} t_n = 0$, then $\lim_{n\to\infty} \phi(t_n) = 0$.

And, let ϕ: $[0, \infty) \to [0, \infty)$ be a non-decreasing and continuous function satisfying

(ϕ_1) $\phi(t) > 0$ for $t > 0$ and $\phi(0) = 0$;
(ϕ_2) ϕ is subadditive, that is, for every $\mu_1, \mu_2 \in [0, \infty)$, $\phi(\mu_1 + \mu_2) \leq \phi(\mu_1) + \phi(\mu_2)$;
(ϕ_3) for all $t \in (0, \infty)$, $\lim_{n\to\infty} t_n = 0$ if and only if $\lim_{n\to\infty} \phi(t_n) = 0$.

We state the notion of cyclic weaker $\phi \circ \phi$-contraction, as follows:

Definition 4 *Let (X, d) be a metric space, $m \in \mathbb{N}$, $A_1, A_2,\ldots, A_m$ nonempty subsets of X and $X = \cup_{i=1}^m A_i$. An operator $f: X \to X$ is called a cyclic weaker $\phi \circ \phi$-contraction if*

(i) $X = \cup_{i=1}^m A_i$ is a cyclic representation of X with respect to f;
(ii) for any $x \in A_i$, $y \in A_{i+1}$, $i = 1, 2,\ldots, m$,

$$\varphi(d(fx, fy)) \leq \phi(\varphi(d(x, y))),$$

where $A_{m+1} = A_1$.

Theorem 3 *Let (X, d) be a complete metric space, $m \in \mathbb{N}$, $A_1, A_2, \ldots, A_m$ nonempty subsets of X and $X = \cup_{i=1}^m A_i$. Let $f: X \to X$ be a cyclic weaker $\phi \circ \phi$-contraction. Then, f has a unique fixed point $z \in \cap_{i=1}^m A_i$.*

Proof. Given x_0 and let $x_{n+1} = fx_n = f^{n+1}x_0$, for $n \in \mathbb{N}\cup\{0\}$. If there exists $n_0 \in \mathbb{N} \cup \{0\}$ such that $x_{n_0+1} = x_{n_0}$, then we finished the proof. Suppose that $x_{n+1} \neq x_n$ for any $n \in \mathbb{N} \cup \{0\}$. Notice that, for any $n > 0$, there exists $i_n \in \{1,2,\ldots,m\}$ such that $x_{n-1} \in A_{i_n}$ and $x_n \in A_{i_n+1}$. Since $f: X \to X$ is a cyclic weaker $\phi \circ \phi$-contraction, we have that for all $n \in \mathbb{N}$

$$\begin{aligned}\varphi(d(x_n, x_{n+1})) &= \varphi(d(fx_{n-1}, fx_n)) \\ &\leq \phi(\varphi(d(x_{n-1}, x_n))),\end{aligned}$$

and so

$$\begin{aligned}\varphi(d(x_n, x_{n+1})) &\leq \phi(\varphi(d(x_{n-1}, x_n))) \\ &\leq \phi(\phi(\varphi(d(x_{n-2}, x_{n-1}))) = \phi^2(\varphi((d(x_{n-2}, x_{n-1}))) \\ &\leq \ldots\ldots \\ &\leq \phi^n(\varphi(d(x_0, x_1))).\end{aligned}$$

Since $\{\phi^n(\phi(d(x_0, x1)))\}_{n\in\mathbb{N}}$ is decreasing, it must converge to some $\eta \geq 0$. We claim that $\eta = 0$. On the contrary, assume that $\eta > 0$. Then by the definition of weaker Meir-Keeler function ϕ, there exists $\delta > 0$ such that for $x_0, x_1 \in X$ with $\eta \leq \phi(d(x_0, x_1)) < \delta + \eta$, there exists $n_0 \in \mathbb{N}$ such that $\phi^{n_0}(\varphi(d(x_0, x_1))) < \eta$. Since $\lim_{n\to\infty} \phi^n(\phi(d(x_0, x_1))) = \eta$, there exists $p_0 \in \mathbb{N}$ such that $\eta \leq \phi^p(\phi(d(x_0, x_1)) < \delta + \eta$, for all $p \geq p_0$. Thus, we conclude that $\phi^{p_0+n_0}(\varphi(d(x_0, x_1))) < \eta$. So we get a contradiction. Therefore $\lim_{n\to\infty} \phi^n(\phi(d(x_0, x_1))) = 0$, that is,

$$\lim_{n\to\infty} \varphi(d(x_n, x_{n+1})) = 0.$$

Next, we claim that $\{x_n\}$ is a Cauchy sequence. We claim that the following result holds:

