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Implicit eigenvalue problems for maximal monotone operators

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Abstract

We study the implicit eigenvalue problem of the form

$$0 \in Tx + C(\lambda, x),$$

where T is a maximal monotone multi-valued operator and the operator C satisfies condition (S_+) or $(\tilde{S}_+)$. In a regularization method by the duality operator, we use the degree theories of Kartsatos and Skrypnik upon conditions of C as well as Browder's degree. There are two cases to consider: One is that C is demicontinuous and bounded with condition (S_+); and the other is that C is quasibounded and densely defined with condition $(\tilde{S}_+)$. Moreover, the eigenvalue problem $0 \in Tx + \lambda Cx$ is also discussed.

1 Introduction and preliminaries

A general eigenvalue theory for maximal monotone operators has been developed in various ways with applications to partial differential equations; see [5–7, 10–12]. A key tool was topological degrees for appropriate classes of operators in, *e.g.*, [2–4, 8, 9, 14–16] and the method of approach was in many cases to use regularization by means of the duality operator, while in [6, 11] the eigenvalue problem was solved by a transformed equation in terms of the approximant without using the regularization method.

Let X be a real reflexive Banach space with dual space X^* and $T : D(T) \subset X \to 2^{X^*}$ be a maximal monotone multi-valued operator. When the resolvents of T or the single-valued operator C are compact, the solvability of the nonlinear inclusion

$$0 \in Tx + \lambda Cx$$

was studied in [5, 6, 10, 12] by applying the Leray-Schauder degree theory for compact operators. More generally, implicit eigenvalue problems were considered in [7, 10], where the single-valued and compact case was dealt with in [7]. In direction of [10], we study the implicit eigenvalue problem of the form

$$0 \in Tx + C(\lambda, x),$$

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where $C : [0, \Lambda] \times M \to X^*$ satisfies condition (S_+) or $(\tilde{S}_+)$. As in [12], we adopt property $(\mathcal{P})$ about the solvability of the related equation

$$T_s x + C(\lambda, x) + \varepsilon J_\psi x = 0,$$

where T_s is the Brezis-Crandall-Pazy approximant introduced in [1] and J_ψ is the (normalized) duality operator with a gauge function ψ.

In the present paper, we divide our investigation into two cases to apply suitable degree theory. One case deals with demicontinuous bounded operators satisfying condition (S_+) with the aid of the most elementary degree theory of Skrypnik [14], in comparison with [10]. The other case is concerned with quasibounded densely defined operators satisfying condition $(\tilde{S}_+)$, where the degree theory of Kartsatos and Skrypnik [8, 9] for densely defined operators is used. In more concrete situations, the eigenvalue problem $0 \in Tx + \lambda Cx$ is discussed. We point out that Browder's degree in [3] for the reduced simple operator $T_s + \varepsilon J_\psi$ under the homotopy plays a crucial role in the proof of our results presented here.

Let X be a real Banach space with its dual X^*, Ω a nonempty subset of X, and Y another real Banach space. An operator $F : \Omega \to Y$ is said to be *bounded* if F maps bounded subsets of Ω into bounded subsets of Y. F is said to be *demicontinuous* if for every $x_0 \in \Omega$ and for every sequence $\{x_n\}$ in Ω with $x_n \to x_0$, we have $Fx_n \rightharpoonup Fx_0$. Here the symbol $\to$ ($\rightharpoonup$) stands for strong (weak) convergence.

A multi-valued operator $T : D(T) \subset X \to 2^{X^*}$ is said to be *monotone* if

$$\langle u^* - v^*, x - y \rangle \geq 0 \quad \text{for every } x, y \in D(T) \text{ and every } u^* \in Tx, v^* \in Ty,$$

where $D(T) = \{x \in X : Tx \neq \emptyset\}$ denotes the *effective domain* of T. T is said to be *maximal monotone* if it is monotone and it follows from $(x, u^*) \in X \times X^*$ and

$$\langle u^* - v^*, x - y \rangle \geq 0 \quad \text{for every } y \in D(T) \text{ and every } v^* \in Ty$$

that $x \in D(T)$ and $u^* \in Tx$.

We say that an operator $T : D(T) \subset X \to X^*$ satisfies condition (S_+) if for every sequence $\{x_n\}$ in $D(T)$ with $x_n \rightharpoonup x_0$ and

$$\limsup_{n \to \infty} \langle Tx_n, x_n - x_0 \rangle \leq 0,$$

we have $x_n \to x_0$.

We say that $T : D(T) \subset X \to X^*$ satisfies condition $(\tilde{S}_+)$ if for every sequence $\{x_n\}$ in $D(T)$ with $x_n \rightharpoonup x_0$, $Tx_n \rightharpoonup h_0^*$ and

$$\limsup_{n \to \infty} \langle Tx_n, x_n - x_0 \rangle \leq 0,$$

we have $x_n \to x_0$, $x_0 \in D(T)$ and $Tx_0 = h_0^*$.

We say that a multi-valued operator $T : D(T) \subset X \to 2^{X^*}$ satisfies condition (S_q) on a set $M \subset D(T)$ if for every sequence $\{x_n\}$ in M with $x_n \rightharpoonup x_0$ and every sequence $\{u_n^*\}$ in X^* with $u_n^* \to u^*$ where $u_n^* \in Tx_n$, we have $x_n \to x_0$.

Throughout this paper, X will always be an infinite dimensional real reflexive Banach space which has been renormed so that X and its dual X^* are locally uniformly convex.

A function $\psi : [0, \infty) \to [0, \infty)$ is called a *gauge function* if ψ is continuous, strictly increasing, $\psi(0) = 0$ and $\psi(t) \to \infty$ as $t \to \infty$. An operator $J_\psi : X \to X^*$ is called a *duality operator* with a gauge function ψ if

$$\langle J_\psi x, x \rangle = \psi(\|x\|) \|x\| \quad \text{and} \quad \|J_\psi x\| = \psi(\|x\|) \quad \text{for } x \in X.$$

If ψ is the identity map I, then $J := J_I$ is called a *normalized duality operator*. It is known in [13] that J_ψ is continuous, bounded, surjective, strictly monotone, maximal monotone and satisfies condition (S_+).

