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Common fixed-point results in uniformly convex Banach spaces

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Abstract
We introduce a condition on mappings, namely condition (K). In a uniformly convex Banach space, the condition is weaker than quasi-nonexpansiveness and weaker than asymptotic nonexpansiveness. We also present the existence theorem of common fixed points for a commuting pair consisting of a mapping satisfying condition (K) and a multivalued mapping satisfying conditions (E) and (C_λ) for some $\lambda \in (0, 1)$.

Keywords: common fixed point; quasi-nonexpansive mapping; generalized nonexpansive mapping; uniformly convex Banach space

1 Introduction

In 1978, Itoh and Takahashi [1] established the existence of common fixed points of a quasi-nonexpansive mapping and a multivalued nonexpansive mapping by an elementary constructive method in a Hilbert space. In 2006, Dhompongsa *et al.* [2] obtained a common fixed point result for a commuting pair of single-valued and multivalued nonexpansive mappings in uniformly convex Banach spaces. The analogy result in CAT(0) spaces was also proved by Dhompongsa *et al.* [3]. Since then, Shahzad and Markin [4] studied an invariant approximation problem and provided sufficient conditions for the existence of $z \in E \subseteq X$ such that $d(z, y) = \operatorname{dist}(y, E)$ and $z = t(z) \in T(z)$, where $y \in X$, t and T are commuting nonexpansive mappings on E. In 2009, Shahzad [5] also obtained a common fixed point and invariant approximation result in a CAT(0) space in which t and T are weakly commuting.

Motivated by Suzuki's result [6], Garcia-Falset *et al.* [7] introduced two kinds of generalizations for condition (C), namely conditions (E) and (C_λ) and studied both the existence of fixed points and their asymptotic behavior. Recently, Abkar and Eslamian [8] proved that if E is a nonempty closed convex and bounded subset of a complete CAT(0) space X, $t : E \to E$ is a single-valued quasi-nonexpansive mapping, and $T : E \to KC(E)$ is a multivalued mapping satisfying conditions (E) and (C_λ) for some $\lambda \in (0, 1)$ such that t and T are weakly commuting, then there exists a point $z \in E$ such that $z = t(z) \in T(z)$. This result was extended to the general setting of uniformly convex metric spaces by Laowang and Panyanak [9].

In this paper, we first introduce the following condition.

Definition 1.1 Let t be a mapping on a subset E of a Banach space X. Then t is said to satisfy condition (K) if

1. the fixed point set $\operatorname{Fix}(t)$ is nonempty closed and convex, and

2. for every $x \in \operatorname{Fix}(t)$, a closed convex subset A with $t(A) \subseteq A$, and $y \in A$ such that $\|x - y\| = \operatorname{dist}(x, A)$, we have $y \in \operatorname{Fix}(t)$.

We show that, in a uniformly convex Banach space, condition (K) is weaker than quasi-nonexpansiveness and weaker than asymptotic nonexpansiveness. We also present the existence theorem of common fixed points for a commuting pair consisting of a mapping satisfying condition (K) and a multivalued mapping satisfying conditions (E) and (C_λ) for some $\lambda \in (0, 1)$. Consequently, such a theorem extends many results in the literature.

2 Preliminaries

In this section, we give some preliminaries.

A mapping t on a subset E of a Banach space X is called an asymptotically nonexpansive mapping if for each $n \in \mathbb{N}$, there exists a positive constant $k_n \geq 1$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\left\|t^n(x) - t^n(y)\right\| \leq k_n \|x - y\| \quad \text{for all } x, y \in E.$$

If $k_n \equiv 1$ for all $n \in \mathbb{N}$, then t is called a nonexpansive mapping. We denote by $\operatorname{Fix}(t)$ the set of fixed points of t, *i.e.*, $\operatorname{Fix}(t) = \{x \in E : x = t(x)\}$.

We shall denote by $\operatorname{FB}(E)$ the family of nonempty bounded closed subsets of E and by $\operatorname{KC}(E)$ the family of nonempty compact convex subsets of E. Let $H(\cdot,\cdot)$ be the Hausdorff distance on $\operatorname{FB}(X)$, *i.e.*,

$$H(A, B) = \max\left\{\sup_{a\in A} \operatorname{dist}(a, B), \sup_{b\in B} \operatorname{dist}(b, A)\right\}, \quad A, B \in \operatorname{FB}(X),$$

where $\operatorname{dist}(a, B) = \inf\{\|a - b\| : b \in B\}$ is the distance from the point a to the subset B.