Claim: for each $\varepsilon > 0$, there is $n_0(\varepsilon) \in \mathbb{N}$ such that for all $p, q \geq n_0(\varepsilon)$,

$$\varphi(d(x_p, x_q)) < \varepsilon, \qquad (*)$$

We shall prove (*) by contradiction. Suppose that (*) is false. Then there exists some $\varepsilon > 0$ such that for all $n \in \mathbb{N}$, there are p_n, $q_n \in \mathbb{N}$ with $p_n > q_n \geq n$ satisfying:

(i) $\varphi(d(x_{p_n}, x_{q_n})) \geq \varepsilon$, and
(ii) p_n is the smallest number greater than q_n such that the condition (i) holds.

Since

$$\begin{aligned}\varepsilon &\leq \varphi(d(x_{p_n}, x_{q_n}))\\ &\leq \varphi(d(x_{p_n}, x_{p_n-1}) + d(x_{p_n-1}, x_{q_n}))\\ &\leq \varphi(d(x_{p_n}, x_{p_n-1})) + \varphi(d(x_{p_n-1}, x_{q_n}))\\ &\leq \varphi(d(x_{p_n}, x_{p_n-1})) + \varepsilon,\end{aligned}$$

hence we conclude $\lim_{p\to\infty}\varphi(d(x_{p_n}, x_{q_n})) = \varepsilon$. Since ϕ is subadditive and nondecreasing, we conclude

$$\begin{aligned}\varphi(d(x_{p_n}, x_{q_n})) &\leq \varphi(d(x_{p_n}, x_{q_n+1}) + d(x_{p_n+1}, x_{q_n}))\\ &\leq \varphi(d(x_{p_n}, x_{q_n+1})) + \varphi(d(x_{p_n+1}, x_{q_n})),\end{aligned}$$

and so

$$\begin{aligned}\varphi(d(x_{p_n}, x_{q_n})) - \varphi(d(x_{p_n}, x_{p_n+1})) &\leq \varphi(d(x_{p_n+1}, x_{q_n}))\\ &\leq \varphi(d(x_{p_n}, x_{p_n+1}) + d(x_{p_n}, x_{q_n}))\\ &\leq \varphi(d(x_{p_n}, x_{p_n+1})) + \varphi(d(x_{p_n}, x_{q_n})).\end{aligned}$$

Letting $n \to \infty$, we also have

$$\lim_{n\to\infty} \varphi(d(x_{p_n+1}, x_{q_n})) = \varepsilon.$$

Thus, there exists i, $0 \leq i \leq m - 1$ such that $p_n - q_n + i = 1 \bmod m$ for infinitely many n. If $i = 0$, then we have that for such n,

$$\begin{aligned}\varepsilon &\leq \varphi(d(x_{p_n}, x_{q_n}))\\ &\leq \varphi(d(x_{p_n}, x_{p_n+1}) + d(x_{p_n+1}, x_{q_{n+1}}) + d(x_{q_n+1}, x_{q_n}))\\ &\leq \varphi(d(x_{p_n}, x_{p_n+1})) + \varphi(d(x_{p_n+1}, x_{q_n+1})) + \varphi(d(x_{q_n+1}, x_{q_n}))\\ &= \varphi(d(x_{p_n}, x_{p_n+1})) + \varphi(d(fx_{p_n}, fx_{q_n})) + \varphi(d(x_{q_n+1}, x_{q_n}))\\ &\leq \varphi(d(x_{p_n}, x_{p_n+1})) + \phi(\varphi(d(x_{p_n}, x_{q_n}))) + \varphi(d(x_{q_n+1}, x_{q_n})).\end{aligned}$$