Given a maximal monotone operator $T : D(T) \subset X \to 2^{X^*}$, $x \in X$ and $s > 0$, there exist unique elements $x_s \in D(T)$ and $u_s^* \in T x_s$ such that

$$J(x_s - x) + s u_s^* = 0.$$

Two operators $J_s : X \to D(T)$ and $T_s : X \to X^*$ defined by

$$J_s x := x_s \quad \text{and} \quad T_s x := u_s^* \quad \text{for } x \in X$$

are called the *Brezis-Crandall-Pazy approximants*. It is known in [1] that J_s is continuous and bounded and T_s is demicontinuous, bounded, and maximal monotone. It is easy to see that $J_s = I - s J^{-1} T_s$ and $T_s x \in T J_s x$ for $x \in X$.

Let $C : [0, \Lambda] \times M \to X^*$ be an operator, where M is a subset of X. Then $C(t, x)$ is said to be *continuous in t uniformly* with respect to $x \in M$ if for every $t_0 \in [0, \Lambda]$ and for every sequence $\{t_n\}$ in $[0, \Lambda]$ with $t_n \to t_0$, we have $C(t_n, x) \to C(t_0, x)$ uniformly with respect to $x \in M$.

We say that C satisfies condition (S_+) if for every $\lambda \in (0, \Lambda]$ and for every sequence $\{x_n\}$ in M with $x_n \rightharpoonup x_0$ and

$$\limsup_{n \to \infty} \langle C(\lambda, x_n), x_n - x_0 \rangle \le 0,$$

we have $x_n \to x_0$.

We say that C satisfies condition $(\tilde{S}_+)$ if for every $\lambda \in (0, \Lambda]$ and for every sequence $\{x_n\}$ in M with $x_n \rightharpoonup x_0$, $C(\lambda, x_n) \rightharpoonup h_0^*$ and

$$\limsup_{n \to \infty} \langle C(\lambda, x_n), x_n - x_0 \rangle \le 0,$$

we have $x_n \to x_0$, $x_0 \in M$ and $C(\lambda, x_0) = h_0^*$.

We often need the following demiclosedness property of maximal monotone operators given in [17].

Lemma 1.1 *Let $T : D(T) \subset X \to 2^{X^*}$ be a maximal monotone multi-valued operator. Then for every sequence $\{x_n\}$ in $D(T)$, $x_n \to x$ in X and $u_n^* \rightharpoonup u^*$ in X^*, where $u_n^* \in T x_n$, imply that $x \in D(T)$ and $u^* \in Tx$.*

2 Implicit eigenvalue problem about demicontinuous operators

In this section, we are concerned with the implicit eigenvalue problem for perturbed maximal monotone operators in reflexive Banach spaces by applying the degree theories of Skrypnik and Browder for nonlinear operators of monotone type.

In what follows, for a bounded subset Ω of X, let $\overline{\Omega}$ and $\partial\Omega$ denote the closure and the boundary of Ω in X, respectively. Following Browder [2, 3], a homotopy $H : [0,1] \times \overline{\Omega} \to X^*$ is said to be of class (S_+) if the following condition holds:

For every sequence $\{u_j\}$ in $\overline{\Omega}$ with $u_j \rightharpoonup u_0$ and every sequence $\{t_j\}$ in $[0,1]$ with $t_j \to t_0$ such that

$$\limsup_{j\to\infty} \langle H(t_j, u_j), u_j - u_0 \rangle \le 0,$$

we have $u_j \to u_0$ and $H(t_j, u_j) \rightharpoonup H(t_0, u_0)$.

Kartsatos and Skrypnik [10] obtained the following result by using Browder's degree in [4] for multi-valued operators. As in [12], we adopt property $(\mathcal{P})$ in terms of the Brezis-Crandall-Pazy approximant so that we can apply the most essential degree theory of Skrypnik [14] for single-valued operators to prove in a direct method.

Theorem 2.1 *Let Ω be a bounded open set in X with $0 \in \Omega$. Let $T : D(T) \subset X \to 2^{X^*}$ be a maximal monotone multi-valued operator with $0 \in D(T)$ and $0 \in T(0)$. Let Λ, s_0, and ε_0 be three positive numbers. Suppose that $C : [0, \Lambda] \times \overline{\Omega} \to X^*$ is demicontinuous, bounded and satisfies condition (S_+) such that $C(0, x) = 0$ for all $x \in \overline{\Omega}$ and $C(t, x)$ is continuous in t uniformly with respect to $x \in \overline{\Omega}$.*

(a) *For a given $\varepsilon > 0$, assume the following property:*

    $(\mathcal{P})$ For every $s \in (0, s_0)$, there exists a $\lambda \in (0, \Lambda]$ such that the equation

$$T_s x + C(\lambda, x) + \varepsilon J x = 0$$

has no solution in Ω. Then there exists a $(\lambda_\varepsilon, x_\varepsilon) \in (0, \Lambda] \times (D(T) \cap \partial\Omega)$ such that

$$0 \in T x_\varepsilon + C(\lambda_\varepsilon, x_\varepsilon) + \varepsilon J x_\varepsilon.$$

(b) *If $0 \notin T(D(T) \cap \partial\Omega)$, T satisfies condition (S_q) on $D(T) \cap \partial\Omega$, and property $(\mathcal{P})$ is fulfilled for all $\varepsilon \in (0, \varepsilon_0]$, then there exists a $(\lambda_0, x_0) \in (0, \Lambda] \times (D(T) \cap \partial\Omega)$ such that*

$$0 \in T x_0 + C(\lambda_0, x_0).$$

Proof (a) We first claim that for any $s \in (0, s_0)$, there exists a $(\lambda_0, x_0) \in (0, \Lambda] \times \partial\Omega$ such that

$$T_s x_0 + C(\lambda_0, x_0) + \varepsilon J x_0 = 0. \tag{2.1}$$

Assume on the contrary that for some $s \in (0, s_0)$ and for every $\lambda \in (0, \Lambda]$, the following holds:

$$T_s x + C(\lambda, x) + \varepsilon J x \ne 0 \quad \text{for all } x \in \partial\Omega. \tag{2.2}$$

Since $(T_s + \varepsilon J)(0) = 0$ and $T_s + \varepsilon J$ is injective by the strict monotonicity of J, we have $T_s x + C(0, x) + \varepsilon J x \neq 0$ for all $x \in \partial\Omega$. Thus, (2.2) holds for all $\lambda \in [0, \Lambda]$.

