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Some fixed point theorems in complex valued metric spaces

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Abstract

Owning the concept of complex valued metric spaces introduced by Azam *et al.*, we prove several fixed point theorems for mappings satisfying certain point-dependent contractive conditions. The main results announced by Sintunavarat and Kumam (J. Inequal. Appl. 2012:84, 2012), Rouzkard and Imdad (Comput. Math. Appl., 2012, doi:10.1016/j.camwa.2012.02.063), and Dass and Gupta (Indian J. Pure Appl. Math. 6(12):1455-1458, 1975) are deduced from our results under weaker assumptions.

1 Introduction

The concept of a complex valued metric space which is a generalization of the classical metric space was recently introduced by Azam, Fisher and Khan (see [1]). To mention this, let us recall a natural relation \preceq on \mathbb{C} , the set of complex numbers as follows: for $z_1, z_2 \in \mathbb{C}$

$$\begin{aligned} z_1 \preceq z_2 &\stackrel{\text{def}}{\iff} \operatorname{Re}(z_1) \leq \operatorname{Re}(z_2) \quad \text{and} \quad \operatorname{Im}(z_1) \leq \operatorname{Im}(z_2), \\ z_1 \prec z_2 &\stackrel{\text{def}}{\iff} \operatorname{Re}(z_1) < \operatorname{Re}(z_2) \quad \text{and} \quad \operatorname{Im}(z_1) < \operatorname{Im}(z_2). \end{aligned}$$

Definition 1.1 Let X be a nonempty set. A mapping $d : X \times X \rightarrow \mathbb{C}$ is called a *complex valued metric* on X if the following conditions are satisfied:

- (CM1) $0 \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0 \iff x = y$;
- (CM2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
- (CM3) $d(x, y) \preceq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

In this case, we say that (X, d) is a *complex valued metric space*.

It is obvious that this concept is a generalization of the classical metric. In fact, if $d : X \times X \rightarrow \mathbb{R}$ satisfies (CM1)-(CM3), then this d is a *metric* in the classical sense; that is, the following conditions are satisfied:

- (M1) $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0 \iff x = y$;
- (M2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
- (M3) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

The following definition is an analogue of several concepts in the classical theory of metric spaces and they are discussed in [1]. There are also other interesting types of generalization of metric spaces; for example, see [2, 3].

Definition 1.2 Suppose that (X, d) is a complex valued metric space.

- We say that a sequence $\{x_n\}$ is a *Cauchy sequence* if for every $0 < c \in \mathbb{C}$ there exists an integer N such that $d(x_n, x_m) < c$ for all $n, m \geq N$.
- We say that $\{x_n\}$ *converges to an element* $x \in X$ if for every $0 < c \in \mathbb{C}$ there exists an integer N such that $d(x_n, x) < c$ for all $n \geq N$. In this case, we write $x_n \xrightarrow{d} x$.
- We say that (X, d) is *complete* if every Cauchy sequence in X converges to a point in X .

The following fact is summarized from Azam, Fisher and Khan's paper [1]. In fact, (b) and (c) of Proposition 1.3 are their Lemmas 2 and 3.

Proposition 1.3 *Let (X, d) be a complex value metric space. Suppose that $d = d_1 + id_2$ where $d_1, d_2 : X \times X \rightarrow \mathbb{R}$, that is, $d_1 = \text{Re}(d)$ and $d_2 = \text{Im}(d)$. Then the following assertions hold.*

- $|d| = (d_1^2 + d_2^2)^{1/2} : X \times X \rightarrow \mathbb{R}$ is a (classical) metric on X .
- If $\{x_n\}$ is a sequence in X and $x \in X$, then $x_n \xrightarrow{d} x$ if and only if $x_n \xrightarrow{|d|} x$.
- (X, d) is complete if and only if $(X, |d|)$ is complete.

The following common fixed point theorem was also proved by Azam, Fisher and Khan. This can be viewed as a generalization of the well-known Banach fixed point theorem.

Theorem 1.4 ([1]) *Let (X, d) be a complete complex valued metric space, and let λ, μ be nonnegative real numbers such that $\lambda + \mu < 1$. Suppose that $S, T : X \rightarrow X$ are mappings satisfying:*

$$d(Sx, Ty) \preceq \lambda d(x, y) + \frac{\mu d(x, Sx)d(y, Ty)}{1 + d(x, y)} \quad \forall x, y \in X. \tag{1}$$

Then S and T have a unique common fixed point.