A multivalued mapping $T : E \to \operatorname{FB}(X)$ is said to be *nonexpansive* if

$$H\big(T(x), T(y)\big) \leq \|x - y\| \quad \text{for all } x, y \in E.$$

Definition 2.1 A multivalued mapping $T : X \to \operatorname{FB}(X)$ is said to satisfy condition (E_μ) provided that

$$\operatorname{dist}\big(x, T(y)\big) \leq \mu \operatorname{dist}\big(x, T(x)\big) + \|x - y\|, \quad \forall x, y \in X.$$

We say that T satisfies condition (E) whenever T satisfies (E_μ) for some $\mu \geq 1$.

Definition 2.2 A multivalued mapping $T : X \to \operatorname{FB}(X)$ is said to satisfy condition (C_λ) for some $\lambda \in (0, 1)$ provided that

$$\lambda \operatorname{dist}\big(x, T(x)\big) \leq \|x - y\| \quad \Rightarrow \quad H\big(T(x), T(y)\big) \leq \|x - y\|, \quad \forall x, y \in X.$$

A point x is called a fixed point for a multivalued mapping T if $x \in T(x)$. A single valued mapping $t : E \to E$ and a multivalued mapping $T : E \to \operatorname{FB}(E)$ are said to be commute if $t(T(x)) \subseteq T(t(x))$ for all $x \in E$.

A Banach space X is said to be strictly convex if

$$\|x+y\| < 2$$

for all $x, y \in X$ with $\|x\| = \|y\| = 1$ and $x \neq y$. We recall that a Banach space X is called uniformly convex (Clarkson [10]) if, for each $\varepsilon > 0$, there is a $\delta > 0$ such that if $\|x\| = \|y\| = 1$ then

$$\left\| \frac{(x+y)}{2} \right\| \leq 1 - \delta.$$

It is obvious that uniform convexity implies strict convexity.

In 1991, Xu [11] proved the characterization of uniform convexity as follows.

Theorem 2.3 [11] *A Banach space X is uniformly convex if and only if for each fixed number $r > 0$, there exists a continuous function $\varphi : [0,\infty) \to [0,\infty)$, $\varphi(s) = 0 \Leftrightarrow s = 0$, such that*

$$\left\|\lambda x + (1-\lambda)y\right\|^2 \leq \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\varphi\big(\|x-y\|\big)$$

for all $\lambda \in [0,1]$ and all $x, y \in X$ such that $\|x\| \leq r$ and $\|y\| \leq r$.

Let E be a nonempty closed and convex subset of a Banach space X and $\{x_n\}$ be a bounded sequence in X. For $x \in X$, define the asymptotic radius of $\{x_n\}$ at x as the number

$$r\big(x, \{x_n\}\big) = \limsup_{n\to\infty} \|x_n - x\|.$$

Let

$$r \equiv r\big(E, \{x_n\}\big) := \inf\big\{r\big(x, \{x_n\}\big) : x \in E\big\}$$

and

$$A \equiv A\big(E, \{x_n\}\big) := \big\{x \in E : r\big(x, \{x_n\}\big) = r\big\}.$$

The number r and the set A are, respectively, called the asymptotic radius and asymptotic center of $\{x_n\}$ relative to E. It is known that $A(E, \{x_n\})$ is as nonempty, weakly compact and convex as E is [12]. The sequence $\{x_n\}$ is called regular relative to E if $r(E, \{x_n\}) = r(E, \{x_{n_k}\})$ for each subsequence $\{x_{n_k}\}$ of $\{x_n\}$.

Goebel [13] and Lim [14] proved the following lemma.

Lemma 2.4 *Let $\{x_n\}$ be a bounded sequence in X, and let E be a nonempty closed convex subset of X. Then $\{x_n\}$ has a subsequence which is regular relative to E.*

The following result was proved by Goebel and Kirk [15].

Lemma 2.5 *Let $\{z_n\}$ and $\{w_n\}$ be bounded sequences in a Banach space X, and let $0 < \lambda < 1$. If, for every natural number n, we have $z_{n+1} = \lambda w_n + (1-\lambda)z_n$ and $\|w_{n+1} - w_n\| \leq \|z_{n+1} - z_n\|$, then $\lim_{n\to\infty} \|w_n - z_n\| = 0$.*

3 Main results

We recall that a mapping t on a subset E of a Banach space X is called quasi-nonexpansive [16] if $\|x - t(y)\| \leq \|x - y\|$ for all $y \in E$ and $x \in \mathrm{Fix}(t)$. From the definition, we can see that nonexpansive mappings with a fixed point are quasi-nonexpansive.

We first show that a quasi-nonexpansive mapping defined on a strictly convex Banach space X satisfies condition (K).