Consider a homotopy $H : [0, 1] \times \overline{\Omega} \to X^*$ defined by

$$H(t, x) = T_s x + C(t\Lambda, x) + \varepsilon J x \quad \text{for } (t, x) \in [0, 1] \times \overline{\Omega}.$$

Then H is of class (S_+). To prove this, let $\{u_j\}$ be a sequence in $\overline{\Omega}$ with $u_j \rightharpoonup u_0$ and $\{t_j\}$ be a sequence in $[0, 1]$ with $t_j \to t_0$ such that

$$\limsup_{j \to \infty} \langle H(t_j, u_j), u_j - u_0 \rangle \leq 0. \tag{2.3}$$

Since T_s and J are monotone, it follows from

$$\langle H(t_j, u_j), u_j - u_0 \rangle = \langle T_s u_j, u_j - u_0 \rangle + \langle C(t_j \Lambda, u_j), u_j - u_0 \rangle + \varepsilon \langle J u_j, u_j - u_0 \rangle$$

that

$$\langle H(t_j, u_j), u_j - u_0 \rangle \geq \langle T_s u_0, u_j - u_0 \rangle + \langle C(t_j \Lambda, u_j), u_j - u_0 \rangle + \varepsilon \langle J u_j, u_j - u_0 \rangle \tag{2.4}$$

and

$$\langle H(t_j, u_j), u_j - u_0 \rangle \geq \langle T_s u_0, u_j - u_0 \rangle + \langle C(t_j \Lambda, u_j), u_j - u_0 \rangle + \varepsilon \langle J u_0, u_j - u_0 \rangle. \tag{2.5}$$

By (2.3) and (2.5), we have

$$\limsup_{j \to \infty} \langle C(t_j \Lambda, u_j), u_j - u_0 \rangle \leq 0. \tag{2.6}$$

There are two cases to consider. If $t_0 = 0$, then $C(t_j \Lambda, u_j) \to 0$ and so

$$\lim_{j \to \infty} \langle C(t_j \Lambda, u_j), u_j - u_0 \rangle = 0. \tag{2.7}$$

Using (2.3), (2.4), and (2.7), we obtain

$$\limsup_{j \to \infty} \langle J u_j, u_j - u_0 \rangle \leq 0.$$

Since J satisfies condition (S_+), we have

$$u_j \to u_0,$$

which implies by the demicontinuity of T_s and C and the continuity of J

$$T_s u_j \rightharpoonup T_s u_0, \qquad C(t_j \Lambda, u_j) \rightharpoonup C(0, u_0), \quad \text{and} \quad J u_j \to J u_0.$$

This means that $H(t_j, u_j) \rightharpoonup H(0, u_0)$. Now, let $t_0 > 0$. We have

$$\langle C(t_j \Lambda, u_j), u_j - u_0 \rangle = \langle C(t_j \Lambda, u_j) - C(t_0 \Lambda, u_j), u_j - u_0 \rangle + \langle C(t_0 \Lambda, u_j), u_j - u_0 \rangle,$$

which implies

$$\limsup_{j\to\infty} \langle C(t_0\Lambda, u_j), u_j - u_0 \rangle$$

$$\leq \limsup_{j\to\infty} \langle C(t_j\Lambda, u_j), u_j - u_0 \rangle + \limsup_{j\to\infty} \left[-\langle C(t_j\Lambda, u_j) - C(t_0\Lambda, u_j), u_j - u_0 \rangle \right]$$

and hence by (2.6)

$$\limsup_{j\to\infty} \langle C(t_0\Lambda, u_j), u_j - u_0 \rangle \leq \lim_{j\to\infty} \left\| C(t_j\Lambda, u_j) - C(t_0\Lambda, u_j) \right\| \| u_j - u_0 \| = 0.$$

Since C satisfies condition (S_+), we get $u_j \to u_0$ from which $T_s u_j \rightharpoonup T_s u_0$, $C(t_j\Lambda, u_j) \rightharpoonup C(t_0\Lambda, u_0)$, and $Ju_j \to Ju_0$. Consequently, $H(t_j, u_j) \rightharpoonup H(t_0, u_0)$. We have just shown that the homotopy H is of class (S_+).

We are now ready to apply the degree theory of Skrypnik [14, 15]. If d_S denotes the Skrypnik degree, the homotopy invariance property of d_S implies that

$$d_S\big(T_s + C(\lambda, \cdot) + \varepsilon J, \Omega, 0\big) = d_S\big(H(t, \cdot), \Omega, 0\big)$$

$$= d_S(T_s + \varepsilon J, \Omega, 0)$$

$$= 1$$

for all $\lambda \in [0, \Lambda]$. The last equality follows from Theorem 3 in [3], based on the fact that $T_s + \varepsilon J$ is demicontinuous, bounded, injective, and satisfies condition (S_+) and $\langle T_s x + \varepsilon Jx, x \rangle \geq 0$ for all $x \in \partial\Omega$. For every $\lambda \in (0, \Lambda]$, the existence property of d_S implies that there is a point x in Ω such that

$$T_s x + C(\lambda, x) + \varepsilon Jx = 0,$$

which contradicts property $(\mathcal{P})$. Therefore, the first claim (2.1) is true.

