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Modified block iterative procedure for solving the common solution of fixed point problems for two countable families of total quasi-ϕ-asymptotically nonexpansive mappings with applications

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Abstract

In this paper, we introduce a new iterative procedure which is constructed by the modified block hybrid projection method for solving a common solution of fixed point problems for two countable families of uniformly total quasi-ϕ-asymptotically nonexpansive and uniformly Lipschitz continuous mappings. Under suitable conditions, some strong convergence theorems are established in a uniformly smooth and strictly convex Banach space with the Kadec-Klee property. Finally, we apply the problem of a strong convergence theorem concerning maximal monotone operators in Banach spaces.

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1 Introduction

Throughout this paper, we assume that E is a real Banach space, E^* is the dual space of E. Let C be a nonempty, closed, and convex subset of E and $\langle \cdot, \cdot \rangle$ be the pairing between E and E^*. We denote the strong convergence and weak convergence of a sequence $\{x_n\}$ by $x_n \to x$ and $x_n \rightharpoonup x$, respectively, and $J : E \to 2^{E^*}$ is the *normalized duality mapping* defined by

$$Jx = \{ f \in E^* : \langle x, f \rangle = \|x\|^2 = \|f\|^2 \}, \quad \forall x \in E. \tag{1.1}$$

In the sequel, we use $F(T)$ to denote the set of fixed points of a mapping T and use $\mathbb{R}$ and $\mathbb{R}^+$ to denote the set of all real numbers and the set of all nonnegative real numbers, respectively.

Definition 1.1 Let E be a Banach space.
(1) E is said to be *strictly convex* if $\frac{\|x+y\|}{2} < 1$ for all $x, y \in U_E = \{z \in E : \|z\| = 1\}$ with $x \neq y$.
(2) E is said to be *uniformly convex* if for each $\varepsilon \in (0, 2]$, there exists $\delta > 0$ such that $\frac{\|x+y\|}{2} \leq 1 - \delta$ for all $x, y \in U_E$ with $\|x - y\| > \varepsilon$.

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(3) E is said to be *smooth* if the limit (1.2)

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{1.2}$$

exists for each $x, y \in U_E$.

(4) E is said to be *uniformly smooth* if the limit (1.2) is attained uniformly for all $x, y \in U_E$.

Remark 1.2 The basic properties below hold (see [1, 2]).

(1) If E is a real uniformly smooth Banach space, then J is uniformly continuous on each bounded subset of E.

(2) If E is a strictly convex reflexive Banach space, then J^{-1} is hemicontinuous, that is, J^{-1} is norm-to-weak*-continuous.

(3) If E is a smooth and strictly convex reflexive Banach space, then J is single-valued, one-to-one, and onto.

(4) A Banach space E is uniformly smooth if and only if E^* is uniformly convex.

(5) Each uniformly convex Banach space E has the *Kadec-Klee property*; that is, for any sequence $\{x_n\} \subset E$, if $\{x_n\} \rightharpoonup x \in E$ and $\|x_n\| \to \|x\|$, then $x_n \to x$.

(6) A Banach space E is strictly convex if and only if J is strictly monotone; that is,

$$\langle x - y, x^* - y^* \rangle > 0, \quad \text{whenever } x, y \in E, x \neq y, \text{ and } x^* \in Jx, y^* \in Jy.$$

(7) Both uniformly smooth Banach spaces and uniformly convex Banach spaces are reflexive.

(8) E^* is uniformly convex, then J is uniformly norm-to-norm continuous on each bounded subset of E.

Let E be a smooth and strictly convex reflexive Banach space, and let C be a nonempty, closed, and convex subset of E. We assume that the *Lyapunov functional* $\phi : E \times E \to \mathbb{R}^+$ is defined by [3, 4]

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2, \quad \forall x, y \in E.$$

From the definition of ϕ, it is easy to see that

$$\left(\|x\| - \|y\| \right)^2 \leq \phi(x, y) \leq \left(\|x\| + \|y\| \right)^2. \tag{1.3}$$

Let C be a nonempty, closed, and convex subset of E. For each $x \in E$, the *generalized projection* [3] $\Pi_C : E \to C$ is defined by

$$\Pi_C(x) = \arg \min_{y \in C} \phi(x, y).$$

Lemma 1.3 [3, 4] *If C is a nonempty, closed, and convex subset of a smooth and strictly convex reflexive real Banach space E, then*

(1) *for $x \in E$ and $u \in C$, one has*

$$u = \Pi_C(x) \quad \Leftrightarrow \quad \langle u - y, Jx - Ju \rangle \geq 0, \quad \forall y \in C.$$

(2) $\phi(x, \Pi_C(y)) + \phi(\Pi_C(y), y) \le \phi(x, y)$, $\forall x \in C, y \in E$.

(3) $\phi(x, y) = 0$ *if and only if* $x = y$, $\forall x, y \in C$.

Remark 1.4 If E is a real Hilbert space H, then $\phi(x, y) = \|x - y\|^2$ and $\Pi_C = P_C$ (the metric projection of H onto C).

Definition 1.5 Let E be a smooth, strictly convex, and reflexive real Banach space, C be a nonempty, closed, and convex subset of E, $T : C \to C$ be a mapping, and Fix(T) be the set of fixed points of T.

(1) A point $p \in C$ is said to be an *asymptotic fixed point* of T if there exists a sequence $\{x_n\} \subset C$ such that $x_n \rightharpoonup p$ and $\|x_n - Tx_n\| \to 0$. We denote the set of all asymptotic fixed points of T by $\widehat{F}(T)$.