In this paper, we continue the study of fixed point theorems in complex valued metric spaces. The obtained results are generalizations of recent results proved by Sintunavarat and Kumam [4], Rouzkard and Imdad [5]. Moreover, we improve several assumptions on the involved mappings. It should be noted that there are also some different fixed point theorems recently proved in [6].

2 Main result

Throughout the paper, let (X, d) be a complete complex valued metric space and $S, T : X \rightarrow X$.

Proposition 2.1 *Let $x_0 \in X$ and define the sequence $\{x_n\}$ by*

$$\begin{aligned} x_{2n+1} &= Sx_{2n}, \\ x_{2n+2} &= Tx_{2n+1}, \quad \text{for all } n = 0, 1, 2, \dots \end{aligned} \tag{2}$$

Assume that there exists a mapping $\lambda : X \times X \rightarrow [0, 1)$ satisfying

$$\lambda(TSx, y) \leq \lambda(x, y) \quad \text{and} \quad \lambda(x, STy) \leq \lambda(x, y) \quad \text{for all } x, y \in X.$$

Then $\lambda(x_{2n}, y) \leq \lambda(x_0, y)$ and $\lambda(x, x_{2n+1}) \leq \lambda(x, x_1)$ for all $x, y \in X$ and $n = 0, 1, 2, \dots$

Proof Let $x, y \in X$ and $n = 0, 1, 2, \dots$. Then we have

$$\lambda(x_{2n}, y) = \lambda(TSx_{2n-2}, y) \leq \lambda(x_{2n-2}, y) = \lambda(TSx_{2n-4}, y) \leq \dots \leq \lambda(x_0, y).$$

Similarly, we have

$$\lambda(x, x_{2n+1}) = \lambda(x, STx_{2n-1}) \leq \lambda(x, x_{2n-1}) = \lambda(x, STx_{2n-3}) \leq \dots \leq \lambda(x, x_1). \quad \square$$

Lemma 2.2 Let $\lambda, \mu : X \times X \rightarrow [0, 1)$ and $x, y \in X$. If S and T satisfy

$$\begin{aligned} d(Sx, TSx) &\lesssim \lambda(x, Sx)d(x, Sx) + \mu(x, Sx) \frac{d(x, Sx)d(Sx, TSx)}{1 + d(x, Sx)}, \\ d(STy, Ty) &\lesssim \lambda(Ty, y)d(Ty, y) + \mu(Ty, y) \frac{d(Ty, STy)d(y, Ty)}{1 + d(Ty, y)}, \end{aligned}$$

then

$$\begin{aligned} |d(Sx, TSx)| &\leq \lambda(x, Sx)|d(x, Sx)| + \mu(x, Sx)|d(Sx, TSx)|, \\ |d(STy, Ty)| &\leq \lambda(Ty, y)|d(Ty, y)| + \mu(Ty, y)|d(Ty, STy)|, \end{aligned}$$

respectively.

Proof We can write

$$\begin{aligned} |d(Sx, TSx)| &\leq \left| \lambda(x, Sx)d(x, Sx) + \mu(x, Sx) \frac{d(x, Sx)d(Sx, TSx)}{1 + d(x, Sx)} \right| \\ &\leq \lambda(x, Sx)|d(x, Sx)| + \mu(x, Sx) \left| \frac{d(x, Sx)}{1 + d(x, Sx)} \right| |d(Sx, TSx)| \\ &\leq \lambda(x, Sx)|d(x, Sx)| + \mu(x, Sx)|d(Sx, TSx)|. \end{aligned} \quad (3)$$

Similarly, we get

$$\begin{aligned} |d(STy, Ty)| &\leq \left| \lambda(Ty, y)d(Ty, y) + \mu(Ty, y) \frac{d(Ty, STy)d(y, Ty)}{1 + d(Ty, y)} \right| \\ &\leq \lambda(Ty, y)|d(Ty, y)| + \mu(Ty, y) \left| \frac{d(y, Ty)}{1 + d(y, Ty)} \right| |d(Ty, STy)| \\ &\leq \lambda(Ty, y)|d(y, Ty)| + \mu(Ty, y)|d(STy, Ty)|. \end{aligned} \quad (4)$$

\square

Lemma 2.3 Let $\{x_n\}$ be a sequence in X and $h \in [0, 1)$. If $a_n = |d(x_n, x_{n+1})|$ satisfies

$$a_n \leq ha_{n-1}, \quad \text{for all } n \in \mathbb{N}, \quad (5)$$

then $\{x_n\}$ is a Cauchy sequence.