In view of (2.1), let $\{s_n\}$ be a sequence in $(0, s_0)$ with $s_n \to 0$ and $\{(\lambda_n, x_n)\}$ be the corresponding sequence in $(0, \Lambda] \times \partial\Omega$ such that

$$T_{s_n} x_n + C(\lambda_n, x_n) + \varepsilon Jx_n = 0. \tag{2.8}$$

Without loss of generality, we may suppose that

$$\lambda_n \to \lambda_0, \qquad x_n \rightharpoonup x_0, \qquad C(\lambda_n, x_n) \rightharpoonup c^* \quad \text{and} \quad Jx_n \rightharpoonup j^*,$$

where $\lambda_0 \in [0, \Lambda]$, $x_0 \in X$, $c^* \in X^*$ and $j^* \in X^*$. To arrive at the conclusion of (a) in the next step, we may suppose that $\lambda_0 \in (0, \Lambda]$. In fact, when $\lambda_0 = 0$, we know that $0 \notin Tx + C(0, x) + \varepsilon Jx$ for all $x \in D(T) \cap \partial\Omega$ because $T + \varepsilon J$ is injective on $D(T) \cap \overline{\Omega}$ and $0 \in (T + \varepsilon J)(D(T) \cap \Omega)$. For our aim, we will now show that

$$\limsup_{n\to\infty} \langle C(\lambda_n, x_n), x_n - x_0 \rangle \leq 0. \tag{2.9}$$

Assume on the contrary that there exists a subsequence of $\{x_n\}$, denoted again by $\{x_n\}$, such that

$$\lim_{n\to\infty}\langle C(\lambda_n, x_n), x_n - x_0\rangle > 0.$$

Note by (2.8) and the monotonicity of J that

$$\langle T_{s_n}x_n, x_n - x_0\rangle \leq -\langle C(\lambda_n, x_n), x_n - x_0\rangle - \varepsilon\langle Jx_0, x_n - x_0\rangle$$

and so

$$\limsup_{n\to\infty}\langle T_{s_n}x_n, x_n - x_0\rangle < 0.$$

Hence, it follows from $T_{s_n}x_n \rightharpoonup -c^* - \varepsilon j^*$ that

$$\limsup_{n\to\infty}\langle T_{s_n}x_n, x_n\rangle = \limsup_{n\to\infty}\left[\langle T_{s_n}x_n, x_n - x_0\rangle + \langle T_{s_n}x_n, x_0\rangle\right]$$

$$< \langle -c^* - \varepsilon j^*, x_0\rangle. \tag{2.10}$$

Let $x \in D(T)$ and $u^* \in Tx$ be arbitrary. Since T is monotone, $T_{s_n}x_n \in TJ_{s_n}x_n$, and $J_{s_n} = I - s_n J^{-1}T_{s_n}$, we have

$$\langle T_{s_n}x_n - u^*, x_n - s_n J^{-1}T_{s_n}x_n - x\rangle \geq 0,$$

which implies

$$\langle T_{s_n}x_n, x_n\rangle \geq s_n\|T_{s_n}x_n\|^2 + \langle T_{s_n}x_n, x\rangle + \langle u^*, x_n - x\rangle - s_n\langle u^*, J^{-1}T_{s_n}x_n\rangle.$$

Since $\{T_{s_n}x_n\}$ and J^{-1} are bounded and $s_n \to 0$, we have

$$\liminf_{n\to\infty}\langle T_{s_n}x_n, x_n\rangle \geq \langle -c^* - \varepsilon j^*, x\rangle + \langle u^*, x_0 - x\rangle. \tag{2.11}$$

Combining (2.10) and (2.11), we get

$$\langle -c^* - \varepsilon j^* - u^*, x_0 - x\rangle > 0 \quad \text{for all } x \in D(T) \text{ and } u^* \in Tx. \tag{2.12}$$

Since T is maximal monotone, we have $x_0 \in D(T)$ and $-c^* - \varepsilon j^* \in Tx_0$. Letting $x = x_0 \in D(T)$ in (2.12) yields a contradiction. Thus, (2.9) holds.

Since $C(t, x)$ is continuous in t uniformly with respect to $x \in \overline{\Omega}$ and $\lambda_n \to \lambda_0$, we have by (2.9)

$$\limsup_{n\to\infty}\langle C(\lambda_0, x_n), x_n - x_0\rangle \leq 0, \tag{2.13}$$

by observing that

$$\langle C(\lambda_n, x_n), x_n - x_0\rangle = \langle C(\lambda_n, x_n) - C(\lambda_0, x_n), x_n - x_0\rangle + \langle C(\lambda_0, x_n), x_n - x_0\rangle.$$

Since C satisfies condition (S_+), it follows from (2.13) that

$$x_n \to x_0 \in \partial\Omega,$$

which implies by (2.8)

$$T_{s_n}x_n \rightharpoonup -C(\lambda_0, x_0) - \varepsilon Jx_0$$

and as above

$$J_{s_n}x_n = x_n - s_n J^{-1}T_{s_n}x_n \to x_0.$$

Since T is maximal monotone and $T_{s_n}x_n \in TJ_{s_n}x_n$, Lemma 1.1 implies that $x_0 \in D(T)$ and

$$0 \in Tx_0 + C(\lambda_0, x_0) + \varepsilon Jx_0.$$

(b) According to the statement (a), for a sequence $\{\varepsilon_n\}$ in $(0, \varepsilon_0]$ with $\varepsilon_n \to 0$, we can choose sequences $\{\lambda_n\}$ in $(0, \Lambda]$, $\{x_n\}$ in $D(T) \cap \partial\Omega$, and $\{u_n^*\}$ in X^* with $u_n^* \in Tx_n$ such that

