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Coupled fixed and coincidence points for monotone operators in partial metric spaces

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Abstract

In this paper, we prove some coupled fixed point results for (ϕ,φ)-weakly contractive mappings in ordered partial metric spaces. As an application, we establish coupled coincidence results without any type of commutativity of the concerned maps. Consequently, the results of Luong and Thuan (Nonlinear Anal. 74:983-992, 2011), Alotaibi and Alsulami (Fixed Point Theory Appl. 2011:44, 2011) and many others are extended to the class of ordered partial metric spaces.

Keywords: coupled fixed point; partial metric space; comparison functions; coupled coincidence point

1 Introduction

The Banach contraction principle is the most celebrated fixed point theorem. Afterward many authors obtained various important extensions of this principle (see [1]). The concept of partial metric spaces was introduced by Matthews [2] in 1994. A partial metric space is a generalized metric space in which each object does not necessarily have to have a zero distance from itself. A motivation behind introducing the concept of a partial metric was to obtain appropriate mathematical models in the theory of computation and, in particular, to give a modified version of the Banach contraction principle [3, 4]. Subsequently, several authors studied the problem of existence and uniqueness of a fixed point for mappings satisfying different contractive conditions on partial metric spaces (*e.g.*, [5–7]).

Recently, Bhaskar and Lakshmikantham [8] presented coupled fixed point theorems for contractions in partially ordered metric spaces. Luong and Thuan [9] presented nice generalizations of these results. Alotaibi and Alsulami [10] further extended the work of Luong and Thuan to coupled coincidences. For more related work on coupled coincidences we refer the readers to recent work in [11–16]. Our main aim in this paper is to extend Luong and Thuan [9] results to ordered partial metric spaces. We shall also establish coupled coincidence results and show that main results in [10] hold in ordered partial metric spaces without the compatibility of maps.

2 Basic concepts

We start by recalling some definitions and properties of partial metric spaces.

Definition 2.1 A *partial metric* on a nonempty set X is a function $p : X \times X \longrightarrow \mathbb{R}^+$ such that for all $x, y, z \in X$,

p1. $x = y \Leftrightarrow p(x,x) = p(x,y) = p(y,y)$.
p2. $p(x,x) \leq p(x,y)$.
p3. $p(x,y) = p(y,x)$.
p4. $p(x,z) \leq p(x,y) + p(y,z) - p(y,y)$.

A partial metric space is a pair (X,p) such that X is a nonempty set and p is a partial metric on X.

From the above definition, if $p(x,y) = 0$, then $x = y$. But if $x = y$, $p(x,y)$ may not be 0 in general. A trivial example of a partial metric space is the pair $(\mathbb{R}^+,p)$, where $p : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is defined as $p(x,y) = \max\{x,y\}$. For some more examples of partial metric spaces, we refer to [4, 6].

Each partial metric p on X generates a T_0 topology τ_p on X which has as a base the family of open p-balls $\{B_p(x,\epsilon) : x \in X, \epsilon > 0\}$, where $B_p(x,\epsilon) = \{y \in X : p(x,y) < p(x,x) + \epsilon\}$ for all $x \in X$ and $\epsilon > 0$. A sequence $\{x_n\}$ in X converges to a point $x \in X$, with respect to τ_p if and only if $p(x,x) = \lim_{n\to\infty} p(x,x_n)$. A sequence $\{x_n\}$ in X is called a Cauchy sequence if $\lim_{n,m\to\infty} p(x_n,x_m)$ exists and is finite.

If p is a partial metric on X, then the function $p^s : X \times X \longrightarrow \mathbb{R}^+$ given by

$$p^s(x,y) = 2p(x,y) - p(x,x) - p(y,y)$$

is a metric on X.

Lemma 2.1 [2, 7] *Let (X,p) be a partial metric space. Then*

(a) *$\{x_n\}$ is a Cauchy sequence in (X,p) if and only if it is a Cauchy sequence in the metric space (X,p^s).*
(b) *(X,p) is complete if and only if the metric space (X,p^s) is complete. Furthermore, $\lim_{n\to\infty} p^s(x_n,x) = 0$ if and only if*

$$p(x,x) = \lim_{n\to\infty} p(x_n,x) = \lim_{n,m\to\infty} p(x_n,x_m).$$

Let (X,p) be a partial metric. We endow the product space $X \times X$ with the partial metric q defined as follows:

$$\text{for } (x,y),(u,v) \in X \times X, \quad q\big((x,y),(u,v)\big) = p(x,u) + p(y,v).$$

A mapping $F : X \times X \to X$ is said to be continuous at $(x,y) \in X \times X$ if for each $\epsilon > 0$, there exists $\delta > 0$ such that $F(B_q((x,y),\delta)) \subseteq B_p(F(x,y),\epsilon)$.