Letting $n \to \infty$. Then by, we have

$$\varepsilon \leq 0 + \lim_{n\to\infty} \phi(\varphi(d(x_{p_n}, x_{q_n}))) + 0 < \varepsilon,$$

a contradiction. Therefore $\lim_{n\to\infty}\varphi(d(x_{p_n}, x_{q_n})) = 0$, by the condition (ϕ_3), we also have $\lim_{n\to\infty}d(x_{p_n}, x_{q_n}) = 0$. The case $i \neq 0$ is similar. Thus, $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists $\nu \in \cup_{i=1}^{m} A_i$ such that $\lim_{n\to\infty} x_n = \nu$. Now for all $i = 0, 1, 2, \ldots, m - 1$, $\{fx_{mn-i}\}$ is a sequence in A_i and also all converge to ν. Since A_i is clsoed for all $i = 1, 2, \ldots, m$, we conclude $\nu \in \cup_{i=1}^{m} A_i$, and also we conclude that $\cap_{i=1}^{m} A_i \neq \phi$. Since

$$\begin{aligned}\varphi(d(\nu, f\nu)) &= \lim_{n\to\infty} \varphi(d(fx_{mn}, f\nu))\\ &\leq \lim_{n\to\infty} \phi(\varphi(d(fx_{mn-1}, \nu))) = 0,\end{aligned}$$

hence $\phi(d(v, fv)) = 0$, that is, $d(v, fv) = 0$, v is a fixed point of f.

Finally, to prove the uniqueness of the fixed point, let μ be another fixed point of f. By the cyclic character of f, we have $\mu, \nu \in \cap_{i=1}^{n} A_i$. Since f is a cyclic weaker $\phi \circ \phi$-contraction, we have

$$\begin{aligned}\varphi(d(\nu,\mu)) = \varphi(d(\nu, f\mu)) &= \lim_{n\to\infty} \varphi(d(fx_{mn}, f\mu)) \\ &\le \lim_{n\to\infty} \phi(\varphi(d(fx_{mn-1}, \mu))) \\ &< \varphi(d(\nu,\mu)),\end{aligned}$$

and this is a contradiction unless $\phi(d(v, \mu)) = 0$, that is, $\mu = v$. Thus v is a unique fixed point of f.

Example 1 *Let $X = \mathbb{R}^3$ and we define d: $X \times X \to [0,\infty)$ by $d(x,y) = |x_1-y_1| + |x_2-y_2| + |x_3-y_3|$, for $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3) \in X$, and let $A = \{(x, 0,0): x \in \mathbb{R}\}$, $B = \{(0,y,0): y \in \mathbb{R}\}$, $C = \{(0,0, z): z \in \mathbb{R}\}$ be three subsets of X. Define f: $A \cup B \cup C \to A \cup B \cup C$ by*

$$f((x,0,0)) = \left(0, \frac{1}{4}x, 0\right); \quad \textit{for all } x \in \mathbb{R};$$
$$f((0,y,0)) = \left(0, 0, \frac{1}{4}y\right); \quad \textit{for all } y \in \mathbb{R};$$
$$f((0,0,z)) = \left(\frac{1}{4}z, 0, 0\right); \quad \textit{for all } z \in \mathbb{R}.$$

We define ϕ: $[0, \infty) \to [0, \infty)$ by

$$\phi(t) = \frac{1}{3}t \textit{ for } t \in [0,\infty),$$

and ϕ: $[0, \infty) \to [0, \infty)$ by

$$\varphi(t) = \frac{1}{2}t \textit{ for } t \in [0,\infty).$$

Then f is a cyclic weaker $\phi \circ \phi$-contraction and $(0, 0, 0)$ is the unique fixed point.

3 Fixed point theory for the cyclic weaker (ϕ, ϕ-contractions

The main purpose of this section is to present a generalization of Theorem 2. In the section, we let ϕ: $[0, \infty) \to [0, \infty)$ be a weaker Meir-Keeler function satisfying the following conditions:

(ϕ_1) $\phi(t) > 0$ for $t > 0$ and $\phi(0) = 0$;
(ϕ_2) for all $t \in (0, \infty)$, $\{\phi^n(t)\}_{n\in\mathbb{N}}$ is decreasing;
(ϕ_3) for $t_n \in [0, \infty)$, if $\lim_{n\to\infty} t_n = \gamma$, then $\lim_{n\to\infty} \phi(t_n) \le \gamma$.

And, let ϕ: $[0, \infty) \to [0, \infty)$ be a non-decreasing and continuous function satisfying $\phi(t) > 0$ for $t > 0$ and $\phi(0) = 0$.