$$u_n^* + C(\lambda_n, x_n) + \varepsilon_n Jx_n = 0. \tag{2.14}$$

We may suppose that

$$\lambda_n \to \lambda_0, \qquad x_n \rightharpoonup x_0, \qquad C(\lambda_n, x_n) \rightharpoonup c^* \quad \text{and} \quad Jx_n \rightharpoonup j^*,$$

where $\lambda_0 \in [0, \Lambda]$, $x_0 \in X$, $c^* \in X^*$, and $j^* \in X^*$. Note that $\lambda_0 \in (0, \Lambda]$. Indeed, if $\lambda_0 = 0$, then (2.14) implies $u_n^* \to 0$ which gives by condition (S_q) on $D(T) \cap \partial\Omega$, $x_n \to x_0 \in \partial\Omega$ and therefore $x_0 \in D(T)$ and $0 \in Tx_0$, which contradicts the hypothesis that $0 \notin T(D(T) \cap \partial\Omega)$.

To show that $x_n \to x_0$, we first claim that

$$\limsup_{n \to \infty} \langle C(\lambda_n, x_n), x_n - x_0 \rangle \leq 0. \tag{2.15}$$

Assume the contrary. So, we may suppose that

$$\lim_{n \to \infty} \langle C(\lambda_n, x_n), x_n - x_0 \rangle > 0.$$

Hence, it follows from (2.14) that $u_n^* \rightharpoonup -c^*$ and

$$\limsup_{n \to \infty} \langle u_n^*, x_n - x_0 \rangle < 0,$$

which imply

$$\limsup_{n \to \infty} \langle u_n^*, x_n \rangle < \langle -c^*, x_0 \rangle. \tag{2.16}$$

For every $x \in D(T)$ and every $u^* \in Tx$, we obtain from the monotonicity of T that

$$\liminf_{n\to\infty}\langle u_n^*, x_n \rangle \geq \liminf_{n\to\infty}\left[\langle u_n^*, x\rangle + \langle u^*, x_n - x\rangle\right]$$
$$\geq \langle -c^*, x\rangle + \langle u^*, x_0 - x\rangle,$$

which implies along with (2.16)

$$\langle -c^* - u^*, x_0 - x\rangle > 0. \tag{2.17}$$

Since T is maximal monotone, we have $x_0 \in D(T)$ and $-c^* \in Tx_0$. Letting $x = x_0$ in (2.17), we have a contradiction. Thus, (2.15) is true.

As above, we can deduce from (2.15) that

$$\limsup_{n\to\infty}\langle C(\lambda_0, x_n), x_n - x_0\rangle \leq 0.$$

Since C satisfies condition (S_+) and is demicontinuous, we obtain from (2.14) that

$$x_n \to x_0 \quad \text{and} \quad u_n^* \rightharpoonup -C(\lambda_0, x_0).$$

We conclude that $x_0 \in D(T) \cap \partial\Omega$ and

$$0 \in Tx_0 + C(\lambda_0, x_0).$$

This completes the proof. □

As a consequence of Theorem 2.1, we have the following result. When C is a compact operator, it was proved by Li and Huang [12] with the aid of the Leray-Schauder degree for compact operators.

Corollary 2.2 *Let T, Ω, Λ, s_0, and ε_0 be as in Theorem 2.1. Suppose that $C : \overline{\Omega} \to X^*$ is a demicontinuous bounded operator which satisfies condition (S_+).*

(a) *For a given $\varepsilon > 0$, assume the following property:*

(P) *For every $s \in (0, s_0)$, there exists a $\lambda \in (0, \Lambda]$ such that the equation*

$$T_s x + \lambda C x + \varepsilon J x = 0$$

has no solution in Ω. Then there exists a $(\lambda_\varepsilon, x_\varepsilon) \in (0, \Lambda] \times (D(T) \cap \partial\Omega)$ such that

$$0 \in Tx_\varepsilon + \lambda_\varepsilon C x_\varepsilon + \varepsilon J x_\varepsilon.$$

(b) *If $0 \notin T(D(T) \cap \partial\Omega)$, T satisfies condition (S_q) on $D(T) \cap \partial\Omega$, and property (P) is fulfilled for all $\varepsilon \in (0, \varepsilon_0]$, then there exists a $(\lambda_0, x_0) \in (0, \Lambda] \times (D(T) \cap \partial\Omega)$ such that*

$$0 \in Tx_0 + \lambda_0 C x_0.$$

Proof Define an operator $\tilde{C} : [0, \Lambda] \times \overline{\Omega} \to X^*$ by

$$\tilde{C}(\lambda, x) = \lambda C x \quad \text{for } (\lambda, x) \in [0, \Lambda] \times \overline{\Omega}.$$

By hypotheses on C, the operator $\tilde{C}$ is obviously demicontinuous, bounded, and satisfies condition (S_+). Moreover, $\tilde{C}(t, x)$ is continuous in t uniformly with respect to $x \in \overline{\Omega}$ because $C(\overline{\Omega})$ is bounded. Apply Theorem 2.1 with $C = \tilde{C}$. □

3 Implicit eigenvalue problem about densely defined operators

In this section, we study the implicit eigenvalue problem for densely defined perturbations of maximal monotone operators, based on the degree theories of Kartsatos and Skrypnik.

An operator $C : [0, \Lambda] \times D(C) \to X^*$ is said to be *uniformly quasibounded* if for every $S > 0$ there exists a constant $K(S) > 0$ such that for all $\lambda \in [0, \Lambda]$ and all $u \in D(C)$ with $\|u\| \leq S$ and $\langle C(\lambda, u), u \rangle \leq 0$, we have $\|C(\lambda, u)\| \leq K(S)$.

In a regularization method by the duality operator, we establish a new result on the existence of eigenvalues, by applying topological degree for densely defined operators in [8, 9].