Definition 2.2 (Mixed monotone property) Let $(X,\preceq)$ be a partially ordered set and $F : X \times X \to X$. We say that the mapping F has the mixed monotone property if F is monotone non-decreasing in its first argument and is monotone non-increasing in its second argument. That is, for any $x,y \in X$,

$$x_1,x_2 \in X, \quad x_1 \preceq x_2 \quad \Rightarrow \quad F(x_1,y) \preceq F(x_2,y) \tag{1}$$

and

$$y_1,y_2 \in X, \quad y_1 \preceq y_2 \quad \Rightarrow \quad F(x,y_1) \succeq F(x,y_2). \tag{2}$$

Definition 2.3 [11] Let $F : X \times X \to X$. We say that $(x, y) \in X \times X$ is a coupled fixed point of F if $F(x, y) = x$ and $F(y, x) = y$.

Definition 2.4 (Mixed g-monotone property [11]) Let $(X, \preceq)$ be a partially ordered set and $F : X \times X \to X$. We say that the mapping F has the mixed g-monotone property if F is monotone g-non-decreasing in its first argument and is monotone g-non-increasing in its second argument. That is, for any $x, y \in X$,

$$x_1, x_2 \in X, \quad gx_1 \preceq gx_2 \quad \Rightarrow \quad F(x_1, y) \preceq F(x_2, y)$$

and

$$y_1, y_2 \in X, \quad gy_1 \preceq gy_2 \quad \Rightarrow \quad F(x, y_1) \succeq F(x, y_2).$$

Definition 2.5 [11] Let $F : X \times X \to X$ and $g : X \to X$. We say that $(x, y) \in X \times X$ is a coupled coincidence point of F and g if $F(x, y) = gx$ and $F(y, x) = gy$.

3 Coupled fixed point results

Let Φ denote all functions $\phi : [0, \infty) \to [0, \infty)$ which satisfy

(ϕ1) ϕ is continuous and non-decreasing,
(ϕ2) $\phi(t) = 0$ if and only if $t = 0$,
(ϕ3) $\phi(t + s) \leq \phi(t) + \phi(s)$, $\forall t, s \in [0, \infty)$,
(ϕ4) $\phi(\alpha x) \leq \alpha\phi(x)$ for $\alpha \in (0, \infty)$,

and let Ψ denote all functions $\psi : [0, \infty) \to (0, \infty)$ which satisfy $\lim_{t \to r} \psi(t) > 0$ for all $r > 0$ and $\lim_{t \to 0^+} \psi(t) = 0$.

Now, we state and prove our main result.

Theorem 3.1 *Let $(X, \preceq)$ be a partially ordered set and suppose there is a partial metric p on X such that (X, d) is a complete partial metric space. Let $F : X \times X \to X$ be a mapping having the mixed monotone property on X. Assume that there exist two elements $x_0, y_0 \in X$ with*

$$x_0 \preceq F(x_0, y_0) \quad \textit{and} \quad y_0 \succeq F(y_0, x_0).$$

Suppose there exist $\phi \in \Phi$ and $\psi \in \Psi$ such that

$$\phi\big(p\big(F(x, y), F(u, v)\big)\big) \leq \frac{1}{2}\phi\big(p(x, u) + p(y, v)\big) - \psi\left(\frac{p(x, u) + p(y, v)}{2}\right) \tag{3}$$

for all $x, y, u, v \in X$ with $x \succeq u$ and $y \preceq v$. Suppose either

(a) *F is continuous or*
(b) *X has the following property:*
  (i) *if a non-decreasing sequence $\{x_n\} \to x$, then $x_n \preceq x$, for all n,*
  (ii) *if a non-increasing sequence $\{y_n\} \to y$, then $y \preceq y_n$, for all n.*

Then there exist $x, y \in X$ *such that*

$$x = F(x, y) \quad \text{and} \quad y = F(y, x),$$

that is, F *has a coupled fixed point in* X.

Proof Let $x_0, y_0 \in X$ be such that $x_0 \preceq F(x_0, y_0)$ and $y_0 \succeq F(y_0, x_0)$. We construct sequences $\{x_n\}$ and $\{y_n\}$ in X as

$$x_{n+1} = F(x_n, y_n) \quad \text{and} \quad y_{n+1} = F(y_n, x_n) \quad \text{for all } n \geq 0. \tag{4}$$

We are to prove that

$$x_n \preceq x_{n+1} \quad \text{for all } n \geq 0 \tag{5}$$

and

$$y_n \succeq y_{n+1} \quad \text{for all } n \geq 0. \tag{6}$$

For this we shall use mathematical induction.

Let $n = 0$. Since $x_0 \preceq F(x_0, y_0)$ and $y_0 \succeq F(y_0, x_0)$ and as $x_1 = F(x_0, y_0)$ and $y_1 = F(y_0, x_0)$, we have $x_0 \preceq x_1$ and $y_0 \succeq y_1$. Thus (5) and (6) hold for $n = 0$.