(2) A point $p \in C$ is said to be a *strong asymptotic fixed point* of T if there exists a sequence $\{x_n\} \subset C$ such that $x_n \to p$ and $\|x_n - Tx_n\| \to 0$. We denote the set of all strong asymptotic fixed points of T by $\widetilde{F}(T)$.

Definition 1.6 Let E be a smooth, strictly convex, and reflexive real Banach space, and let C be a nonempty, closed, and convex subset of E.

(1) A mapping $T : C \to C$ is said to be *closed* if for each $\{x_n\} \subset C$ with $x_n \to x$ and $Tx_n \to y$, then $Tx = y$.

(2) A mapping $T : C \to C$ is said to be *relatively nonexpansive* [5, 6] if Fix(T) $\ne \emptyset$, Fix(T) = $\widehat{F}(T)$, and

$$\phi(p, Tx) \le \phi(p, x), \quad \forall x \in C, p \in \text{Fix}(T).$$

(3) A mapping $T : C \to C$ is said to be *weak relatively nonexpansive* [7] if Fix(T) $\ne \emptyset$, Fix(T) = $\widetilde{F}(T)$, and

$$\phi(p, Tx) \le \phi(p, x), \quad \forall x \in C, p \in \text{Fix}(T).$$

(4) A mapping $T : C \to C$ is said to be *quasi-ϕ-nonexpansive* (*relatively quasi-nonexpansive*) if Fix(T) $\ne \emptyset$ and

$$\phi(p, Tx) \le \phi(p, x), \quad \forall x \in C, p \in \text{Fix}(T).$$

(5) A mapping $T : C \to C$ is said to be *quasi-ϕ-asymptotically nonexpansive* (*asymptotically relatively nonexpansive*) if Fix(T) $\ne \emptyset$ and there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ such that

$$\phi(p, T^n x) \le k_n \phi(p, x), \quad \forall x \in C, p \in \text{Fix}(T), \text{ and } \forall n \ge 1.$$

(6) A mapping $T : C \to C$ is said to be *total quasi-ϕ-asymptotically nonexpansive* if Fix(T) $\ne \phi$ and there exists nonnegative real sequences $\{v_n\}$ and $\{\mu_n\}$ with $v_n \to 0$, $\mu_n \to 0$ (as $n \to \infty$), and a strictly increasing continuous function $\zeta : \mathbb{R}^+ \to \mathbb{R}^+$ with $\zeta(0) = 0$ such that

$$\phi(p, T^n x) \le \phi(p, x) + v_n \zeta(\phi(p, x)) + \mu_n, \quad \forall x \in C, p \in \text{Fix}(T), \text{ and } \forall n \ge 1.$$

Remark 1.7 From Definition 1.6, it is easy to know that

(1) every relatively nonexpansive mapping is closed;

(2) every quasi-ϕ-asymptotically nonexpansive mapping is a total quasi-ϕ-asymptotically nonexpansive mapping, but the converse is not true;

(3) every quasi-ϕ-nonexpansive mapping is a quasi-ϕ-asymptotically nonexpansive mapping with $\{k_n = 1\}$, but the converse is not true;

(4) every weak relatively nonexpansive mapping is a quasi-ϕ-nonexpansive mapping because it does not require the condition $\mathrm{Fix}(T) = \widetilde{F}(T)$, but the converse is not true;

(5) every relatively nonexpansive mapping is a weak relatively nonexpansive mapping, but the converse is not true.

Regarding the iterative methods of nonlinear operator equations for relatively nonexpansive mappings, in 2005 Matsushita and Takahashi [5] and in 2008 Plubtieng and Ungchittrakool [6] proved the following result, respectively.

Theorem MT *Let E be a uniformly convex and uniformly smooth Banach space, let C be a nonempty closed and convex subset of E. Let $T : C \to C$ be a relatively nonexpansive mapping, and let $\{\alpha_n\}$ be a real sequence in $[0, 1)$ with $\limsup_{n \to \infty} \alpha_n < 1$. Let $\{x_n\}$ be a sequence defined by*

$$
\begin{cases}
x_0 \in C \quad \text{chosen arbitrarily,} \\
y_n = J^{-1}(\alpha_n J x_0 + (1 - \alpha_n) J T x_n), \\
C_n = \{z \in C : \phi(z, y_n) \le \phi(z, x_n)\}, \\
Q_n = \{z \in C : \langle x_n - z, J x_0 - J x_n \rangle \ge 0\}, \\
x_{n+1} = \Pi_{C_n \cap Q_n}(x_0), \quad \forall n \ge 0,
\end{cases}
\tag{1.4}
$$

where J is the duality mapping on E. If $\mathrm{Fix}(T) \ne \emptyset$, then the sequence $\{x_n\}$ converges strongly to $\Pi_{\mathrm{Fix}(T)}(x_0)$, where $\Pi_{\mathrm{Fix}(T)}(\cdot)$ is the generalized projection from C onto $\mathrm{Fix}(T)$.

Theorem PU *Let E be a uniformly convex and uniformly smooth Banach space, let C be a nonempty, closed, and convex subset of E. Let $T, S : C \to C$ be two relatively nonexpansive mappings with $\Omega := \mathrm{Fix}(T) \cap \mathrm{Fix}(S) \ne \emptyset$. Let $\{x_n\}$ be a sequence defined by*

$$
\begin{cases}
x_0 \in C \quad \text{chosen arbitrarily,} \\
y_n = J^{-1}(\alpha_n J x_0 + (1 - \alpha_n) J T z_n), \\
z_n = J^{-1}(\beta_n^{(1)} J x_n + \beta_n^{(2)} J T x_n + \beta_n^{(3)} J S x_n), \\
H_n = \{z \in C : \phi(z, y_n) \le \phi(z, x_n) + \alpha_n(\|x\|^2 + 2\langle z, J x_n - J x_0 \rangle)\}, \\
W_n = \{z \in C : \langle x_n - z, J x_0 - J x_n \rangle \ge 0\}, \\
x_{n+1} = \Pi_{H_n \cap W_n}(x_0), \quad \forall n \ge 0,
\end{cases}
\tag{1.5}
$$

with the following restrictions:

(1) $0 < \alpha_n < 1$ *and* $\limsup_{n \to \infty} \alpha_n < 1$;

(2) $0 \le \beta_n^{(1)}, \beta_n^{(2)}, \beta_n^{(3)} \le 1$, $\lim_{n \to \infty} \beta_n^{(1)} = 0$, *and* $\liminf_{n \to \infty} \beta_n^{(2)} \beta_n^{(3)} > 0$.