Proposition 3.1 *Let E be a strictly convex subset of a Banach space. If $t : E \to E$ is a quasi-nonexpansive mapping, then t satisfies condition (K).*

Proof It is known that $\mathrm{Fix}(t)$ is nonempty closed and convex [17, Theorem 1]. Let $x \in \mathrm{Fix}(t)$ and A be a closed convex subset with $t(A) \subseteq A$. Let $y \in A$ be such that $\|x - y\| = \mathrm{dist}(x, A)$. Quasi-nonexpansiveness of t implies that

$$\left\|x - t(y)\right\| \leq \|x - y\|.$$

Since E is strictly convex and $t(A) \subseteq A$, it must be the case that $t(y) = y$. Therefore, t satisfies condition (K). □

An asymptotically nonexpansive mapping defined on a uniformly convex Banach space also satisfies condition (K).

Proposition 3.2 *Let X be a uniformly convex Banach space and E be a nonempty subset of X. If $t : E \to E$ is an asymptotically nonexpansive mapping with $\mathrm{Fix}(t) \neq \emptyset$, then t satisfies condition (K).*

Proof The set $\mathrm{Fix}(t)$ is closed and convex [18, Theorem 2]. Let $x \in \mathrm{Fix}(t)$, A be a closed convex subset of E with $t(A) \subseteq A$, and $y \in A$ be such that $\|x - y\| = \mathrm{dist}(x, A)$. By Theorem 2.3, there exists a continuous function ψ such that for all integers $l, m \geq 1$,

$$\begin{aligned} \left\|x - \left(\frac{t^l(y) + t^m(y)}{2}\right)\right\|^2 &\leq \frac{1}{2}\left\|x - t^l(y)\right\|^2 + \frac{1}{2}\left\|x - t^m(y)\right\|^2 - \frac{1}{4}\varphi\left(\left\|t^l(y) - t^m(y)\right\|\right) \\ &\leq \frac{1}{2}k_l^2\|x - y\|^2 + \frac{1}{2}k_m^2\|x - y\|^2 - \frac{1}{4}\varphi\left(\left\|t^l(y) - t^m(y)\right\|\right). \end{aligned} \tag{3.1}$$

Since $\|x - y\| = \mathrm{dist}(x, A)$ and A is convex, we have

$$\|x - y\|^2 \leq \left\|x - \left(\frac{t^l(y) + t^m(y)}{2}\right)\right\|^2.$$

Thus,

$$\varphi\left(\left\|t^l(y) - t^m(y)\right\|\right) \leq 4\left(\left(k_l^2 + k_m^2\right)/2 - 1\right)\|x - y\|^2.$$

Since t is asymptotically nonexpansive, the right-hand side of the inequality tends to zero as l, m tend to infinity. Hence, $\{t^i(y)\}$ is a Cauchy sequence. Let $\lim_{i\to\infty} t^i(y) = z \in A$. We have

$$\left\|t(z) - t^{i+1}(y)\right\| \leq k_1\left\|z - t^i(y)\right\|.$$

By letting $i \to \infty$, we can conclude that $\|t(z) - z\| \leq 0$, that is $z \in \mathrm{Fix}(t)$. Now, letting $l, m \to \infty$ in (3.1) yields

$$\|x - z\|^2 \leq \|x - y\|^2.$$

Since $z \in A$ and $\|x - y\| = \mathrm{dist}(x, A)$, thus $y = z$. Therefore, $y \in \mathrm{Fix}(t)$. □

The following example shows that the class of mappings satisfying condition (K) is strictly wider than the class of quasi-nonexpansive mappings and asymptotically nonexpansive mappings.

Example 3.3 Let f be a function on $[0, 1]$ defined by

$$f(x) = \begin{cases} x, & x \leq \frac{1}{2}; \\ 0, & x > \frac{1}{2}. \end{cases}$$

Then f is neither quasi-nonexpansive nor asymptotically nonexpansive. However, f satisfies condition (K).

We are now in a position to state our main theorem.

Theorem 3.4 *Let E be a nonempty bounded closed convex subset of a uniformly convex Banach space X. Let $t : E \to E$ be a mapping satisfying condition (K), and let $T : E \to \mathrm{KC}(E)$ be a multivalued mapping satisfying conditions (E) and (C_λ) for some $\lambda \in (0, 1)$. If t and T are commute, then they have a common fixed point, that is, there exists a point $z \in E$ such that $z = t(z) \in T(z)$.*

Proof Commutative property of t and T implies that $t(T(x)) \subseteq T(x)$ for all $x \in \mathrm{Fix}(t)$. Then we have $\mathrm{Fix}(t) \cap T(x) \neq \emptyset$ for all $x \in \mathrm{Fix}(t)$ since t satisfies condition (K).