We now state the notion of cyclic weaker (ϕ, ϕ)-contraction, as follows:

Definition 5 *Let (X, d) be a metric space, $m \in \mathbb{N}$, $A_1, A_2,..., A_m$ nonempty subsets of X and $X = \cup_{i=1}^{m} A_i$. An operator f: $X \to X$ is called a cyclic weaker (ϕ,ϕ)-contraction if*

(i) $X = \cup_{i=1}^{m} A_i$ is a cyclic representation of X with respect to f;

(ii) for any $x \in A_i$, $y \in A_{i+1}$, $i = 1, 2,..., m$,

$$d(fx, fy) \leq \phi(d(x, y)) - \varphi(d(x, y)),$$

where $A_{m+1} = A_1$.

Theorem 4 *Let (X, d) be a complete metric space, $m \in \mathbb{N}$, $A_1, A_2,..., A_m$ nonempty subsets of X and $X = \cup_{i=1}^{m} A_i$. Let f: $X \to X$ be a cyclic weaker (ϕ, φ)-contraction. Then f has a unique fixed point $z \in \cap_{i=1}^{m} A_i$.*

Proof. Given x_0 and let $x_{n+1} = fx_n = f^{n+1} x_0$, for $n \in \mathbb{N} \cup \{0\}$. If there exists $n \in \mathbb{N} \cup \{0\}$ such that $x_{n_0+1} = x_{n_0}$, then we finished the proof. Suppose that $x_{n+1} \neq x_n$ for any $n \in \mathbb{N} \cup \{0\}$. Notice that, for any $n > 0$, there exists $i_n \in \{1,2,...,m\}$ such that $x_{n-1} \in A_{i_n}$ and $x_n \in A_{i_n+1}$. Since f: $X \to X$ is a cyclic weaker (ϕ, φ)-contraction, we have that $n \in \mathbb{N}$

$$\begin{aligned} d(x_n, x_{n+1}) &= d(fx_{n-1}, fx_n) \\ &\leq \phi(d(x_{n-1}, x_n)) - \varphi(d(x_{n-1}, x_n)) \\ &\leq \phi(d(x_{n-1}, x_n)), \end{aligned}$$

and so

$$\begin{aligned} d(x_n, x_{n+1}) &\leq \phi(d(x_{n-1}, x_n)) \\ &\leq \phi(\phi(d(x_{n-2}, x_{n-1})) = \phi^2(d(x_{n-2}, x_{n-1})) \\ &\leq \cdots\cdots \\ &\leq \phi^n(d(x_0, x_1)). \end{aligned}$$

Since $\{\phi^n (d(x_0, x_1))\}_{n\in \mathbb{N}}$ is decreasing, it must converge to some $\eta \geq 0$. We claim that $\eta = 0$. On the contrary, assume that $\eta > 0$. Then by the definition of weaker Meir-Keeler function ϕ, there exists $\delta > 0$ such that for $x_0, x_1 \in X$ with $\eta \leq d(x_0, x_1) < \delta + \eta$, there exists $n_0 \in \mathbb{N}$ such that $\phi^{n_0}(d(x_0, x_1)) < \eta$. Since $\lim_{n\to\infty}$, $\phi^n (d(x_0, x_1)) = \eta$, there exists $p_0 \in \mathbb{N}$ such that $\eta \leq \phi^p (d(x_0, x_1)) < \delta + \eta$, for all $p \geq p_0$. Thus, we conclude that $\phi^{p_0+n_0}(d(x_0, x_1)) < \eta$. So we get a contradiction. Therefore $\lim_{n\to\infty} \phi^n(d(x_0, x_1)) = 0$, that is,

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0.$$

Next, we claim that $\{x_n\}$ is a Cauchy sequence. We claim that the following result holds:

Claim: For every $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that if $p, q \geq n$ with $p\text{-}q = 1 \bmod m$, then $d(x_p, x_q) < \varepsilon$.