Proof Let $h \in [0, 1)$. Then

$$a_n \leq ha_{n-1} \leq h^2 a_{n-2} \leq \dots \leq h^n a_0, \quad \text{for all } n \in \mathbb{N}.$$

For $m, n \in \mathbb{N}$ such that $m > n$, we have

$$\begin{aligned} |d(x_n, x_m)| &\leq a_n + a_{n+1} + \dots + a_{m-1} \\ &\leq h^n (1 + h + h^2 + \dots + h^{m-n-1}) a_0 \\ &\leq \frac{h^n}{1-h} a_0. \end{aligned}$$

Thus, we have $|d(x_n, x_m)| \rightarrow 0$ as $n \rightarrow \infty$, and hence $\{x_n\}$ is a Cauchy sequence. \square

Theorem 2.4 *Let (X, d) be a complete complex valued metric space and $S, T : X \rightarrow X$. If there exist mappings $\lambda, \mu, \gamma : X \times X \rightarrow [0, 1)$ such that for all $x, y \in X$:*

- (a) $\lambda(TSx, y) \leq \lambda(x, y)$ and $\lambda(x, STy) \leq \lambda(x, y)$,
 $\mu(TSx, y) \leq \mu(x, y)$ and $\mu(x, STy) \leq \mu(x, y)$,
 $\gamma(TSx, y) \leq \gamma(x, y)$ and $\gamma(x, STy) \leq \gamma(x, y)$;
- (b) $\lambda(x, y) + \mu(x, y) + \gamma(x, y) < 1$;
- (c)

$$d(Sx, Ty) \lesssim \lambda(x, y)d(x, y) + \mu(x, y) \frac{d(x, Sx)d(y, Ty)}{1 + d(x, y)} + \gamma(x, y) \frac{d(y, Sx)d(x, Ty)}{1 + d(x, y)}. \quad (6)$$

Then S and T have a unique common fixed point.

Proof Let $x, y \in X$. From (6), we consider

$$\begin{aligned} d(Sx, TSx) &\lesssim \lambda(x, Sx)d(x, Sx) + \mu(x, Sx) \frac{d(x, Sx)d(Sx, TSx)}{1 + d(x, Sx)} \\ &\quad + \gamma(x, Sx) \frac{d(Sx, Sx)d(x, TSx)}{1 + d(x, Sx)} \\ &= \lambda(x, Sx)d(x, Sx) + \mu(x, Sx) \frac{d(x, Sx)d(Sx, TSx)}{1 + d(x, Sx)}. \end{aligned}$$

From Lemma 2.2, we have

$$|d(Sx, TSx)| \leq \lambda(x, Sx)|d(x, Sx)| + \mu(x, Sx)|d(Sx, TSx)|. \quad (7)$$

Similarly, we get

$$\begin{aligned} d(STy, Ty) &\lesssim \lambda(Ty, y)d(Ty, y) + \mu(Ty, y) \frac{d(Ty, STy)d(y, Ty)}{1 + d(Ty, y)} \\ &\quad + \gamma(Ty, y) \frac{d(y, STy)d(Ty, Ty)}{1 + d(Ty, y)} \\ &= \lambda(Ty, y)d(Ty, y) + \mu(Ty, y) \frac{d(Ty, STy)d(y, Ty)}{1 + d(Ty, y)}. \end{aligned}$$

From Lemma 2.2, we have

$$|d(STy, Ty)| \leq \lambda(Ty, y)|d(Ty, y)| + \mu(Ty, y)|d(Ty, STy)|. \quad (8)$$

Let $x_0 \in X$ and the sequence $\{x_n\}$ be defined by (2). We show that $\{x_n\}$ is a Cauchy sequence. From Proposition 2.1, (7), (8) and for all $k = 0, 1, 2, \dots$, we have

$$\begin{aligned} |d(x_{2k+1}, x_{2k})| &= |d(STx_{2k-1}, Tx_{2k-1})| \\ &\leq \lambda(Tx_{2k-1}, x_{2k-1})|d(Tx_{2k-1}, x_{2k-1})| \\ &\quad + \mu(Tx_{2k-1}, x_{2k-1})|d(Tx_{2k-1}, STx_{2k-1})| \\ &= \lambda(x_{2k}, x_{2k-1})|d(x_{2k-1}, x_{2k})| + \mu(x_{2k}, x_{2k-1})|d(x_{2k}, x_{2k+1})| \\ &\leq \lambda(x_0, x_{2k-1})|d(x_{2k-1}, x_{2k})| + \mu(x_0, x_{2k-1})|d(x_{2k+1}, x_{2k})| \\ &\leq \lambda(x_0, x_1)|d(x_{2k-1}, x_{2k})| + \mu(x_0, x_1)|d(x_{2k+1}, x_{2k})|, \end{aligned}$$

which implies that

$$|d(x_{2k+1}, x_{2k})| \leq \frac{\lambda(x_0, x_1)}{1 - \mu(x_0, x_1)} |d(x_{2k-1}, x_{2k})|. \tag{9}$$