Then the sequence $\{x_n\}$ converges strongly to $\Pi_\Omega(x_0)$, where $\Pi_\Omega(\cdot)$ is the generalized projection from C onto Ω.

In 2010, Su *et al.* [7] introduced the concept of a countable family of weak relatively nonexpansive mappings and proved the following theorem which extends and improves Theorem MT and Theorem PU.

Theorem SXZ *Let E be a uniformly convex and uniformly smooth real Banach space, let C be a nonempty, closed, and convex subset of E. Let $\{T_n\}_{n=0}^{\infty}, \{S_n\}_{n=0}^{\infty} : C \to C$ be two countable families of weak relatively nonexpansive mappings such that $\Omega := (\bigcap_{n=0}^{\infty} \mathrm{Fix}(T_n)) \cap (\bigcap_{n=0}^{\infty} \mathrm{Fix}(S_n)) \neq \emptyset$.*

Let $\{x_n\}$ be a sequence defined by

$$
\begin{cases}
x_0 \in C \quad \text{chosen arbitrarily}, \\
y_n = J^{-1}(\alpha_n J x_n + (1 - \alpha_n) J z_n), \\
z_n = J^{-1}(\beta_n^{(1)} J x_n + \beta_n^{(2)} J T_n x_n + \beta_n^{(3)} J S_n x_n), \\
C_n = \{z \in C_{n-1} \cap Q_{n-1} : \phi(z, y_n) \leq \phi(z, x_n)\}, \\
C_0 = \{z \in C : \phi(z, y_0) \leq \phi(z, x_0)\}, \\
Q_n = \{z \in C_{n-1} \cap Q_{n-1} : \langle x_n - z, J x_0 - J x_n \rangle \geq 0\}, \\
Q_0 = C, \\
x_{n+1} = \Pi_{C_n \cap Q_n}(x_0), \quad \forall n \geq 0,
\end{cases}
\tag{1.6}
$$

with the conditions:

  (1) $0 \leq \alpha_n \leq \alpha < 1$ *for some* $\alpha \in (0, 1)$;

  (2) $\beta_n^{(1)}, \beta_n^{(2)}, \beta_n^{(3)} \in (0, 1)$ *such that* $\beta_n^{(1)} + \beta_n^{(2)} + \beta_n^{(3)} = 1$ *for each* $n \geq 1$;

  (3) $\liminf_{n \to \infty} \beta_n^{(1)} \beta_n^{(2)} > 0$ *and* $\liminf_{n \to \infty} \beta_n^{(2)} \beta_n^{(3)} > 0$.

Then the sequence $\{x_n\}$ converges strongly to $\Pi_{\Omega}(x_0)$, where $\Pi_{\Omega}(\cdot)$ is the generalized projection from C onto Ω.

In 2011, Chang *et al.* [8] proved some approximation theorems for common fixed points of countable families of total quasi-ϕ-asymptotically nonexpansive mappings which contain several kinds of mappings as their special cases in Banach spaces. Next, Chang *et al.* [9] modified the Halpern-type iteration algorithm for a total quasi-ϕ-asymptotically nonexpansive mapping to have the strong convergence under a limit condition only in the framework of Banach spaces. Recently, Chang, Lee, and Chan [10] introduced a block hybrid projection algorithm for solving the convex feasibility problem and the generalized equilibrium problems for an infinite family of total quasi-ϕ-asymptotically nonexpansive mappings and they proved strong convergence theorems in a uniformly smooth and strictly convex Banach space with the Kadec-Klee property.

Theorem CLCY *Let E be a uniformly smooth and strictly convex Banach space with the Kadec-Klee property, let C be a nonempty, closed, and convex subset of E, and let $\{T_i\}_{i=1}^{\infty} : C \to C$ be a countable family of closed, uniformly L_i-Lipschitz continuous, and uniformly total quasi-ϕ-asymptotically nonexpansive mappings with nonnegative real sequences $\{\nu_n\}, \{\mu_n\}$, and a strictly increasing continuous function $\zeta : \mathbb{R}^+ \to \mathbb{R}^+$ such that*

$\mu_1 = 0$, $\nu_n \to 0$, $\mu_n \to 0$ *(as $n \to \infty$), and $\zeta(0) = 0$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_0 \in C \quad \text{chosen arbitrarily}, C_0 = C, \\ z_n = J^{-1}(\alpha_n J x_n + (1 - \alpha_n) J z_n), \\ y_n = J^{-1}(\beta_{n,0} J x_n + \sum_{i=1}^{\infty} \beta_{n,i} J T_i^n x_n), \\ C_{n+1} = \{v \in C_n : \phi(v, y_n) \le \phi(v, x_n) + \xi_n\}, \\ x_{n+1} = \Pi_{C_{n+1}}(x_0), \quad \forall n \ge 0, \end{cases} \tag{1.7}$$

where $\xi_n = \nu_n \sup_{p \in \Omega} \zeta(\phi(p, x_n)) + \mu_n$, $\Pi_{C_{n+1}}$ is the generalized projection of E onto C_{n+1}. Let $\{\beta_{n,0}\}$, $\{\beta_{n,i}\}$, and $\{\alpha_n\}$ be sequences in $[0, 1]$ satisfying the following conditions:

  (1) *for each $n \ge 0$, $\beta_{n,0} + \sum_{i=1}^{\infty} \beta_{n,i} = 1$;*
  (2) $\liminf_{n \to \infty} \beta_{n,0} \beta_{n,i} > 0$ *for all $i \ge 1$;*
  (3) $0 \le \alpha_n \le \alpha < 1$ *for some $\alpha \in (0, 1)$.*

If $\Omega := \bigcap_{i=1}^{\infty} \text{Fix}(T_i)$ is a nonempty and bounded subset of C, then the sequence $\{x_n\}$ converges strongly to $\Pi_{\Omega}(x_0)$.

Theorem CLCZ *Let E be a uniformly smooth and strictly convex Banach space with the Kadec-Klee property, let C be a nonempty, closed, and convex subset of E, and let $\{T_m\}_{m=1}^{\infty} : C \to C$ be a countable family of closed, uniformly L_m-Lipschitz continuous, and uniformly total quasi-ϕ-asymptotically nonexpansive mappings with nonnegative real sequences $\{\nu_n\}$, $\{\mu_n\}$, and a strictly increasing continuous function $\zeta : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\mu_1 = 0$, $\nu_n \to 0$, $\mu_n \to 0$ (as $n \to \infty$), and $\zeta(0) = 0$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_1 \in E \quad \text{chosen arbitrarily}, C_1 = C, \\ y_{n,m} = J^{-1}(\alpha_n J x_1 + (1 - \alpha_n) J T_m^n x_n), \quad m \ge 1, \\ C_{n+1} = \{z \in C_n : \sup_{m \ge 1} \phi(z, y_{n,m}) \le \alpha_n \phi(z, x_1) + (1 - \alpha) \phi(z, x_n) + \xi_n\}, \\ x_{n+1} = \Pi_{C_{n+1}}(x_1), \quad \forall n \ge 1, \end{cases} \tag{1.8}$$

where $\xi_n = \nu_n \sup_{p \in \Omega} \zeta(\phi(p, x_n)) + \mu_n$, and $\Pi_{C_{n+1}}$ is the generalized projection of E onto C_{n+1}. If $\Omega := \bigcap_{m=1}^{\infty} \text{Fix}(T_m)$ is a nonempty and bounded subset of C, then the sequence $\{x_n\}$ converges strongly to $\Pi_{\Omega}(x_1)$.

Theorem CLC *Let E be a uniformly smooth and strictly convex Banach space with the Kadec-Klee property, let C be a nonempty, closed, and convex subset of E, and let $\{S_i\}_{i=1}^{\infty} : C \to C$ be a countable family of closed, uniformly L_i-Lipschitz continuous, and uniformly total quasi-ϕ-asymptotically nonexpansive mappings with nonnegative real sequences $\{\nu_n\}$, $\{\mu_n\}$, and a strictly increasing continuous function $\zeta : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\mu_1 = 0$, $\nu_n \to 0$, $\mu_n \to 0$ (as $n \to \infty$), and $\zeta(0) = 0$. Let $B_i : C \to E^*$ ($i = 1, 2, 3, \ldots, N$) be a finite family of continuous and monotone mappings. Let $\psi_i : C \to \mathbb{R}$ ($i = 1, 2, 3, \ldots, N$) be a finite family of lower semi-continuous and convex functions, and let $F_i : C \times C \to \mathbb{R}$ ($i = 1, 2, 3, \ldots, N$) be a finite family of bifunctions satisfying the conditions (A1)-(A4). Suppose that $\Omega := (\bigcap_{i=1}^{N} \mathfrak{F}_i) \cap (\bigcap_{i=1}^{\infty} \text{Fix}(S_i))$ is a nonempty and bounded subset of C, where $\mathfrak{F}_i$ ($i = 1, 2, 3, \ldots, N$) is the set of the following generalized mixed quasi-equilibrium problems:*

$$F_i(u, y) + \langle B_i u, y - u \rangle + \psi_i(y) - \psi_i(u) \ge 0, \quad \forall y \in C, i = 1, 2, 3, \ldots, N.$$

Let $\{x_n\}$ be a sequence generated by

$$
\begin{cases}
x_0 \in C \quad \text{chosen arbitrarily}, C_0 = C, \\
y_n = J^{-1}(\beta_n J x_n + (1 - \beta_n) J z_n), \\
z_n = J^{-1}(\alpha_{n,0} J x_n + \sum_{i=1}^{\infty} \alpha_{n,i} J S_i^n x_n), \\
u_n^{(i)} = K_{f_i, r_i} K_{f_{i-1}, r_{i-1}} \cdots K_{f_2, r_2} K_{f_1, r_1} y_n, \quad i = 1, 2, 3, \ldots, N, \\
C_{n+1} = \{v \in C_n : \max_{i=1,2,3,\ldots,N} \phi(v, u_n^{(i)}) \le \phi(v, x_n) + \xi_n\}, \\
x_{n+1} = \Pi_{C_{n+1}}(x_0), \quad \forall n \ge 0,
\end{cases}
\tag{1.9}
$$

where $\xi_n = \nu_n \sup_{p \in \Omega} \zeta(\phi(u, x_n)) + \mu_n, \forall n \ge 1$, $\Pi_{C_{n+1}}$ is the generalized projection of E onto C_{n+1}. Let $\alpha_{n,i}$, $\{\beta_n\}$ be sequences in $[0,1]$ satisfying the following conditions:

(1) for each $n \ge 0$, $\sum_{i=0}^{\infty} \alpha_{n,i} = 1$;

(2) $\liminf_{n \to \infty} (1 - \beta_n) \alpha_{n,0} \alpha_{n,i} > 0$ for all $i \ge 1$.