Suppose the above statement is false. Then there exists $\epsilon > 0$ such that for any $n \in \mathbb{N}$, there are $p_n, q_n \in \mathbb{N}$ with $p_n > q_n \geq n$ with $p_n - q_n = 1 \bmod m$ satisfying

$$d(x_{q_n}, x_{p_n}) \geq \varepsilon.$$

Now, we let $n > 2m$. Then corresponding to $q_n \geq n$ use, we can choose p_n in such a way, that it is the smallest integer with $p_n > q_n \geq n$ satisfying $p_n - q_n = 1 \bmod m$ and $d(x_{q_n}, x_{p_n}) \geq \varepsilon$. Therefore $d(x_{q_n}, x_{p_n-m}) \leq \varepsilon$ and

$$\begin{aligned} \varepsilon &\leq d(x_{q_n}, x_{p_n}) \\ &\leq d(x_{q_n}, x_{p_n-m}) + \sum_{i=1}^{m} d(x_{p_{n-i}}, x_{p_{n-i+1}}) \\ &< \varepsilon + \sum_{i=1}^{m} d(x_{p_{n-i}}, x_{p_{n-i+1}}). \end{aligned}$$

Letting $n \to \infty$, we obtain that

$$\lim_{n\to\infty} d(x_{q_n}, x_{p_n}) = \varepsilon.$$

On the other hand, we can conclude that

$$\begin{aligned}\varepsilon &\le d(x_{q_n}, x_{p_n})\\ &\le d(x_{q_n}, x_{q_{n+1}}) + d(x_{q_{n+1}}, x_{p_{n+1}}) + d(x_{x_{p_{n+1}}}, {}_{p_n})\\ &\le d(x_{q_n}, x_{q_{n+1}}) + d(x_{q_{n+1}}, x_{q_n}) + d(x_{q_n}, x_{p_n}) + d(x_{p_n}, x_{p_{n+1}}) + d(x_{x_{p_{n+1}}}, {}_{pn}).\end{aligned}$$

Letting $n \to \infty$, we obtain that

$$\lim_{n\to\infty} d(x_{q_{n+1}}, x_{p_{n+1}}) = \varepsilon.$$

Since x_{q_n} and x_{p_n} lie in different adjacently labeled sets A_i and A_{i+1} for certain $1 \le i \le m$, by using the fact that f is a cyclic weaker (ϕ, ϕ)-contraction, we have

$$d(x_{q_{n+1}}, x_{p_{n+1}}) = d(fx_{q_n}, fx_{p_n}) \le \phi(d(x_{q_n}, x_{p_n})) - \varphi(d(x_{q_n}, x_{p_n})).$$

Letting $n \to \infty$, by using the condition ϕ_3 of the function ϕ, we obtain that

$$\varepsilon \le \varepsilon - \varphi(\varepsilon),$$

and consequently, $\phi(\epsilon) = 0$. By the definition of the function ϕ, we get $\epsilon = 0$ which is contraction. Therefore, our claim is proved.

In the sequel, we shall show that $\{x_n\}$ is a Cauchy sequence. Let $\varepsilon > 0$ be given. By our claim, there exists $n_1 \in \mathbb{N}$ such that if $p, q \ge n_1$ with $p - q = 1 \bmod m$, then

$$d(x_p, x_q) \le \frac{\varepsilon}{2}.$$

Since $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$, there exists $n_2 \in \mathbb{N}$such that

$$d(x_n, x_{n+1}) \le \frac{\varepsilon}{2m},$$

for any $n \ge n_2$.

Let $p, q \ge \max\{n_1, n_2\}$ and $p > q$. Then there exists $k \in \{1, 2, ..., m\}$ such that $p - q = k \bmod m$. Therefore, $p - q + j = 1 \bmod m$ for $j = m - k + 1$, and so we have

$$\begin{aligned}d(x_q, x_p) &\le d(x_q, x_{p+j}) + d(x_{p+j}, x_{p+j-1}) + \cdots + d(x_{p+1}, x_p)\\ &\le \frac{\varepsilon}{2} + j \times \frac{\varepsilon}{2m}\\ &\le \frac{\varepsilon}{2} + m \times \frac{\varepsilon}{2m} = \varepsilon.\end{aligned}$$

Thus, $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists $\nu \in \cup_{i=1}^m A_i$ such that $\lim_{n\to\infty} x_n = \nu$. Since $X = \cup_{i=1}^m A_i$ is a cyclic representation of X with respect to f, the sequence $\{x_n\}$ has infinite terms in each A_i for $i \in \{1,2,...,m\}$. Now for all $i = 1,2,..., m$, we may take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ with $x_{n_k} \in A_{i-1}$ and also all converge to ν. Since

$$\begin{aligned}d(x_{n_{k+1}}, f\nu) &= d(fx_{n_k}, f\nu)\\ &\le \phi(d(x_{n_k}, \nu)) - \varphi(d(x_{n_k}, \nu))\\ &\le \phi(d(x_{n_k}, \nu)).\end{aligned}$$

Letting $k \to \infty$, we have

$$d(\nu, f\nu) \leq 0,$$

and so $\nu = f\nu$.