Suppose now that (5) and (6) hold for some fixed $n \geq 0$, then, since $x_n \preceq x_{n+1}$ and $y_n \succeq y_{n+1}$, we have

$$x_{n+2} = F(x_{n+1}, y_{n+1}) \succeq F(x_n, y_{n+1}) \succeq F(x_n, y_n) = x_{n+1} \tag{7}$$

and

$$y_{n+2} = F(y_{n+1}, x_{n+1}) \preceq F(y_n, x_{n+1}) \preceq F(y_n, x_n) = y_{n+1}. \tag{8}$$

Using (7) and (8), we get

$$x_{n+1} \preceq x_{n+2} \quad \text{and} \quad y_{n+1} \succeq y_{n+2}.$$

Hence, by the induction method we conclude that (5) and (6) hold for all $n \geq 0$. Therefore,

$$x_0 \preceq x_1 \preceq x_2 \preceq \cdots \preceq x_n \preceq x_{n+1} \preceq \cdots \tag{9}$$

and

$$y_0 \succeq y_1 \succeq y_2 \succeq \cdots \succeq y_n \succeq y_{n+1} \succeq \cdots. \tag{10}$$

Since $x_n \succeq x_{n-1}$ and $y_n \preceq y_{n-1}$, using (3) and (4), we have

$$\begin{aligned} \phi\big(p(x_{n+1}, x_n)\big) &= \phi\big(p\big(F(x_n, y_n), F(x_{n-1}, y_{n-1})\big)\big) \\ &\leq \frac{1}{2}\phi\big(p(x_n, x_{n-1}) + p(y_n, y_{n-1})\big) - \psi\left(\frac{p(x_n, x_{n-1}) + p(y_n, y_{n-1})}{2}\right). \end{aligned} \tag{11}$$

Similarly, since $y_{n-1} \succeq y_n$ and $x_{n-1} \preceq x_n$, using (3) and (4), we also have

$$\begin{aligned}\phi\big(p(y_n, y_{n+1})\big) &= \phi\big(p\big(F(y_{n-1}, x_{n-1}), F(y_n, x_n)\big)\big) \\ &\leq \frac{1}{2}\phi\big(p(y_{n-1}, y_n) + p(x_{n-1}, x_n)\big) - \psi\left(\frac{p(y_{n-1}, y_n) + p(x_{n-1}, x_n)}{2}\right).\end{aligned} \tag{12}$$

Using (11) and (12), we have

$$\begin{aligned}\phi\big(p(x_{n+1}, x_n)\big) + \phi\big(p(y_{n+1}, y_n)\big) &\leq \phi\big(p(x_n, x_{n-1}) + p(y_n, y_{n-1})\big) \\ &\quad - 2\psi\left(\frac{p(x_n, x_{n-1}) + p(y_n, y_{n-1})}{2}\right).\end{aligned} \tag{13}$$

By property (ϕ3), we have

$$\phi\big(p(x_{n+1}, x_n) + p(y_{n+1}, y_n)\big) \leq \phi\big(p(x_{n+1}, x_n)\big) + \phi\big(p(y_{n+1}, y_n)\big). \tag{14}$$

Using (13) and (14), we have

$$\begin{aligned}\phi\big(p(x_{n+1}, x_n) + p(y_{n+1}, y_n)\big) &\leq \phi\big(p(x_n, x_{n-1}) + p(y_n, y_{n-1})\big) \\ &\quad - 2\psi\left(\frac{p(x_n, x_{n-1}) + p(y_n, y_{n-1})}{2}\right),\end{aligned} \tag{15}$$

which implies, since ψ is a non-negative function,

$$\phi\big(p(x_{n+1}, x_n) + p(y_{n+1}, y_n)\big) \leq \phi\big(p(x_n, x_{n-1}) + p(y_n, y_{n-1})\big).$$

Using the fact that ϕ is non-decreasing, we get

$$p(x_{n+1}, x_n) + p(y_{n+1}, y_n) \leq p(x_n, x_{n-1}) + p(y_n, y_{n-1}).$$

Set

$$\delta_n = p(x_{n+1}, x_n) + p(y_{n+1}, y_n).$$

Now, we show that $\delta_n \to 0$ as $n \to \infty$. It is clear that the sequence $\{\delta_n\}$ is decreasing. Therefore, there is some $\delta \geq 0$ such that

$$\lim_{n\to\infty} \delta_n = \lim_{n\to\infty} \big[p(x_{n+1}, x_n) + p(y_{n+1}, y_n)\big] = \delta. \tag{16}$$

We shall prove that $\delta = 0$. Suppose, to the contrary, that $\delta > 0$. Then taking the limit as $n \to \infty$ (equivalently, $\delta_n \to \delta$) of both sides of (15) and remembering $\lim_{t\to r} \psi(t) > 0$ for all $r > 0$ and ϕ is continuous, we have

$$\begin{aligned}\phi(\delta) &= \lim_{n\to\infty} \phi(\delta_n) \leq \lim_{n\to\infty} \left[\phi(\delta_{n-1}) - 2\psi\left(\frac{\delta_{n-1}}{2}\right)\right] \\ &= \phi(\delta) - 2 \lim_{\delta_{n-1}\to\delta} \psi\left(\frac{\delta_{n-1}}{2}\right) < \phi(\delta),\end{aligned}$$

a contradiction. Thus $\delta = 0$, that is,

$$\lim_{n\to\infty} \delta_n = \lim_{n\to\infty} \bigl[p(x_{n+1}, x_n) + p(y_{n+1}, y_n)\bigr] = 0. \tag{17}$$