Theorem 3.1 *Let Ω be a bounded open set in X with $0 \in \Omega$ and L a dense subspace of X. Let $T : D(T) \subset X \to 2^{X^*}$ be a maximal monotone multi-valued operator with $0 \in D(T)$ and $0 \in T(0)$. Let Λ, s_0, and ε_0 be positive numbers. Assume that $C : [0, \Lambda] \times D(C) \to X^*$ is a single-valued operator with $L \subset D(C) \subset X$ such that $C(0, x) = 0$ for all $x \in D(C)$ and $C(t, x)$ is continuous in t uniformly with respect to $x \in D(C)$. Assume further that*

(c1) *C is uniformly quasibounded,*

(c2) *C satisfies condition $(\tilde{S}_+)$, and*

(c3) *for every $\lambda \in [0, \Lambda]$ and for every $F \in \mathcal{F}(L)$ and $v \in L$, the function $c(\lambda, F, v) : F \to \mathbb{R}$, $c(\lambda, F, v)(u) = \langle C(\lambda, u), v \rangle$, is continuous on F, where $\mathcal{F}(L)$ denotes the set of all finite-dimensional subspaces of L.*

Then the following statements hold:

(a) *For a given $\varepsilon > 0$, assume the following property*:

   $(\mathcal{P})$ *For every $s \in (0, s_0)$, there exists a $\lambda \in (0, \Lambda]$ such that the equation*

   $$T_s x + C(\lambda, x) + \varepsilon J_\psi x = 0$$

   has no solution in $D(C) \cap \Omega$. Then there is a $(\lambda_\varepsilon, x_\varepsilon) \in (0, \Lambda] \times (D(T) \cap D(C) \cap \partial\Omega)$ such that

   $$0 \in T x_\varepsilon + C(\lambda_\varepsilon, x_\varepsilon) + \varepsilon J_\psi x_\varepsilon.$$

(b) *If $0 \notin T(D(T) \cap \partial\Omega)$, T satisfies condition (S_q) on $D(T) \cap \partial\Omega$, and property $(\mathcal{P})$ is fulfilled for all $\varepsilon \in (0, \varepsilon_0]$, then there exists a $(\lambda_0, x_0) \in (0, \Lambda] \times (D(T) \cap D(C) \cap \partial\Omega)$ such that*

   $$0 \in T x_0 + C(\lambda_0, x_0).$$

Proof (a) First, we prove that for every $s \in (0, s_0)$, there exists a $(\lambda_0, x_0) \in (0, \Lambda] \times (D(C) \cap \partial\Omega)$ such that

$$T_s x_0 + C(\lambda_0, x_0) + \varepsilon J_\psi x_0 = 0. \tag{3.1}$$

Assume on the contrary that for some $s \in (0, s_0)$ and for every $\lambda \in (0, \Lambda]$, the following holds:

$$T_s x + C(\lambda, x) + \varepsilon J_\psi x \neq 0 \quad \text{for all } x \in D(C) \cap \partial\Omega. \tag{3.2}$$

Then (3.2) holds for all $\lambda \in [0, \Lambda]$. For $\lambda = 0$, the assertion is obvious.

For $t \in [0, 1]$, we set $T^t = T_s$ and $C^t = C(t\Lambda, \cdot) + \varepsilon J_\psi$. To show that $\{T^t + C^t\}$ is an admissible homotopy in the sense of Definition 2.4 in [9], we have to check the following conditions on two families $\{T^t\}$ and $\{C^t\}$. In fact, conditions on $\{T^t\}$ are automatically satisfied, with T^t independent of t, due to the monotonicity of T_s and $T_s(0) = 0$.

(c_1^t) $\{C^t\}$ is uniformly strongly quasibounded with respect to T^t, *i.e.*, for every $\ell > 0$ there exists a constant $K(\ell) > 0$ such that for all $u \in L$ with $\|u\| \leq \ell$ and all $t \in [0, 1]$,

$$\langle T^t u + C^t u, u \rangle \leq 0 \quad \text{implies} \quad \|C^t u\| \leq K(\ell).$$

This follows trivially from (c1) and the fact that T_s and J_ψ are monotone and J_ψ is bounded.

(c_2^t) For every pair of sequences $\{t_j\}$ in $[0, 1]$ and $\{u_j\}$ in L such that $t_j \to t_0$, $u_j \rightharpoonup u_0$, $C^{t_j} u_j \rightharpoonup h_0^*$ and

$$\limsup_{j \to \infty} \langle C^{t_j} u_j, u_j - u_0 \rangle \leq 0, \qquad \langle T^{t_j} u_j + C^{t_j} u_j, u_j \rangle \leq 0, \tag{3.3}$$

where $t_0 \in [0, 1]$, $u_0 \in X$, and $h_0^* \in X^*$, we have $u_j \to u_0$, $u_0 \in D(C)$ and $C^{t_0} u_0 = h_0^*$.

For this, there are two cases to consider. If $t_0 = 0$, then the second inequality in (3.3) implies

$$\varepsilon \psi(\|u_j\|) \|u_j\| = \varepsilon \langle J_\psi u_j, u_j \rangle \leq \langle T_s u_j, u_j \rangle + \varepsilon \langle J_\psi u_j, u_j \rangle \leq -\langle C(t_j \Lambda, u_j), u_j \rangle \to 0$$

and hence $u_j \to 0$, $u_0 = 0 \in D(C)$, and $C^0 u_0 = 0 = h_0^*$. Now let $t_0 > 0$. From the first inequality in (3.3) it follows that

$$\langle C^{t_j} u_j, u_j - u_0 \rangle \geq \langle C(t_j \Lambda, u_j), u_j - u_0 \rangle + \varepsilon \langle J_\psi u_0, u_j - u_0 \rangle$$

and so

$$\limsup_{j \to \infty} \langle C(t_j \Lambda, u_j), u_j - u_0 \rangle \leq 0.$$