Then the sequence $\{x_n\}$ converges strongly to $\Pi_\Omega(x_0)$.

In this paper, motivated and inspired by the previously mentioned results, we introduce a new iterative procedure by the modified block hybrid projection method for solving a common solution of fixed point problems for two countable families of uniformly total quasi-ϕ-asymptotically nonexpansive and uniformly Lipschitz continuous mappings in a uniformly smooth and strictly convex Banach space with the Kadec-Klee property. Then, we prove a strong convergence theorem of the iterative procedure generated by these conditions. The results obtained in this paper extend and improve several recent results in this area.

2 Preliminaries

Definition 2.1 Let C be a nonempty, closed, and convex subset of a real Banach space E.

(1) A mapping $T : C \to C$ is said to be *nonexpansive* if

$$
\|Tx - Ty\| \le \|x - y\|, \quad \forall x, y \in C.
$$

(2) A mapping $T : C \to C$ is said to be *uniformly L-Lipschitz continuous* if there exists a constant $L > 0$ such that

$$
\|T^n x - T^n y\| \le L \|x - y\|, \quad \forall x, y \in C, \forall n \ge 1.
$$

Definition 2.2 [11] Let C be a nonempty, closed, and convex subset of a real Banach space E.

(1) A countable family of mappings $\{T_i\}_{i=1}^{\infty}$ is said to be a *uniformly quasi-ϕ-asymptotically nonexpansive mapping* if $\bigcap_{i=1}^{\infty} \text{Fix}(T_i) \ne \emptyset$ and there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ such that for each $i \ge 1$,

$$
\phi(p, T_i^n x) \le k_n \phi(p, x), \quad \forall x \in C, p \in \bigcap_{i=1}^{\infty} \text{Fix}(T_i), \text{ and } \forall n \ge 1.
$$

(2) A countable family of mappings $\{T_i\}_{i=1}^{\infty}$ is said to be a *uniformly total quasi-ϕ-asymptotically nonexpansive mapping* if $\bigcap_{i=1}^{\infty} \text{Fix}(T_i) \ne \emptyset$ and there exist

nonnegative real sequences $\{v_n\}$ and $\{\mu_n\}$ with $v_n \to 0$, $\mu_n \to 0$ (as $n \to \infty$), and a strictly increasing continuous function $\zeta : \mathbb{R}^+ \to \mathbb{R}^+$ with $\zeta(0) = 0$ such that for each $i \geq 1$,

$$\phi(p, T_i^n x) \leq \phi(p, x) + v_n \zeta(\phi(p, x)) + \mu_n, \quad \forall x \in C, p \in \bigcap_{i=1}^{\infty} \text{Fix}(T_i), \text{ and } \forall n \geq 1.$$

Lemma 2.3 [4] *Let E be a uniformly smooth and strictly convex real Banach space, and let $\{x_n\}$ and $\{y_n\}$ be two sequences of E. If $\phi(x_n, y_n) \to 0$ and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $\|x_n - y_n\| \to 0$.*

Lemma 2.4 [8] *Let E be a uniformly smooth and strictly convex real Banach space with the Kadec-Klee property, and let C be a nonempty, closed, and convex subset of E. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in C and $p \in E$. If $x_n \to p$ and $\phi(x_n, y_n) \to 0$, then $y_n \to p$.*

Lemma 2.5 [8] *Let E be a uniformly smooth and strictly convex real Banach space with the Kadec-Klee property, and let C be a nonempty, closed, and convex subset of E. Let $T : C \to C$ be a closed and total quasi-ϕ-asymptotically nonexpansive mapping with nonnegative real sequences $\{v_n\}$, $\{\mu_n\}$, and a strictly increasing continuous function $\zeta : \mathbb{R}^+ \to \mathbb{R}^+$ such that $v_n \to 0$, $\mu_n \to 0$ (as $n \to \infty$), and $\zeta(0) = 0$. If $\mu_1 = 0$, then the fixed point set $\text{Fix}(T)$ is a closed and convex subset of C.*

Lemma 2.6 [11] *Let E be a uniformly convex Banach space, $r > 0$ be a positive number, and $B_r(0)$ be a closed ball of E. Then for any given sequence $\{x_n\}_{n=1}^{\infty} \subset B_r(0)$ and for any given $\{\lambda_n\}_{n=1}^{\infty} \subset (0,1)$ with $\sum_{n=1}^{\infty} \lambda_n = 1$, there exists a continuous, strictly increasing, and convex function $g : [0, 2r) \to [0, \infty)$ with $g(0) = 0$ such that for any positive integers i, j with $i < j$,*

$$\left\| \sum_{n=1}^{\infty} \lambda_n x_n \right\|^2 \leq \sum_{n=1}^{\infty} \lambda_n \|x_n\|^2 - \lambda_i \lambda_j g(\|x_i - x_j\|).$$

3 Main results

In this section, we shall use the modified block hybrid projection method to study a common solution of fixed point problems for two countable families of closed and L_i, ℓ_i-Lipschitz continuous and uniformly total quasi-ϕ-asymptotically nonexpansive mappings in Banach spaces. For the purpose, we assume the following hypotheses.