Let

$$\delta_n^s = p^s(x_n, x_{n+1}) + p^s(y_n, y_{n+1})$$

for all $n \in \mathbb{N}$. From the definition of p^s, it is clear that $\delta_n^s \leq 2\delta_n$ for all $n \in \mathbb{N}$. Using (17), we get

$$\lim_{n\to+\infty} \delta_n^s = \lim_{n\to+\infty} p^s(x_n, x_{n+1}) + p^s(y_n, y_{n+1}) = 0.$$

Now, we prove that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in the partial metric space (X, p). From Lemma 2.1, it is sufficient to prove that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in the metric space (X, p^s). Suppose, to the contrary, that at least one of $\{x_n\}$ or $\{y_n\}$ is not a Cauchy sequence. Then there exists an $\epsilon > 0$ for which we can find subsequences $\{x_{n(k)}\}$, $\{x_{m(k)}\}$ of $\{x_n\}$ and $\{y_{n(k)}\}$, $\{y_{m(k)}\}$ of $\{y_n\}$ with $n(k) > m(k) \geq k$ such that

$$p^s(x_{n(k)}, x_{m(k)}) + p^s(y_{n(k)}, y_{m(k)}) \geq \epsilon. \tag{18}$$

Further, corresponding to $m(k)$, we can choose $n(k)$ in such a way that it is the smallest integer with $n(k) > m(k)$ and satisfying (18). Then

$$p^s(x_{n(k)-1}, x_{m(k)}) + p^s(y_{n(k)-1}, y_{m(k)}) < \epsilon. \tag{19}$$

Using (18), (19) and the triangle inequality, we have

$$\begin{aligned}
\epsilon \leq r_k^s &:= p^s(x_{n(k)}, x_{m(k)}) + p^s(y_{n(k)}, y_{m(k)}) \\
&\leq p^s(x_{n(k)}, x_{n(k)-1}) + p^s(x_{n(k)-1}, x_{m(k)}) + p^s(y_{n(k)}, y_{n(k)-1}) + p^s(y_{n(k)-1}, y_{m(k)}) \\
&\leq p^s(x_{n(k)}, x_{n(k)-1}) + p^s(y_{n(k)}, y_{n(k)-1}) + \epsilon.
\end{aligned}$$

Letting $k \to \infty$ and using (17), we get

$$\lim_{k\to\infty} r_k^s = \lim_{k\to\infty} \bigl[p^s(x_{n(k)}, x_{m(k)}) + p^s(y_{n(k)}, y_{m(k)})\bigr] = \epsilon. \tag{20}$$

By the triangle inequality,

$$\begin{aligned}
r_k^s &= p^s(x_{n(k)}, x_{m(k)}) + p^s(y_{n(k)}, y_{m(k)}) \\
&\leq p^s(x_{n(k)}, x_{n(k)+1}) + p^s(x_{n(k)+1}, x_{m(k)+1}) + p^s(x_{m(k)+1}, x_{m(k)}) \\
&\quad + p^s(y_{n(k)}, y_{n(k)+1}) + p^s(y_{n(k)+1}, y_{m(k)+1}) + p^s(y_{m(k)+1}, y_{m(k)}) \\
&= \delta_{n(k)}^s + \delta_{m(k)}^s + p^s(x_{n(k)+1}, x_{m(k)+1}) + p^s(y_{n(k)+1}, y_{m(k)+1}).
\end{aligned}$$

Using the properties of ϕ, we have

$$\begin{aligned}\phi\bigl(r_k^s\bigr) &\le \phi\bigl(\delta_{n(k)}^s + \delta_{m(k)}^s + p^s(x_{n(k)+1}, x_{m(k)+1}) + p^s(y_{n(k)+1}, y_{m(k)+1})\bigr)\\ &\le \phi\bigl(\delta_{n(k)}^s + \delta_{m(k)}^s\bigr) + \phi\bigl(p^s(x_{n(k)+1}, x_{m(k)+1})\bigr) + \phi\bigl(p^s(y_{n(k)+1}, y_{m(k)+1})\bigr). \end{aligned} \tag{21}$$

Now, let

$$r_k = p(x_{n(k)}, x_{m(k)}) + p(y_{n(k)}, y_{m(k)}).$$

By the definition of r_k^s, we have

$$\begin{aligned} r_k^s &= p^s(x_{n(k)}, x_{m(k)}) + p^s(y_{n(k)}, y_{m(k)})\\ &= 2p(x_{n(k)}, x_{m(k)}) - p(x_{n(k)}, x_{n(k)}) - p(x_{m(k)}, x_{m(k)})\\ &\quad + 2p(y_{n(k)}, y_{m(k)}) - p(y_{n(k)}, y_{n(k)}) - p(y_{m(k)}, y_{m(k)})\\ &= 2r_k - p(x_{n(k)}, x_{n(k)}) - p(x_{m(k)}, x_{m(k)})\\ &\quad - p(y_{n(k)}, y_{n(k)}) - p(y_{m(k)}, y_{m(k)}). \end{aligned} \tag{22}$$

In view of property (p2) and (17), we have

$$\begin{aligned} \lim_{k\to+\infty} p(x_{n(k)}, x_{n(k)}) &= \lim_{k\to+\infty} p(x_{m(k)}, x_{m(k)})\\ &= \lim_{k\to+\infty} p(y_{n(k)}, y_{n(k)})\\ &= \lim_{k\to+\infty} p(y_{m(k)}, y_{m(k)}) = 0. \end{aligned}$$