Finally, to prove the uniqueness of the fixed point, let μ be the another fixed point of f. By the cyclic character of f, we have $\mu, \nu \in \cap_{i=1}^{n} A_i$. Since f is a cyclic weaker (ϕ, φ)-contraction, we have

$$\begin{aligned} d(\nu, \mu) &= d(\nu, f\mu) \\ &= \lim_{n\to\infty} d(x_{n_{k+1}}, f\mu) \\ &= \lim_{n\to\infty} d(fx_{n_k}, f\mu) \\ &\leq \lim_{n\to\infty} [\phi(d(x_{n_k}, \mu)) - \varphi(d(x_{n_k}, \mu))] \\ &\leq d(\nu, \mu) - \varphi(d(\nu, \mu)), \end{aligned}$$

and we can conclude that

$$\varphi(d(\nu, \mu)) = 0.$$

So we have $\mu = \nu$. We complete the proof.

Example 2 *Let* $X = [-1,1]$ *with the usual metric. Suppose that* $A_1 = [-1,0] = A_3$ *and* $A_2 = [0,1] = A_4$. *Define* f: $X \to X$ *by* $f(x) = \frac{-x}{6}$ *for all* $x \in X$, *and let* ϕ, φ: $[0,\infty) \to [0,\infty)$ *be* $\phi(t) = \frac{1}{2}, \varphi(t) = \frac{t}{4}$. *Then* f *is a cyclic weaker* (ϕ, φ)-*contraction and* 0 *is the unique fixed point.*

Example 3 *Let* $X = \mathbb{R}^+$ *with the metric* $d: X \times X \to \mathbb{R}^+$ *given by*

$$d(x, y) = \max\{x, y\}, \quad \text{for} \quad x, y \in X.$$

Let $A_1 = A_2 = \ldots = A_m = \mathbb{R}^+$. *Define* f: $X \to X$ *by*

$$f(x) = \frac{x^2}{77} \quad \text{for} \quad x \in X,$$

and let ϕ, φ: $[0, \infty) \to [0,\infty)$ *be* $\varphi(t) = \frac{t^3}{2(t+2)}$ *and*

$$\phi(t) = \begin{cases} \dfrac{2t^3}{3t+8}, & \text{if } t \geq 1; \\ \dfrac{t^2}{2}, & \text{if } t < 1. \end{cases}$$

Then f *is a cyclic weaker* (ϕ, φ)-*contraction and* 0 *is the unique fixed point.*

Example 4 *Let* $X = \mathbb{R}^3$ *and we define* d: $X \times X \to [0, \infty)$ *by*

$$d(x, y) = \max\left\{|x_1 - y_1|, |x_2 - y_2|, |x_3 - y_3|,\right\}$$

for $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3) \in X$, *and let* $A = \{(x,0,0): x \in [0,1]\}$, $B = \{(0,y,0): y \in [0,1]\}$, $C = \{(0,0,z): z \in [0,1]\}$ *be three subsets of* X.

Define f: $A \cup B \cup C \to A \cup B \cup C$ *by*

$$f((x,0,0)) = \left(0, \frac{1}{8}x^2, 0\right); \quad \text{for all } x \in [0,1];$$

$$f((0,y,0)) = \left(0, 0, \frac{1}{8}y^2\right); \quad \text{for all } y \in [0,1];$$

$$f((0,0,z)) = \left(\frac{1}{8}z^2, 0, 0\right); \quad \text{for all } z \in [0,1].$$

We define ϕ: $[0, \infty) \to [0,\infty)$ *by*

$$\phi(t) = \frac{t^2}{t+1} \text{ for } t \in [0,\infty),$$

and ϕ: $[0, \infty) \to [0,\infty)$ *by*

$$\varphi(t) = \frac{t^2}{t+2} \text{ for } t \in [0,\infty).$$

Then f is a cyclic weaker (ϕ, ϕ)*-contraction and* (0,0,0) *is the unique fixed point.*

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Competing interests
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