Combining this with

$$\langle C(t_j \Lambda, u_j), u_j - u_0 \rangle = \langle C(t_j \Lambda, u_j) - C(t_0 \Lambda, u_j), u_j - u_0 \rangle + \langle C(t_0 \Lambda, u_j), u_j - u_0 \rangle,$$

we have

$$\limsup_{j \to \infty} \langle C(t_0 \Lambda, u_j), u_j - u_0 \rangle \leq \limsup_{j \to \infty} \left[-\langle C(t_j \Lambda, u_j) - C(t_0 \Lambda, u_j), u_j - u_0 \rangle \right]$$

$$\leq \lim_{j \to \infty} \left\| C(t_j \Lambda, u_j) - C(t_0 \Lambda, u_j) \right\| \|u_j - u_0\| = 0.$$

In view of $C^{t_j} u_j \rightharpoonup h_0^*$, we can find a subsequence of $\{u_j\}$, denoted again by $\{u_j\}$, such that $C(t_j \Lambda, u_j) \rightharpoonup h_1^*$ and $J_\psi u_j \rightharpoonup h_2^*$ for some $h_1^*, h_2^* \in X^*$. Note that $C(t_0 \Lambda, u_j) \rightharpoonup h_1^*$ and $h_0^* = h_1^* + \varepsilon h_2^*$. Hence (c2) implies that $u_j \to u_0$, $u_0 \in D(C)$, and $C(t_0 \Lambda, u_0) = h_1^*$ and so $C^{t_0} u_0 = C(t_0 \Lambda, u_0) + \varepsilon J_\psi u_0 = h_0^*$. Thus, condition ($c_2^t$) is satisfied in both cases.

(c_3^t) For every $F \in \mathcal{F}(L)$ and $v \in L$, the function $\tilde{c}(F, v) : [0,1] \times F \to \mathbb{R}$, $\tilde{c}(F, v)(t, u) = \langle C^t u, v \rangle$, is continuous.

Actually, $\tilde{c}(F, v)$ is continuous on $[0,1] \times F$ because $c(t\Lambda, F, v)$ in the notation of (c3) is continuous on F and J_ψ is continuous on X. Consequently, we have shown that $\{T^t + C^t\}$, $t \in [0,1]$, is an admissible homotopy.

Following Kartsatos and Skrypnik [8, 9], we can use the homotopy invariance property of the degree d with respect to the bounded open set Ω as follows:

$$d\left(T_s + C(\lambda, \cdot) + \varepsilon J_\psi, \Omega, 0\right) = d(T_s + \varepsilon J_\psi, \Omega, 0)$$

$$= 1$$

for all $\lambda \in [0, \Lambda]$. The latter follows from Theorem 3 in [3], noticing that $T_s + \varepsilon J_\psi$ is demi-continuous, bounded, strictly monotone and satisfies condition (S_+). For every $\lambda \in (0, \Lambda]$, the existence property of the degree implies that

$$T_s x + C(\lambda, x) + \varepsilon J_\psi x = 0 \quad \text{for some } x \in D(C) \cap \Omega,$$

which contradicts property $(\mathcal{P})$. Therefore, the first assertion (3.1) is true.

According to assertion (3.1), let $\{s_n\}$ be a sequence in $(0, s_0)$ with $s_n \to 0$ and $\{(\lambda_n, x_n)\}$ be a sequence in $(0, \Lambda] \times (D(C) \cap \partial \Omega)$ such that

$$T_{s_n} x_n + C(\lambda_n, x_n) + \varepsilon J_\psi x_n = 0. \tag{3.4}$$

Without loss of generality, we may suppose that

$$\lambda_n \to \lambda_0, \qquad x_n \rightharpoonup x_0, \qquad C(\lambda_n, x_n) \rightharpoonup c^* \quad \text{and} \quad J_\psi x_n \rightharpoonup j^*,$$

where $\lambda_0 \in [0, \Lambda]$, $x_0 \in X$, $c^* \in X^*$ and $j^* \in X^*$. We may consider the case that $\lambda_0 \in (0, \Lambda]$ as the conclusion of (a) does not hold for $\lambda_0 = 0$. In order to apply the condition $(\tilde{S}_+)$, we first show that

$$\limsup_{n \to \infty} \langle C(\lambda_n, x_n), x_n - x_0 \rangle \leq 0. \tag{3.5}$$

Assume on the contrary that there exists a subsequence of $\{x_n\}$, denoted again by $\{x_n\}$, such that

$$\lim_{n \to \infty} \langle C(\lambda_n, x_n), x_n - x_0 \rangle > 0.$$

Note by (3.4) and the monotonicity of J_ψ that

$$\langle T_{s_n} x_n, x_n - x_0 \rangle \leq -\langle C(\lambda_n, x_n), x_n - x_0 \rangle - \varepsilon \langle J_\psi x_0, x_n - x_0 \rangle$$

and so

$$\limsup_{n \to \infty} \langle T_{s_n} x_n, x_n - x_0 \rangle < 0.$$

Hence, it follows from $T_{s_n} x_n \rightharpoonup -c^* - \varepsilon j^*$ that

$$\limsup_{n \to \infty} \langle T_{s_n} x_n, x_n \rangle < \langle -c^* - \varepsilon j^*, x_0 \rangle. \tag{3.6}$$

For every $x \in D(T)$ and every $u^* \in Tx$, it is shown as in the proof of part (a) of Theorem 2.1 that

$$\liminf_{n \to \infty} \langle T_{s_n} x_n, x_n \rangle \geq \langle -c^* - \varepsilon j^*, x \rangle + \langle u^*, x_0 - x \rangle,$$

which implies in view of (3.6)

$$\langle -c^* - \varepsilon j^* - u^*, x_0 - x \rangle > 0. \tag{3.7}$$

Since T is maximal monotone, we have $x_0 \in D(T)$ and $-c^* - \varepsilon j^* \in Tx_0$. Letting $x = x_0$ in (3.7), we have a contradiction. Thus, (3.5) holds.