Therefore, letting $k \to +\infty$ in (22) and using (20), we get

$$\lim_{k\to+\infty} r_k = \frac{\epsilon}{2}.$$

Since $x_{n(k)} \succeq x_{m(k)}$ and $y_{n(k)} \preceq y_{m(k)}$, we have

$$\begin{aligned} \phi\bigl(p^s(x_{n(k)+1}, x_{m(k)+1})\bigr) &\le \phi\bigl(2p(x_{n(k)+1}, x_{m(k)+1})\bigr)\\ &\le 2\phi\bigl(p(x_{n(k)+1}, x_{m(k)+1})\bigr)\\ &= 2\phi\bigl(p\bigl(F(x_{n(k)}, y_{n(k)})\bigr), p\bigl(F(x_{m(k)}, y_{m(k)})\bigr)\bigr)\\ &\le \phi\bigl(p(x_{n(k)}, x_{m(k)}) + p(y_{n(k)}, y_{m(k)})\bigr)\\ &\quad - 2\psi\left(\frac{p(x_{n(k)}, x_{m(k)}) + p(y_{n(k)}, y_{m(k)})}{2}\right)\\ &= \phi(r_k) - 2\psi\left(\frac{r_k}{2}\right). \end{aligned} \tag{23}$$

Similarly,

$$\begin{aligned} \phi\bigl(p^s(y_{n(k)+1}, y_{m(k)+1})\bigr) &\le \phi\bigl(2p(y_{n(k)+1}, y_{m(k)+1})\bigr)\\ &\le 2\phi\bigl(p(y_{n(k)+1}, y_{m(k)+1})\bigr) \end{aligned}$$

$$\begin{aligned}
&= 2\phi\bigl(p\bigl(F(y_{n(k)}, x_{n(k)})\bigr), p\bigl(F(y_{m(k)}, x_{m(k)})\bigr)\bigr) \\
&\leq \phi\bigl(p(y_{n(k)}, y_{m(k)}) + p(x_{n(k)}, x_{m(k)})\bigr) \\
&\quad - 2\psi\left(\frac{p(y_{n(k)}, y_{m(k)}) + p(x_{n(k)}, x_{m(k)})}{2}\right) \\
&= \phi(r_k) - 2\psi\left(\frac{r_k}{2}\right).
\end{aligned} \tag{24}$$

Adding (23) and (24), we get

$$\phi\bigl(p^s(x_{n(k)+1}, x_{m(k)+1})\bigr) + \phi\bigl(p^s(y_{n(k)+1}, y_{m(k)+1})\bigr) \leq 2\phi(r_k) - 4\psi\left(\frac{r_k}{2}\right).$$

Thus, from (21), we have

$$\phi\bigl(r_k^s\bigr) \leq \phi\bigl(\delta_{n(k)}^s + \delta_{m(k)}^s\bigr) + 2\phi(r_k) - 4\psi\left(\frac{r_k}{2}\right).$$

Letting $k \to +\infty$, and using the properties of ϕ and ψ together with the inequalities established above, we have

$$\begin{aligned}
\phi(\epsilon) &\leq \phi(0) + 2\phi\left(\frac{\epsilon}{2}\right) - 4\lim_{k\to+\infty}\psi\left(\frac{r_k}{2}\right) \leq \phi(\epsilon) - 4\lim_{\frac{r_k}{2}\to\frac{\epsilon}{4}}\psi\left(\frac{r_k}{2}\right) \\
&\leq \phi(\epsilon) - 4\lim_{t\to\frac{\epsilon}{4}}\psi(t) < \phi(\epsilon),
\end{aligned} \tag{25}$$

which is a contradiction. Therefore, $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in the complete metric space (X, p^s). Thus, there are $x, y \in X$ such that

$$\lim_{n\to+\infty} p^s(x_n, x) = \lim_{n\to+\infty} p^s(y_n, y) = 0, \tag{26}$$

which implies that

$$\begin{aligned}
&\lim_{n\to+\infty} F(x_n, y_n) = \lim_{n\to+\infty} x_n = x, \\
&\lim_{n\to+\infty} F(y_n, x_n) = \lim_{n\to+\infty} y_n = y.
\end{aligned} \tag{27}$$

Therefore, from Lemma 2.1, using (17) and the property (p2), we have

$$p(x, x) = \lim_{n\to+\infty} p(x_n, x) = \lim_{n\to+\infty} p(x_n, x_n) = 0, \tag{28}$$

$$p(y, y) = \lim_{n\to+\infty} p(y_n, y) = \lim_{n\to+\infty} p(y_n, y_n) = 0. \tag{29}$$

We now show that $x = F(x, y)$ and $y = F(y, x)$. Suppose that the assumption (a) holds.