Similarly, we get

$$\begin{aligned} |d(x_{2k+2}, x_{2k+1})| &= |d(TSx_{2k}, Sx_{2k})| \\ &\leq \lambda(x_{2k}, Sx_{2k})|d(x_{2k}, Sx_{2k})| + \mu(x_{2k}, Sx_{2k})|d(Sx_{2k}, TSx_{2k})| \\ &= \lambda(x_{2k}, x_{2k+1})|d(x_{2k}, x_{2k+1})| + \mu(x_{2k}, x_{2k+1})|d(x_{2k+1}, x_{2k+2})| \\ &\leq \lambda(x_0, x_{2k+1})|d(x_{2k}, x_{2k+1})| + \mu(x_0, x_{2k+1})|d(x_{2k+2}, x_{2k+1})| \\ &\leq \lambda(x_0, x_1)|d(x_{2k}, x_{2k+1})| + \mu(x_0, x_1)|d(x_{2k+2}, x_{2k+1})|, \end{aligned}$$

which implies that

$$|d(x_{2k+2}, x_{2k+1})| \leq \frac{\lambda(x_0, x_1)}{1 - \mu(x_0, x_1)} |d(x_{2k}, x_{2k+1})|. \tag{10}$$

Let $h = \frac{\lambda(x_0, x_1)}{1 - \mu(x_0, x_1)} < 1$. Then we have

$$|d(x_{n+1}, x_n)| \leq h |d(x_{n-1}, x_n)|, \quad \text{for all } n \in \mathbb{N}. \tag{11}$$

From Lemma 2.3, we have $\{x_n\}$ is a Cauchy sequence in (X, d) . By the completeness of X , there exists $z \in X$ such that $x_n \rightarrow z$ as $n \rightarrow \infty$.

Next, we show that z is a fixed point of S . By (6) and Proposition 2.1, we have

$$\begin{aligned} d(z, Sz) &\lesssim d(z, Tx_{2n+1}) + d(Tx_{2n+1}, Sz) \\ &= d(z, x_{2n+2}) + d(Sz, Tx_{2n+1}) \\ &\lesssim d(z, x_{2n+2}) + \lambda(z, x_{2n+1})d(z, x_{2n+1}) + \mu(z, x_{2n+1}) \frac{d(z, Sz)d(x_{2n+1}, Tx_{2n+1})}{1 + d(z, x_{2n+1})} \\ &\quad + \gamma(z, x_{2n+1}) \frac{d(x_{2n+1}, Sz)d(z, Tx_{2n+1})}{1 + d(z, x_{2n+1})} \end{aligned}$$

$$\begin{aligned} &\lesssim d(z, x_{2n+2}) + \lambda(z, x_1)d(z, x_{2n+1}) + \mu(z, x_1) \frac{d(z, Sz)d(x_{2n+1}, x_{2n+2})}{1 + d(z, x_{2n+1})} \\ &\quad + \gamma(z, x_1) \frac{d(x_{2n+1}, Sz)d(z, x_{2n+2})}{1 + d(z, x_{2n+1})}. \end{aligned}$$

Thus, $d(z, Sz) = 0$ and hence $z = Sz$.

We also show that z is a fixed point of T . By (6), we have

$$\begin{aligned} d(z, Tz) &\lesssim d(z, Sx_{2n}) + d(Sx_{2n}, Tz) \\ &\lesssim d(z, x_{2n+1}) + \lambda(x_{2n}, z)d(x_{2n}, z) + \mu(x_{2n}, z) \frac{d(x_{2n}, Sx_{2n})d(z, Tz)}{1 + d(x_{2n}, z)} \\ &\quad + \gamma(x_{2n}, z) \frac{d(z, Sx_{2n})d(x_{2n}, Ty)}{1 + d(x_{2n}, z)} \\ &\lesssim d(z, x_{2n+1}) + \lambda(x_0, z)d(x_{2n}, z) + \mu(x_0, z) \frac{d(x_{2n}, x_{2n+1})d(z, Tz)}{1 + d(x_{2n}, z)} \\ &\quad + \gamma(x_0, z) \frac{d(z, x_{2n+1})d(x_{2n}, Ty)}{1 + d(x_{2n}, z)}. \end{aligned}$$

Thus, $d(z, Tz) = 0$ and hence $z = Tz$. Therefore, z is a common fixed point of S and T .