As before, we can deduce from (3.5) that

$$\limsup_{n \to \infty} \langle C(\lambda_0, x_n), x_n - x_0 \rangle \leq 0.$$

Since C satisfies condition $(\tilde{S}_+)$, we have

$$x_n \to x_0, \quad x_0 \in D(C), \quad \text{and} \quad C(\lambda_0, x_0) = c^*. \tag{3.8}$$

Combining (3.4) and (3.8), we obtain

$$T_{s_n} x_n \rightharpoonup -C(\lambda_0, x_0) - \varepsilon J_\psi x_0$$

and

$$J_{s_n} x_n \to x_0.$$

Since T is maximal monotone and $T_{s_n} x_n \in T J_{s_n} x_n$, Lemma 1.1 implies that $x_0 \in D(T) \cap D(C) \cap \partial\Omega$ and

$$0 \in Tx_0 + C(\lambda_0, x_0) + \varepsilon J_\psi x_0.$$

(b) In view of (a), for a sequence $\{\varepsilon_n\}$ in $(0, \varepsilon_0]$ with $\varepsilon_n \to 0$, we take sequences $\{\lambda_n\}$ in $(0, \Lambda]$, $\{x_n\}$ in $D(T) \cap D(C) \cap \partial\Omega$ and $\{u_n^*\}$ in X^* with $u_n^* \in Tx_n$ such that

$$u_n^* + C(\lambda_n, x_n) + \varepsilon_n J_\psi x_n = 0. \tag{3.9}$$

We may suppose that

$$\lambda_n \to \lambda_0, \qquad x_n \rightharpoonup x_0, \qquad C(\lambda_n, x_n) \rightharpoonup c^* \quad \text{and} \quad J_\psi x_n \rightharpoonup j^*,$$

where $\lambda_0 \in [0, \Lambda]$, $x_0 \in X$, $c^* \in X^*$ and $j^* \in X^*$. Note that $\lambda_0 \in (0, \Lambda]$. As in the proof of part (b) of Theorem 2.1, a similar argument shows that

$$\limsup_{n \to \infty} \langle C(\lambda_n, x_n), x_n - x_0 \rangle \leq 0 \tag{3.10}$$

and so in a usual way

$$\limsup_{n \to \infty} \langle C(\lambda_0, x_n), x_n - x_0 \rangle \leq 0.$$

Hence, it follows from (c2) that

$$x_n \to x_0 \in \partial\Omega, \quad x_0 \in D(C), \quad \text{and} \quad C(\lambda_0, x_0) = c^*$$

which implies by (3.9)

$$u_n^* \rightharpoonup -C(\lambda_0, x_0).$$

Consequently, we obtain from the maximal monotonicity of T that $x_0 \in D(T)$ and

$$0 \in Tx_0 + C(\lambda_0, x_0).$$

This completes the proof. □

Remark 3.2 In an analogous way to Theorem 3.1, we can observe the eigenvalue problem $0 \in Tx + \lambda Cx$ for quasibounded perturbations of maximal monotone operators; see [10].

References
1. Brezis, H, Crandall, MG, Pazy, A: Perturbations of nonlinear maximal monotone sets in Banach space. Commun. Pure Appl. Math. **23**, 123-144 (1970)
2. Browder, FE: Fixed point theory and nonlinear problems. Bull. Am. Math. Soc. **9**, 1-39 (1983)
3. Browder, FE: Degree of mapping for nonlinear mappings of monotone type. Proc. Natl. Acad. Sci. USA **80**, 1771-1773 (1983)

4. Browder, FE: The degree of mapping and its generalizations. Contemp. Math. **21**, 15-40 (1983)
5. Guan, Z, Kartsatos, AG: On the eigenvalue problem for perturbations of nonlinear accretive and monotone operators in Banach spaces. Nonlinear Anal. **27**, 125-141 (1996)
6. Kartsatos, AG: New results in the perturbation theory of maximal monotone and m-accretive operators in Banach spaces. Trans. Am. Math. Soc. **348**, 1663-1707 (1996)
7. Kartsatos, AG, Skrypnik, IV: Normalized eigenvectors for nonlinear abstract and elliptic operators. J. Differ. Equ. **155**, 443-475 (1999)
8. Kartsatos, AG, Skrypnik, IV: Topological degree theories for densely defined mappings involving operators of type (S_+). Adv. Differ. Equ. **4**, 413-456 (1999)
9. Kartsatos, AG, Skrypnik, IV: A new topological degree theory for densely defined quasibounded $(\tilde{S}_+)$-perturbations of multivalued maximal monotone operators in reflexive Banach spaces. Abstr. Appl. Anal. **2005**, 121-158 (2005)
10. Kartsatos, AG, Skrypnik, IV: On the eigenvalue problem for perturbed nonlinear maximal monotone operators in reflexive Banach spaces. Trans. Am. Math. Soc. **358**, 3851-3881 (2006)
11. Kim, I-S: Some eigenvalue results for maximal monotone operators. Nonlinear Anal. **74**, 6041-6049 (2011)
12. Li, H-X, Huang, F-L: On the nonlinear eigenvalue problem for perturbations of monotone and accretive operators in Banach spaces. Sichuan Daxue Xuebao **37**, 303-309 (2000)
13. Petryshyn, WV: Approximation-Solvability of Nonlinear Functional and Differential Equations. Marcel Dekker, New York (1993)
14. Skrypnik, IV: Nonlinear Higher Order Elliptic Equations. Naukova Dumka, Kiev (1973) (Russian)
15. Skrypnik, IV: Methods for Analysis of Nonlinear Elliptic Boundary Value Problems. Transl., Ser. II., vol. 139. Am. Math. Soc., Providence (1994)
16. Väth, M: Fixed point theorems and fixed point index for countably condensing maps. Topol. Methods Nonlinear Anal. **13**, 341-363 (1999)
17. Zeidler, E: Nonlinear Functional Analysis and Its Applications II/B: Nonlinear Monotone Operators. Springer, New York (1990)