As F is continuous at (x, y), so for any $\epsilon > 0$, there exists $\delta > 0$ such that if $(u, v) \in X \times X$ with $\nu((x, y), (u, v)) < \nu((x, y), (x, y)) + \delta = \delta$, meaning that

$$p(x, u) + p(y, v) < p(x, x) + p(y, y) + \delta = \delta,$$

because $p(x,x) = p(y,y) = 0$. Then we have

$$p\big(F(x,y),F(u,v)\big) < p\big(F(x,y),F(x,y)\big) + \frac{\epsilon}{2}.$$

Since $\lim_{n\to+\infty} p(x_n,x) = \lim_{n\to+\infty} p(y_n,y) = 0$, for $\eta = \min(\frac{\delta}{2},\frac{\epsilon}{2}) > 0$, there exist $n_0, m_0 \in \mathbb{N}$ such that, for $n \geq n_0$, $m \geq m_0$,

$$p(x_n,x) < \eta \quad \text{and} \quad p(y_m,y) < \eta.$$

Then for $n \in \mathbb{N}$, $n \geq \max(n_0,m_0)$, we have $p(x_n,x) + p(y_n,y) < 2\eta < \delta$, so we get

$$p\big(F(x,y),F(x_n,y_n)\big) < p\big(F(x,y),F(x,y)\big) + \frac{\epsilon}{2}. \tag{30}$$

Further, for any $n \geq \max(n_0,m_0)$, by using (30), we have

$$\begin{aligned} p\big(F(x,y),x\big) &\leq p\big(F(x,y),x_{n+1}\big) + p(x_{n+1},x) \\ &= p\big(F(x,y),F(x_n,y_n)\big) + p(x_{n+1},x) \\ &\leq p\big(F(x,y),F(x,y)\big) + \frac{\epsilon}{2} + \eta \\ &\leq p\big(F(x,y),F(x,y)\big) + \epsilon. \end{aligned} \tag{31}$$

On utilizing $p(x,x) = p(y,y) = 0$ in (3), we get

$$\begin{aligned} \phi\big(p\big(F(x,y),F(x,y)\big)\big) &\leq \frac{1}{2}\phi\big(p(x,x)+p(y,y)\big) - \psi\left(\frac{p(x,x)+p(y,y)}{2}\right) \\ &= \frac{1}{2}\phi(0) - \psi(0) = -\psi(0) \leq 0, \end{aligned}$$

which implies $p(F(x,y),F(x,y)) = 0$. Hence, for any $\epsilon > 0$, (31) implies that

$$p\big(F(x,y),x\big) < \epsilon.$$

Thus, we have $F(x,y) = x$. Similarly, we can show that $F(y,x) = y$.

Finally, suppose that (b) holds. By (5), (26) and (27), we have $\{x_n\}$ is a non-decreasing sequence, $x_n \to x$ and $\{y_n\}$ is a non-increasing sequence, $y_n \to y$ as $n \to \infty$. Hence, by the assumption (b), we have for all $n \geq 0$,

$$x_n \preceq x \quad \text{and} \quad y_n \succeq y. \tag{32}$$

By property (p4), we have

$$p\big(x,F(x,y)\big) \leq p(x,x_{n+1}) + p\big(x_{n+1},F(x,y)\big) = p(x,x_{n+1}) + p\big(F(x_n,y_n),F(x,y)\big).$$

Therefore,

$$\begin{aligned} \phi\big(p\big(x,F(x,y)\big)\big) &\leq \phi\big(p(x,x_{n+1})\big) + \phi\big(p\big(F(x_n,y_n),F(x,y)\big)\big) \\ &\leq \phi\big(p(x,x_{n+1})\big) + \frac{1}{2}\phi\big(p(x_n,x)+p(y_n,y)\big) - \psi\left(\frac{p(x_n,x)+p(y_n,y)}{2}\right). \end{aligned}$$

Taking limit as $n \to \infty$ in the above inequality, using (31) and (29) and the properties of ϕ and ψ, we get $\phi(p(x, F(x, y))) = 0$, which implies $p(x, F(x, y)) = 0$. Hence, $x = F(x, y)$. Similarly, we can show that $y = F(y, x)$. Thus F has a coupled fixed point. □

Remark 3.1 Note that the property (ϕ4) is utilized only to get the inequality (25). Thus the conclusion of Theorem 3.1 holds if we drop property (ϕ4) and assume the additivity in (ϕ3), *i.e.*, $\phi(t + s) = \phi(t) + \phi(s)$, $\forall t, s \in [0, \infty)$.

As an immediate consequence of the above theorem, by taking $\phi(t) = t$, we have:

Corollary 3.1 *Let $(X, \preceq)$ be a partially ordered set and suppose there is a partial metric p on X such that (X, d) is a complete partial metric space. Let $F : X \times X \to X$ be a mapping having the mixed monotone property on X. Assume that there exist two elements $x_0, y_0 \in X$ with*

$$x_0 \preceq F(x_0, y_0) \quad \textit{and} \quad y_0 \succeq F(y_0, x_0).$$

Suppose there exist $\phi \in \Phi$ and $\psi \in \Psi$ such that

$$p\bigl(F(x, y), F(u, v)\bigr) \le \frac{1}{2}\bigl(p(x, u) + p(y, v)\bigr) - \psi\left(\frac{p(x, u) + p(y, v)}{2}\right)$$

for all $x, y, u, v \in X$ with $x \succeq u$ and $y \preceq v$. Suppose either
(a) *F is continuous or*
(b) *X has the following property*:
 (i) *if a non-decreasing sequence $\{x_n\} \to x$, then $x_n \preceq x$ for all n,*
 (ii) *if a non-increasing sequence $\{y_n\} \to y$, then $y \preceq y_n$ for all n.*

Then there exist $x, y \in X$ such that

$$x = F(x, y) \quad \textit{and} \quad y = F(y, x),$$

that is, F has a coupled fixed point in X.