(A1) Let $\{T_i\}_{i=1}^{\infty} : C \to C$ be a countable family of closed, uniformly L_i-Lipschitz continuous, and uniformly total quasi-ϕ-asymptotically nonexpansive mappings with nonnegative real sequences $\{v_n\}$, $\{\mu_n\}$ and a strictly increasing function $\zeta : \mathbb{R}^+ \to \mathbb{R}^+$ such that $v_n \to 0$, $\mu_n \to 0$ (as $n \to \infty$), $\mu_1 = 0$, and $\zeta(0) = 0$, and for each $i \geq 1$,

$$\phi(p, T_i^n x) \leq \phi(p, x) + v_n \zeta(\phi(p, x)) + \mu_n, \quad \forall x \in C, p \in \bigcap_{i=1}^{\infty} \text{Fix}(T_i), \text{ and } \forall n \geq 1.$$

(A2) Let $\{S_i\}_{i=1}^{\infty} : C \to C$ be a countable family of closed, uniformly ℓ_i-Lipschitz continuous, and uniformly total quasi-ϕ-asymptotically nonexpansive mappings with nonnegative real sequences $\{\kappa_n\}$, $\{\omega_n\}$ and a strictly increasing function $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$\kappa_n \to 0$, $\omega_n \to 0$ (as $n \to \infty$), $\omega_1 = 0$, and $\rho(0) = 0$, and for each $i \geq 1$,

$$\phi(p, S_i^n x) \leq \phi(p, x) + \kappa_n \rho(\phi(p, x)) + \omega_n, \quad \forall x \in C, p \in \bigcap_{i=1}^{\infty} \mathrm{Fix}(S_i), \text{ and } \forall n \geq 1.$$

Theorem 3.1 *Let E be a uniformly smooth and strictly convex real Banach space with the Kadec-Klee property, let C be a nonempty, closed, and convex subset of E, and let $\{T_i\}_{i=1}^{\infty}$ and $\{S_i\}_{i=1}^{\infty}$ satisfy the above conditions* (A1)-(A2), *respectively.*

Suppose that $\Omega := (\bigcap_{i=1}^{\infty} \mathrm{Fix}(T_i)) \cap (\bigcap_{i=1}^{\infty} \mathrm{Fix}(S_i))$ is nonempty and bounded in C. Let $\{x_n\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty}$, and $\{z_n\}_{n=1}^{\infty}$ be sequences generated by

$$\begin{cases} x_0 \in C \quad \text{chosen arbitrarily, } C_0 = C, \\ z_n = J^{-1}(\alpha_{n,0} J x_n + \sum_{i=1}^{\infty} \alpha_{n,i} J S_i^n y_n), \\ y_n = J^{-1}(\beta_{n,0} J x_n + \sum_{i=1}^{\infty} \beta_{n,i} J T_i^n x_n), \\ C_{n+1} = \{v \in C_n : \phi(v, y_n) \leq \phi(v, x_n) + \xi_n \text{ and } \phi(v, z_n) \leq \phi(v, x_n) + \theta_n\}, \\ x_{n+1} = \Pi_{C_{n+1}}(x_0), \quad \forall n \geq 0, \end{cases} \tag{3.1}$$

where $\xi_n = \nu_n \sup_{p \in \Omega} \zeta(\phi(p, x_n)) + \mu_n$ and $\theta_n = \kappa_n \sup_{p \in \Omega} \rho(\phi(p, x_n)) + \omega_n$.

Let $\{\alpha_{n,i}\}$ and $\{\beta_{n,i}\}$ be coefficient sequences in $[0, 1]$ satisfying the following conditions:

(1) $\sum_{i=0}^{\infty} \alpha_{n,i} = 1$ *and* $\sum_{i=0}^{\infty} \beta_{n,i} = 1$;

(2) $\liminf_{n \to \infty} \alpha_{n,0} \alpha_{n,i} > 0$, *and* $\liminf_{n \to \infty} \beta_{n,0} \beta_{n,i} > 0$, $\forall i \geq 1$.

Then the sequence $\{x_n\}_{n=1}^{\infty}$ converges strongly to some point p^*, where $p^* = \Pi_{\Omega}(x_0)$.

Proof We shall complete this proof of Theorem 3.1 in seven steps.

Step 1. We will show that Ω and C_{n+1} are closed and convex for each $n \geq 0$.

In fact, it follows from Lemma 2.5 that $\mathrm{Fix}(T_i)$ and $\mathrm{Fix}(S_i)$, for any $i \geq 1$, are closed and convex subsets of C. Therefore, Ω is closed and convex in C.

Clearly, $C_0 = C$ is closed and convex. Suppose that C_n is closed and convex for some $n \geq 1$.

By the assumption of C_{n+1},

$$\phi(v, y_n) \leq \phi(v, x_n) + \xi_n$$

is equivalent to

$$\|v\|^2 - 2\langle v, J y_n \rangle + \|y_n\|^2 \leq \|v\|^2 - 2\langle v, J x_n \rangle + \|x_n\|^2 + \xi_n.$$

So that $2\langle v, J x_n \rangle - 2\langle v, J y_n \rangle = 2\langle v, J x_n - J y_n \rangle \leq \|x_n\|^2 - \|y_n\|^2 + \xi_n$.

Again, by the assumption of C_{n+1},

$$\phi(v, z_n) \leq \phi(v, x_n) + \theta_n$$

is equivalent to

$$\|v\|^2 - 2\langle v, J z_n \rangle + \|z_n\|^2 \leq \|v\|^2 - 2\langle v, J x_n \rangle + \|x_n\|^2 + \theta_n.$$

So that $2\langle v, J x_n \rangle - 2\langle v, J z_n \rangle = 2\langle v, J x_n - J z_n \rangle \leq \|x_n\|^2 - \|z_n\|^2 + \theta_n$.