Finally, we show the uniqueness. Suppose that there is $z^* \in X$ such that $z^* = Sz^* = Tz^*$. Then

$$\begin{aligned} d(z, z^*) &= d(Sz, Tz^*) \\ &\lesssim \lambda(z, z^*)d(z, z^*) + \mu(z, z^*) \frac{d(z, Sz)d(z^*, Tz^*)}{1 + d(z, z^*)} + \gamma(z, z^*) \frac{d(z^*, Sz)d(z, Tz^*)}{1 + d(z, z^*)} \\ &= \lambda(z, z^*)d(z, z^*) + \gamma(z, z^*) \frac{d(z^*, z)d(z, z^*)}{1 + d(z, z^*)}. \end{aligned}$$

Therefore, we have

$$\begin{aligned} |d(z, z^*)| &\leq \lambda(z, z^*)|d(z, z^*)| + \gamma(z, z^*)|d(z, z^*)| \left| \frac{d(z, z^*)}{1 + d(z, z^*)} \right| \\ &\leq \lambda(z, z^*)|d(z, z^*)| + \gamma(z, z^*)|d(z, z^*)| \\ &\leq (\lambda(z, z^*) + \gamma(z, z^*))|d(z, z^*)|. \end{aligned}$$

Since $\lambda(z, z^*) + \gamma(z, z^*) < 1$, we have $|d(z, z^*)| = 0$. Thus $z = z^*$. □

By setting $S = T$ in Theorem 2.4, we deduce the following corollary.

Corollary 2.5 *Let (X, d) be a complete complex valued metric space and $T : X \rightarrow X$. If there exist mappings $\lambda, \mu, \gamma : X \times X \rightarrow [0, 1)$ such that for all $x, y \in X$:*

- (a) $\lambda(Tx, y) \leq \lambda(x, y)$ and $\lambda(x, Ty) \leq \lambda(x, y)$,
 $\mu(Tx, y) \leq \mu(x, y)$ and $\mu(x, Ty) \leq \mu(x, y)$,
 $\gamma(Tx, y) \leq \gamma(x, y)$ and $\gamma(x, Ty) \leq \gamma(x, y)$;
- (b) $\lambda(x, y) + \mu(x, y) + \gamma(x, y) < 1$;

(c)

$$d(Tx, Ty) \lesssim \lambda(x, y)d(x, y) + \mu(x, y)\frac{d(x, Tx)d(y, Ty)}{1 + d(x, y)} + \gamma(x, y)\frac{d(y, Tx)d(x, Ty)}{1 + d(x, y)}. \quad (12)$$

Then T has a unique common fixed point.

By choosing $\gamma = 0$ in Theorem 2.4, we deduce the following corollary.

Corollary 2.6 *Let (X, d) be a complete complex valued metric space and $S, T : X \rightarrow X$. If there exist mappings $\lambda, \mu : X \times X \rightarrow [0, 1)$ such that for all $x, y \in X$:*

- (a) $\lambda(TSx, y) \leq \lambda(x, y)$ and $\lambda(x, STy) \leq \lambda(x, y)$,
 $\mu(TSx, y) \leq \mu(x, y)$ and $\mu(x, STy) \leq \mu(x, y)$;
- (b) $\lambda(x, y) + \mu(x, y) < 1$;
- (c)

$$d(Sx, Ty) \lesssim \lambda(x, y)d(x, y) + \mu(x, y)\frac{d(x, Sx)d(y, Ty)}{1 + d(x, y)}. \quad (13)$$

Then S and T have a unique common fixed point.

By choosing $\mu = 0$ in Theorem 2.4, we deduce the following corollary.