Moreover, if we take $\psi(t) = \frac{1-k}{2} t$ where $k \in [0, 1)$ in Corollary 3.1, we get:

Corollary 3.2 *Let $(X, \preceq)$ be a partially ordered set and suppose there is a partial metric p on X such that (X, d) is a complete partial metric space. Let $F : X \times X \to X$ be a mapping having the mixed monotone property on X. Assume that there exist two elements $x_0, y_0 \in X$ with*

$$x_0 \preceq F(x_0, y_0) \quad \textit{and} \quad y_0 \succeq F(y_0, x_0).$$

Suppose there exist $\phi \in \Phi$ and $\psi \in \Psi$ such that

$$p\bigl(F(x, y), F(u, v)\bigr) \le \frac{k}{2}\bigl(p(x, u) + p(y, v)\bigr)$$

for all $x, y, u, v \in X$ with $x \succeq u$ and $y \preceq v$. Suppose either

(a) *F is continuous or*
(b) *X has the following property*:
  (i) *if a non-decreasing sequence* $\{x_n\} \to x$, *then* $x_n \preceq x$ *for all* n,
  (ii) *if a non-increasing sequence* $\{y_n\} \to y$, *then* $y \preceq y_n$ *for all* n.

Then there exist $x, y \in X$ *such that*

$$x = F(x, y) \quad \textit{and} \quad y = F(y, x),$$

that is, F has a coupled fixed point in X.

Recently, Alotaibi and Alsulami [10] extended Luong and Thuan's [9] main result to coupled coincidences using the notion of compatible maps. Here we extend these results to partial metric spaces without the condition of compatible maps. We shall need the following lemma.

Lemma 3.1 (see [16–18]) *Let X be a nonempty set and* $g : X \to X$ *be a mapping. Then there exists a subset* $E \subseteq X$ *such that* $g(E) = g(X)$ *and* $g : E \to X$ *is one-to-one.*

Theorem 3.2 *Let* $(X, \preceq)$ *be a partially ordered set and suppose there is a partial metric p on X such that* (X, d) *is a partial metric space. Let* $g : X \to X$ *and* $F : X \times X \to X$ *be a mapping having the mixed g-monotone property on X such that there exist two elements* $x_0, y_0 \in X$ *with*

$$gx_0 \preceq F(x_0, y_0) \quad \textit{and} \quad gy_0 \succeq F(y_0, x_0).$$

Suppose there exist $\phi \in \Phi$ *and* $\psi \in \Psi$ *such that*

$$\phi\big(p\big(F(x,y), F(u,v)\big)\big) \le \frac{1}{2}\phi\big(p(gx,gu) + p(gy,gv)\big) - \psi\left(\frac{p(gx,gu) + p(gy,gv)}{2}\right) \tag{33}$$

for all $x, y, u, v \in X$ *with* $gx \succeq gu$ *and* $gy \preceq gv$. *Suppose* $F(X \times X) \subseteq g(X)$, *g is continuous and* $g(X)$ *is complete and also suppose either*
(a) *F is continuous or*
(b) *X has the following property*:
  (i) *if a non-decreasing sequence* $\{x_n\} \to x$, *then* $x_n \preceq x$ *for all* n,
  (ii) *if a non-increasing sequence* $\{y_n\} \to y$, *then* $y \preceq y_n$ *for all* n.

Then there exist $x, y \in X$ *such that*

$$gx = F(x, y) \quad \textit{and} \quad gy = F(y, x),$$

that is, F and g have a coupled coincidence point in X.

Proof Using Lemma 3.1, there exists $E \subseteq X$ such that $g(E) = g(X)$ and $g : E \to X$ is one-to-one. We define a mapping $G : g(E) \times g(E) \to X$ by

$$G(gx, gy) = F(x, y), \tag{34}$$

for all $gx, gy \in g(E)$. As g is one-to-one on $g(E)$, so G is well defined. Thus, it follows from (33) and (34) that

$$\begin{aligned}
&\phi\big(p\big(G(gx,gy),G(gu,gv)\big)\big) \\
&\quad = \phi\big(p\big(F(x,y),F(u,v)\big)\big) \\
&\quad \leq \frac{1}{2}\phi\big(p(gx,gu)+p(gy,gv)\big) - \psi\left(\frac{p(gx,gu)+p(gy,gv)}{2}\right)
\end{aligned} \tag{35}$$

for all $gx, gy, gu, gv \in g(X)$ for which $g(x) \preceq g(u)$ and $g(y) \succeq g(v)$. Since F has the mixed g-monotone property, for all $gx, gy \in g(X)$,

$$gx_1, gx_2 \in g(X), \quad g(x_1) \preceq g(x_2) \quad \text{implies } G(gx_1, gy) \preceq G(gx_2, gy) \tag{36}$$

and

$$gy_1, gy_2 \in g(X), \quad g(y_1) \preceq g(y_2) \quad \text{implies } G(gx, gy_1) \succeq G(gx, gy_2), \tag{37}$$

which implies that G has the mixed monotone property. Also, there exist $x_0, y_0 \in X$ such that

$$g(x_0) \preceq F(x_0, y_0) \quad \text{and} \quad g(y_0) \succeq F(y_0, x_0).$$