Hence, $C_{n+1} = \{v \in C_n : 2\langle v, Jx_n - Jy_n\rangle \leq \|x_n\|^2 - \|y_n\|^2 + \xi_n$ and $2\langle v, Jx_n - Jz_n\rangle \leq \|x_n\|^2 - \|z_n\|^2 + \theta_n\}$ is closed and convex.

Step 2. We will show that $\{x_n\}$ is bounded and $\{\phi(x_n, x_0)\}$ is a convergent sequence for all $n \geq 1$.

Indeed, it follows from (3.1) and Lemma 1.3(2) that

$$
\begin{aligned}
\phi(x_n, x_0) &= \phi\big(\Pi_{C_n}(x_0), x_0\big) \\
&\leq \phi(p, x_0) - \phi\big(p, \Pi_{C_n}(x_0)\big) \\
&\leq \phi(p, x_0), \quad \forall n \geq 0, p \in \Omega.
\end{aligned}
$$

This implies that $\{\phi(x_n, x_0)\}$ is bounded. By virtue of (1.3), the sequence $\{x_n\}$ is also bounded.

By the assumption of C_n, we have $C_{n+1} \subset C_n$, $x_n = \Pi_{C_n}(x_0)$ and $x_{n+1} = \Pi_{C_{n+1}}(x_0)$.

This implies that $x_{n+1} \in C_{n+1} \subset C_n$ and

$$
\phi(x_n, x_0) \leq \phi(x_{n+1}, x_0), \quad \forall n \geq 0.
$$

Therefore, $\{\phi(x_n, x_0)\}$ is a convergent sequence. Without loss of generality, we can assume that

$$
\lim_{n \to \infty} \phi(x_n, x_0) = d \geq 0. \tag{3.2}
$$

Step 3. We will show that $\Omega \subset C_n$ for all $n \geq 0$.

It is obvious that $\Omega \subset C_0 = C$. Suppose that $\Omega \subset C_n$ for some $n \geq 1$. For any given $p \in \Omega$, from (3.1) and Lemma 2.6, we compute

$$
\begin{aligned}
\phi(p, y_n) &= \phi\left(p, J^{-1}\left(\beta_{n,0} Jx_n + \sum_{i=1}^{\infty} \beta_{n,i} JT_i^n x_n\right)\right) \\
&= \|p\|^2 - 2\left\langle p, \beta_{n,0} Jx_n + \sum_{i=1}^{\infty} \beta_{n,i} JT_i^n x_n\right\rangle + \left\|\beta_{n,0} Jx_n + \sum_{i=1}^{\infty} \beta_{n,i} JT_i^n x_n\right\|^2 \\
&\leq \|p\|^2 - 2\beta_{n,0}\langle p, Jx_n\rangle - 2\sum_{i=1}^{\infty} \beta_{n,i}\langle p, JT_i^n x_n\rangle \\
&\quad + \beta_{n,0}\|x_n\|^2 + \sum_{i=1}^{\infty} \beta_{n,i}\|T_i^n x_n\|^2 \\
&\quad - \beta_{n,0}\beta_{n,i} g\big(\|Jx_n - JT_i^n x_n\|\big) \\
&= \beta_{n,0}\phi(p, x_n) + (1 - \beta_{n,0})\|p\|^2 \\
&\quad - 2\sum_{i=1}^{\infty} \beta_{n,i}\langle p, JT_i^n x_n\rangle + \sum_{i=1}^{\infty} \beta_{n,i}\|T_i^n x_n\|^2 \\
&\quad - \beta_{n,0}\beta_{n,i} g\big(\|Jx_n - JT_i^n x_n\|\big) \\
&= \beta_{n,0}\phi(p, x_n) + \sum_{i=1}^{\infty} \beta_{n,i}\phi\big(p, T_i^n x_n\big) - \beta_{n,0}\beta_{n,i} g\big(\|Jx_n - JT_i^n x_n\|\big)
\end{aligned}
$$

$$\leq \beta_{n,0}\phi(p,x_n) + \sum_{i=1}^{\infty} \beta_{n,i}\big[\phi(p,x_n) + \nu_n\zeta\big(\phi(p,x_n)\big) + \mu_n\big]$$

$$- \beta_{n,0}\beta_{n,i}g\big(\big\|Jx_n - JT_i^n x_n\big\|\big)$$

$$= \left(\beta_{n,0} + \sum_{i=1}^{\infty}\beta_{n,i}\right)\phi(p,x_n) + \sum_{i=1}^{\infty}\beta_{n,i}\big[\nu_n\zeta\big(\phi(p,x_n)\big) + \mu_n\big]$$

$$- \beta_{n,0}\beta_{n,i}g\big(\big\|Jx_n - JT_i^n x_n\big\|\big)$$

$$\leq \phi(p,x_n) + \nu_n\zeta\big(\phi(p,x_n)\big) + \mu_n - \beta_{n,0}\beta_{n,i}g\big(\big\|Jx_n - JT_i^n x_n\big\|\big)$$

$$\leq \phi(p,x_n) + \nu_n\sup_{p\in\Omega}\zeta\big(\phi(p,x_n)\big) + \mu_n - \beta_{n,0}\beta_{n,i}g\big(\big\|Jx_n - JT_i^n x_n\big\|\big)$$

$$= \phi(p,x_n) + \xi_n - \beta_{n,0}\beta_{n,i}g\big(\big\|Jx_n - JT_i^n x_n\big\|\big). \tag{3.3}$$

It follows that

$$\phi(p,y_n) \leq \phi(p,x_n) + \xi_n, \quad \text{where } \xi_n = \nu_n\sup_{p\in\Omega}\zeta\big(\phi(p,x_n)\big) + \mu_n. \tag{3.4}$$