Corollary 2.7 *Let (X, d) be a complete complex valued metric space and $S, T : X \rightarrow X$. If there exist mappings $\lambda, \gamma : X \times X \rightarrow [0, 1)$ such that for all $x, y \in X$:*

- (a) $\lambda(TSx, y) \leq \lambda(x, y)$ and $\lambda(x, STy) \leq \lambda(x, y)$,
 $\gamma(TSx, y) \leq \gamma(x, y)$ and $\gamma(x, STy) \leq \gamma(x, y)$;
- (b) $\lambda(x, y) + \gamma(x, y) < 1$;
- (c)

$$d(Sx, Ty) \lesssim \lambda(x, y)d(x, y) + \gamma(x, y)\frac{d(y, Sx)d(x, Ty)}{1 + d(x, y)}. \quad (14)$$

Then S and T have a unique common fixed point.

The following result is closely related to Corollary 2.5 with $\gamma = 0$. The real valued metric space version of this result is an extension of Dass and Gupta's result [7].

Theorem 2.8 *Let (X, d) be a complete complex valued metric space and $T : X \rightarrow X$. If there exist mappings $\lambda, \mu : X \times X \rightarrow [0, 1)$ such that for all $x, y \in X$:*

- (a) $\lambda(Tx, y) \leq \lambda(x, y)$ and $\lambda(x, Ty) \leq \lambda(x, y)$,
 $\mu(Tx, y) \leq \mu(x, y)$ and $\mu(x, Ty) \leq \mu(x, y)$;
- (b) $\lambda(x, y) + \mu(x, y) < 1$;
- (c)

$$d(Tx, Ty) \lesssim \lambda(x, y)d(x, y) + \mu(x, y)\frac{d(y, Ty)[1 + d(x, Tx)]}{1 + d(x, y)}. \quad (15)$$

Then T has a unique fixed point.

Proof Let $x_0 \in X$ and the sequence $\{x_n\}$ be defined by

$$x_{n+1} = Tx_n, \quad \text{where } n = 0, 1, 2, \dots \tag{16}$$

We show that $\{x_n\}$ is a Cauchy sequence. From (15), we have

$$\begin{aligned} d(x_{n+1}, x_{n+2}) &= d(Tx_n, Tx_{n+1}) \\ &\lesssim \lambda(x_n, x_{n+1})d(x_n, x_{n+1}) + \mu(x_n, x_{n+1}) \frac{d(x_{n+1}, Tx_{n+1})[1 + d(x_n, Tx_n)]}{1 + d(x_n, x_{n+1})} \\ &= \lambda(x_n, x_{n+1})d(x_n, x_{n+1}) + \mu(x_n, x_{n+1}) \frac{d(x_{n+1}, x_{n+2})[1 + d(x_n, x_{n+1})]}{1 + d(x_n, x_{n+1})} \\ &= \lambda(x_n, x_{n+1})d(x_n, x_{n+1}) + \mu(x_n, x_{n+1})d(x_{n+1}, x_{n+2}). \end{aligned}$$

It follows from (a) that

$$\begin{aligned} d(x_{n+1}, x_{n+2}) &\lesssim \lambda(x_n, x_{n+1})d(x_n, x_{n+1}) + \mu(x_n, x_{n+1})d(x_{n+1}, x_{n+2}) \\ &\lesssim \lambda(x_0, x_{n+1})d(x_n, x_{n+1}) + \mu(x_0, x_{n+1})d(x_{n+1}, x_{n+2}) \\ &\lesssim \lambda(x_0, x_0)d(x_n, x_{n+1}) + \mu(x_0, x_0)d(x_{n+1}, x_{n+2}). \end{aligned}$$

Therefore,

$$|d(x_{n+1}, x_{n+2})| \leq \lambda(x_0, x_0)|d(x_n, x_{n+1})| + \mu(x_0, x_0)|d(x_{n+1}, x_{n+2})|,$$

and hence

$$|d(x_{n+1}, x_{n+2})| \leq \frac{\lambda(x_0, x_0)}{1 - \mu(x_0, x_0)} |d(x_n, x_{n+1})|, \quad \text{for all } n = 0, 1, 2, \dots \tag{17}$$

Let $h = \frac{\lambda(x_0, x_0)}{1 - \mu(x_0, x_0)} < 1$. Then

$$|d(x_{n+1}, x_{n+2})| \leq h |d(x_n, x_{n+1})|, \quad \text{for all } n = 0, 1, 2, \dots$$