This implies there exist $gx_0, gy_0 \in g(X)$ such that

$$g(x_0) \preceq G(gx_0, gy_0) \quad \text{and} \quad g(y_0) \succeq G(gy_0, gx_0).$$

Suppose that the assumption (a) holds. Since F is continuous, G is also continuous. Using Theorem 3.1 with the mapping G, it follows that G has a coupled fixed point $(u,v) \in g(X) \times g(X)$.

Suppose that the assumption (b) holds. We conclude similarly that the mapping G has a coupled fixed point $(u,v) \in g(X) \times g(X)$. Finally, we prove that F and g have a coupled coincidence point. Since (u,v) is a coupled fixed point of G, we get

$$u = G(u,v) \quad \text{and} \quad v = G(v,u). \tag{38}$$

Since $(u,v) \in g(X) \times g(X)$, there exists a point $(u_0, v_0) \in X \times X$ such that

$$u = gu_0 \quad \text{and} \quad v = gv_0. \tag{39}$$

It follows from (38) and (39) that

$$gu_0 = G(gu_0, gv_0) \quad \text{and} \quad gv_0 = G(gv_0, gu_0). \tag{40}$$

Combining (34) and (40), we get

$$gu_0 = F(u_0, v_0) \quad \text{and} \quad gv_0 = F(v_0, u_0). \tag{41}$$

Thus, (u_0, v_0) is a required coupled coincidence point of F and g. This completes the proof. □

The following coupled coincidence point theorems are obtained respectively from Corollaries 3.1 and 3.2 in a similar way as Theorem 3.2 from Theorem 3.1.

Theorem 3.3 *Let $(X, \preceq)$ be a partially ordered set, and suppose there is a partial metric p on X such that (X,p) is a partial metric space. Let $g : X \to X$ and $F : X \times X \to X$ be a mapping having the mixed g-monotone property on X such that there exist two elements $x_0, y_0 \in X$ with*

$$gx_0 \preceq F(x_0, y_0) \quad \text{and} \quad gy_0 \succeq F(y_0, x_0).$$

Suppose there exist $\phi \in \Phi$ and $\psi \in \Psi$ such that

$$\phi\big(p\big(F(x,y), F(u,v)\big)\big) \le \frac{1}{2}\big(p(gx,gu) + p(gy,gv)\big) - \psi\left(\frac{p(gx,gu) + p(gy,gv)}{2}\right) \tag{42}$$

for all $x, y, u, v \in X$ with $gx \succeq gu$ and $gy \preceq gv$. Suppose $F(X \times X) \subseteq g(X)$, g is continuous and $g(X)$ is complete and also suppose either

(a) *F is continuous or*

(b) *X has the following property:*

  (i) *if a non-decreasing sequence $\{x_n\} \to x$, then $x_n \preceq x$ for all n,*

  (ii) *if a non-increasing sequence $\{y_n\} \to y$, then $y \preceq y_n$ for all n.*

Then there exist $x, y \in X$ such that

$$gx = F(x,y) \quad \text{and} \quad gy = F(y,x),$$

that is, F and g have a coupled coincidence point in X.

Theorem 3.4 *Let $(X, \preceq)$ be a partially ordered set and suppose there is a partial metric p on X such that (X,d) is a complete partial metric space. Let $g : X \to X$ and $F : X \times X \to X$ be a mapping having the mixed g-monotone property on X such that there exist two elements $x_0, y_0 \in X$ with*

$$gx_0 \preceq F(x_0, y_0) \quad \text{and} \quad gy_0 \succeq F(y_0, x_0).$$

Suppose there exist $\phi \in \Phi$ and $\psi \in \Psi$ such that

$$\phi\big(p\big(F(x,y), F(u,v)\big)\big) \le \frac{k}{2}\big(p(gx,gu) + p(gy,gv)\big) \tag{43}$$

for all $x, y, u, v \in X$ with $gx \succeq gu$ and $gy \preceq gv$. Suppose $F(X \times X) \subseteq g(X)$, g is continuous and $g(X)$ is complete and also suppose either

(a) *F is continuous or*

(b) *X has the following property:*

  (i) *if a non-decreasing sequence $\{x_n\} \to x$, then $x_n \preceq x$ for all n,*

  (ii) *if a non-increasing sequence $\{y_n\} \to y$, then $y \preceq y_n$ for all n.*

Then there exist $x, y \in X$ *such that*

$$gx = F(x, y) \quad \text{and} \quad gy = F(y, x),$$

that is, F *and* g *have a coupled coincidence point in* X.

Remark 3.2 From the proof of Theorem 3.2 we conclude that Theorems 3.3, 4.4 and 5.4 in [6] hold without the compatibility of the maps (F, g).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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