Now, we find an approximate fixed point sequence in $\mathrm{Fix}(t)$ for T. Take $x_0 \in \mathrm{Fix}(t)$. Since $\mathrm{Fix}(t) \cap T(x_0) \neq \emptyset$, we choose $y_0 \in \mathrm{Fix}(t) \cap T(x_0)$. Define

$$x_1 = (1 - \lambda)x_0 + \lambda y_0.$$

Since $\mathrm{Fix}(t)$ is convex, we have $x_1 \in \mathrm{Fix}(t)$. Let $y_1 \in T(x_1)$ such that $\|y_0 - y_1\| = \mathrm{dist}(y_0, T(x_1))$. We get $y_1 \in \mathrm{Fix}(t)$ since t satisfies condition (K). Put

$$x_2 = (1 - \lambda)x_1 + \lambda y_1.$$

Again, choose $y_2 \in T(x_2)$ such that $\|y_1 - y_2\| = \mathrm{dist}(y_1, T(x_2))$. Similarly, we get $y_2 \in \mathrm{Fix}(t)$. We have a sequence $\{x_n\} \subseteq \mathrm{Fix}(t)$ such that

$$x_{n+1} = (1 - \lambda)x_n + \lambda y_n,$$

where $y_n \in T(x_n) \cap \mathrm{Fix}(t)$ and $\|y_{n-1} - y_n\| = \mathrm{dist}(y_{n-1}, T(x_n))$. For every natural number $n \geq 1$, we have

$$\|x_{n+1} - x_n\| = \lambda \|x_n - y_n\|.$$

It follows that

$$\lambda \operatorname{dist}\bigl(x_n, T(x_n)\bigr) \le \lambda \|x_n - y_n\| = \|x_{n+1} - x_n\|.$$

Since T satisfies condition (C_λ), we have

$$H\bigl(T(x_n), T(x_{n+1})\bigr) \le \|x_n - x_{n+1}\|.$$

Hence, for each $n \ge 1$, we have

$$\|y_n - y_{n+1}\| = \operatorname{dist}\bigl(y_n, T(x_{n+1})\bigr) \le H\bigl(T(x_n), T(x_{n+1})\bigr) \le \|x_n - x_{n+1}\|.$$

We now apply Lemma 2.5 to conclude that $\lim_{n\to\infty} \|x_n - y_n\| = 0$. That is, as $n \to \infty$,

$$\operatorname{dist}\bigl(x_n, T(x_n)\bigr) \le \|x_n - y_n\| \to 0.$$

Passing through a subsequence, if necessary, we can assume that $\{x_n\}$ is regular. Let $A(\operatorname{Fix}(t), \{x_n\}) = \{z\}$. For each $n \ge 1$, we choose $z_n \in T(z)$ such that $\|x_n - z_n\| = \operatorname{dist}(x_n, T(z))$. Since t satisfies condition (K), we have $z_n \in \operatorname{Fix}(t)$. The compactness of $T(z)$ implies that the sequence $\{z_n\}$ has a convergent subsequence $\{z_{n_k}\}$ with the limit point $w \in T(z)$. We also obtain $w \in \operatorname{Fix}(t)$ since $\operatorname{Fix}(t)$ is closed. By the condition (E) of T, we have for some $\mu \ge 1$,

$$\operatorname{dist}\bigl(x_{n_k}, T(z)\bigr) \le \mu \operatorname{dist}\bigl(x_{n_k}, T(x_{n_k})\bigr) + \|x_{n_k} - z\|.$$

Note that

$$\|x_{n_k} - w\| \le \|x_{n_k} - z_{n_k}\| + \|z_{n_k} - w\| \le \mu \operatorname{dist}\bigl(x_{n_k}, T(x_{n_k})\bigr) + \|x_{n_k} - z\| + \|z_{n_k} - w\|.$$

These entail

$$\limsup_{k\to\infty} \|x_{n_k} - w\| \le \limsup_{k\to\infty} \|x_{n_k} - z\|.$$

Since $\{x_n\}$ is regular, and an asymptotic center of a bounded sequence in a uniformly convex Banach space is a singleton set, these show that $z = w \in T(z)$. Hence, $z = t(z) \in T(z)$. □

As a consequence of Proposition 3.1, Proposition 3.2, and Theorem 3.4, we obtain the following corollaries.

Corollary 3.5 *Let E be a nonempty bounded closed convex subset of a uniformly convex Banach space X. Let $t : E \to E$ be a quasi-nonexpansive mapping, and let $T : E \to \mathrm{KC}(E)$ be a multivalued mapping satisfying conditions (E) and (C_λ) for some $\lambda \in (0,1)$. If t and T are commute, then they have a common fixed point, that is, there exists a point $z \in E$ such that $z = t(z) \in T(z)$.*

Corollary 3.6 *Let E be a nonempty bounded closed convex subset of a uniformly convex Banach space X. Let $t : E \to E$ be an asymptotically nonexpansive mapping, and let $T : E \to \mathrm{KC}(E)$ be a multivalued mapping satisfying conditions (E) and (C_λ) for some $\lambda \in (0,1)$. If t and T are commute, then they have a common fixed point, that is, there exists a point $z \in E$ such that $z = t(z) \in T(z)$.*

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