From Lemma 2.3, we have $\{x_n\}$ is a Cauchy sequence in (X, d) . By the completeness of X , there exists $z \in X$ such that $x_n \rightarrow z$ as $n \rightarrow \infty$. Next, we show that z is a fixed point of T . Then

$$\begin{aligned} d(z, Tz) &\lesssim d(z, Tx_n) + d(Tx_n, Tz) \\ &\lesssim d(z, x_{n+1}) + \lambda(x_n, z)d(x_n, z) + \mu(x_n, z) \frac{d(z, Tz)[1 + d(x_n, Tx_n)]}{1 + d(z, x_n)} \\ &\lesssim d(z, x_{n+1}) + \lambda(x_0, z)d(x_n, z) + \mu(x_0, z) \frac{d(z, Tz)[1 + d(x_n, x_{n+1})]}{1 + d(z, x_n)}. \end{aligned}$$

Notice that $\mu(x_0, z) \in [0, 1)$. Therefore, we get $d(z, Tz) \lesssim \mu(x_0, z)d(z, Tz)$, that is, $z = Tz$.

Finally, we show the uniqueness. Suppose that there is $z^* \in X$ such that $z^* = Tz^*$. Then

$$\begin{aligned} d(z, z^*) &= d(Tz, Tz^*) \\ &\lesssim \lambda(z, z^*)d(z, z^*) + \mu(z, z^*) \frac{d(z^*, Tz^*)[1 + d(z, Tz)]}{1 + d(z, z^*)} \\ &= \lambda(z, z^*)d(z, z^*). \end{aligned}$$

Since $\lambda(z, z^*) \in [0, 1]$, we have $d(z, z^*) = 0$, that is, $z = z^*$. This completes the proof. \square

3 Deduced results

3.1 Sintunavarat and Kumam's results

We deduce the main result of [4] as follows.

Theorem 3.1 ([4, Theorem 3.1]) *Let (X, d) be a complete complex valued metric space and $S, T : X \rightarrow X$. If there exist mappings $\Lambda, \Xi : X \rightarrow [0, 1]$ such that for all $x, y \in X$:*

- (i) $\Lambda(Sx) \leq \Lambda(x)$ and $\Xi(Sx) \leq \Xi(x)$;
- (ii) $\Lambda(Tx) \leq \Lambda(x)$ and $\Xi(Tx) \leq \Xi(x)$;
- (iii) $(\Lambda + \Xi)(x) < 1$;
- (iv)

$$d(Sx, Ty) \lesssim \Lambda(x)d(x, y) + \frac{\Xi(x)d(x, Sx)d(y, Ty)}{1 + d(x, y)}.$$

Then S and T have a unique common fixed point.

Proof Define $\lambda, \mu : X \times X \rightarrow [0, 1]$ by

$$\lambda(x, y) = \Lambda(x) \quad \text{and} \quad \mu(x, y) = \Xi(x), \quad \text{for all } x, y \in X. \tag{18}$$

Then for all $x, y \in X$,

- (a) $\lambda(TSx, y) = \Lambda(TSx) \leq \Lambda(Sx) \leq \Lambda(x) = \lambda(x, y)$ and $\lambda(x, STy) = \Lambda(x) = \lambda(x, y)$;
 $\mu(TSx, y) = \Xi(TSx) \leq \Xi(Sx) \leq \Xi(x) = \mu(x, y)$ and $\mu(x, STy) = \Xi(x) = \mu(x, y)$;
- (b) $\lambda(x, y) + \mu(x, y) = \Lambda(x) + \Xi(x) < 1$;
- (c)

$$\begin{aligned} d(Sx, Ty) &\lesssim \Lambda(x)d(x, y) + \frac{\Xi(x)d(x, Sx)d(y, Ty)}{1 + d(x, y)} \\ &= \lambda(x, y)d(x, y) + \mu(x, y) \frac{d(x, Sx)d(y, Ty)}{1 + d(x, y)}. \end{aligned}$$

By Corollary 2.6, S and T have a unique common fixed point. \square

Remark 1 It is worth mentioning that (i) and (ii) of Theorem 3.1 above can be weakened by the condition

$$\Lambda(TSx) \leq \Lambda(x) \quad \text{and} \quad \Xi(TSx) \leq \Xi(x), \quad \text{for all } x \in X.$$

3.2 Rouzkard and Imdad's results

The following corollary is easily obtained from our Theorem 2.4.

Corollary 3.2 *Let (X, d) be a complete complex valued metric space and $S, T : X \rightarrow X$. If there exist mappings $\lambda, \mu, \gamma : X \rightarrow [0, 1]$ such that for all $x, y \in X$:*

- (a) $\lambda(TSx) \leq \lambda(x)$, $\mu(TSx) \leq \mu(x)$ and $\gamma(TSx) \leq \gamma(x)$;
- (b) $\lambda(x) + \mu(x) + \gamma(x) < 1$;
- (c)

$$d(Sx, Ty) \preceq \lambda(x)d(x, y) + \mu(x)\frac{d(x, Sx)d(y, Ty)}{1 + d(x, y)} + \gamma(x)\frac{d(y, Sx)d(x, Ty)}{1 + d(x, y)}. \tag{19}$$

Then S and T have a unique common fixed point.

Proof Define $\lambda, \mu, \gamma : X \times X \rightarrow [0, 1]$ by

$$\lambda(x, y) = \lambda(x), \quad \mu(x, y) = \mu(x) \quad \text{and} \quad \gamma(x, y) = \gamma(x), \quad \text{for all } x, y \in X. \tag{20}$$

Then for all $x, y \in X$,

- (a) $\lambda(TSx, y) = \lambda(TSx) \leq \lambda(x) = \lambda(x, y)$ and $\lambda(x, STy) = \lambda(x) = \lambda(x, y)$;
 $\mu(TSx, y) = \mu(TSx) \leq \mu(x) = \mu(x, y)$ and $\mu(x, STy) = \mu(x) = \mu(x, y)$;
 $\gamma(TSx, y) = \gamma(TSx) \leq \gamma(x) = \gamma(x, y)$ and $\gamma(x, STy) = \gamma(x) = \gamma(x, y)$;
- (b) $\lambda(x, y) + \mu(x, y) + \gamma(x, y) = \lambda(x) + \mu(x) + \gamma(x) < 1$;
- (c)

$$\begin{aligned} d(Sx, Ty) &\preceq \lambda(x)d(x, y) + \mu(x)\frac{d(x, Sx)d(y, Ty)}{1 + d(x, y)} + \gamma(x)\frac{d(y, Sx)d(x, Ty)}{1 + d(x, y)} \\ &= \lambda(x, y)d(x, y) + \mu(x, y)\frac{d(x, Sx)d(y, Ty)}{1 + d(x, y)} + \gamma(x, y)\frac{d(y, Sx)d(x, Ty)}{1 + d(x, y)}. \end{aligned}$$

By Theorem 2.4, S and T have a unique common fixed point. □

Letting $\lambda(\cdot) = \lambda$, $\mu(\cdot) = \mu$ and $\gamma(\cdot) = \gamma$ in Corollary 3.2 gives the following result proved by Rouzkard and Imdad in [5].

Corollary 3.3 ([5]) *If S and T are self-mappings defined on a complete complex valued metric space (X, d) satisfying the condition*

$$d(Sx, Ty) \preceq \lambda d(x, y) + \mu \frac{d(x, Sx)d(y, Ty)}{1 + d(x, y)} + \gamma \frac{d(y, Sx)d(x, Ty)}{1 + d(x, y)}$$

for all $x, y \in X$, where λ, μ, γ are nonnegative reals with $\lambda + \mu + \gamma < 1$, then S and T have a unique common fixed point.

3.3 Dass and Gupta's results

Applying the proof of our Theorem 2.8, we can deduce the following result of Dass and Gupta [7] in the context of real valued metric spaces.

Theorem 3.4 ([7]) *Let (X, d) be a real valued metric space. Let $T : X \rightarrow X$ be such that*

(i)

$$d(Tx, Ty) \leq \lambda d(x, y) + \frac{\mu d(y, Ty)[1 + d(x, Tx)]}{1 + d(x, y)}$$

for all $x, y \in X$, $\lambda > 0$, $\mu > 0$, $\lambda + \mu < 1$, and

(ii) *for some $x_0 \in X$, the sequence of iterates $\{T^n(x_0)\}$ has a subsequence $\{T^{n_k}(x_0)\}$ with*
$$z = \lim_{k \rightarrow \infty} T^{n_k}(x_0).$$

Then z is a unique fixed point of T .

Proof Define $\lambda, \mu : X \times X \rightarrow [0, 1)$ by

$$\lambda(x, y) = \lambda \quad \text{and} \quad \mu(x, y) = \mu, \quad \text{for all } x, y \in X.$$

Then the conditions (a), (b) and (c) of Theorem 2.8 are satisfied. Hence, we have $\{x_n\}$ is a Cauchy sequence in (X, d) . By (ii), the whole sequence $x_n \rightarrow z$ as $n \rightarrow \infty$. It follows again from the proof of Theorem 2.8 that z is a unique fixed point of T as desired. \square

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally and significantly to this research work. All authors read and approved the final manuscript.

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