From (3.1) and Lemma 2.6, we compute

$$\phi(p,z_n) = \phi\left(p, J^{-1}\left(\alpha_{n,0}Jx_n + \sum_{i=1}^{\infty}\alpha_{n,i}JS_i^n y_n\right)\right)$$

$$= \|p\|^2 - 2\left\langle p, \alpha_{n,0}Jx_n + \sum_{i=1}^{\infty}\alpha_{n,i}JS_i^n y_n\right\rangle + \left\|\alpha_{n,0}Jx_n + \sum_{i=1}^{\infty}\alpha_{n,i}JS_i^n y_n\right\|^2$$

$$\leq \|p\|^2 - 2\alpha_{n,0}\langle p, Jx_n\rangle - 2\sum_{i=1}^{\infty}\alpha_{n,i}\langle p, JS_i^n y_n\rangle + \alpha_{n,0}\|x_n\|^2 + \sum_{i=1}^{\infty}\alpha_{n,i}\big\|S_i^n y_n\big\|^2$$

$$- \alpha_{n,0}\alpha_{n,i}g\big(\big\|Jy_n - JS_i^n y_n\big\|\big)$$

$$= \alpha_{n,0}\phi(p,x_n) + (1-\alpha_{n,0})\|p\|^2 - 2\sum_{i=1}^{\infty}\alpha_{n,i}\langle p, JS_i^n y_n\rangle + \sum_{i=1}^{\infty}\alpha_{n,i}\big\|S_i^n y_n\big\|^2$$

$$- \alpha_{n,0}\alpha_{n,i}g\big(\big\|Jy_n - JS_i^n y_n\big\|\big)$$

$$= \alpha_{n,0}\phi(p,x_n) + \sum_{i=1}^{\infty}\alpha_{n,i}\phi(p, S_i^n y_n) - \alpha_{n,0}\alpha_{n,i}g\big(\big\|Jy_n - JS_i^n y_n\big\|\big)$$

$$\leq \alpha_{n,0}\phi(p,x_n) + \sum_{i=1}^{\infty}\alpha_{n,i}\big[\phi(p,x) + \kappa_n\rho\big(\phi(p,x)\big) + \omega_n\big]$$

$$- \alpha_{n,0}\alpha_{n,i}g\big(\big\|Jy_n - JS_i^n y_n\big\|\big)$$

$$= \left(\alpha_{n,0} + \sum_{i=1}^{\infty}\alpha_{n,i}\right)\phi(p,x_n) + \sum_{i=1}^{\infty}\alpha_{n,i}\big[\kappa_n\rho\big(\phi(p,x)\big) + \omega_n\big]$$

$$- \alpha_{n,0}\alpha_{n,i}g\big(\big\|Jy_n - JS_i^n y_n\big\|\big)$$

$$\leq \phi(p,x_n) + \kappa_n\rho\big(\phi(p,x)\big) + \omega_n - \alpha_{n,0}\alpha_{n,i}g\big(\big\|Jy_n - JS_i^n y_n\big\|\big)$$

$$\leq \phi(p,x_n) + \kappa_n\sup_{p\in\Omega}\rho\big(\phi(p,x)\big) + \omega_n - \alpha_{n,0}\alpha_{n,i}g\big(\big\|Jy_n - JS_i^n y_n\big\|\big)$$

$$= \phi(p,x_n) + \theta_n - \alpha_{n,0}\alpha_{n,i}g\big(\big\|Jy_n - JS_i^n y_n\big\|\big). \tag{3.5}$$

It follows that

$$\phi(p, z_n) \leq \phi(p, x_n) + \theta_n, \quad \text{where } \theta_n = \kappa_n \sup_{p \in \Omega} \rho\big(\phi(p, x)\big) + \omega_n. \tag{3.6}$$

By the assumptions of $\{v_n\}$, $\{\mu_n\}$, $\{\kappa_n\}$, and $\{\omega_n\}$, and from (3.3) and (3.5), we obtain

$$\xi_n = v_n \sup_{p \in \Omega} \zeta\big(\phi(p, x_n)\big) + \mu_n \to 0 \quad (\text{as } n \to \infty), \tag{3.7}$$

and

$$\theta_n = \kappa_n \sup_{p \in \Omega} \rho\big(\phi(p, x_n)\big) + \omega_n \to 0 \quad (\text{as } n \to \infty). \tag{3.8}$$

So, we get $p \in C_{n+1}$. This implies that $\Omega \subset C_n$ for all $n \geq 0$, and the sequence $\{x_n\}$ is well defined.

Step 4. We will show that there exists some point $p^* \in C$ such that $x_n \to p^*$.

In fact, since $\{x_n\}$ is bounded and E is reflexive, then there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $x_{n_i} \rightharpoonup p^*$ (some point in C).

Since C_n is closed and convex and $C_{n+1} \subset C_n$, it follows that C_n is weakly closed and $p^* \in C_n$ for each $n \geq 0$.

In view of $x_{n_i} = \Pi_{C_{n_i}}(x_0)$, we have

$$\phi(x_{n_i}, x_0) \leq \phi\big(p^*, x_0\big), \quad \forall n_i \geq 0.$$

Since the norm $\|\cdot\|$ is weakly lower semi-continuous, we have

$$\liminf_{n_i \to \infty} \phi(x_{n_i}, x_0) = \liminf_{n_i \to \infty} \big\{\|x_{n_i}\|^2 - 2\langle x_{n_i}, Jx_0 \rangle + \|x_0\|^2\big\}$$
$$\geq \|p^*\|^2 - 2\langle p^*, Jx_0 \rangle + \|x_0\|^2 = \phi\big(p^*, x_0\big),$$

and so

$$\phi\big(p^*, x_0\big) \leq \liminf_{n_i \to \infty} \phi(x_{n_i}, x_0) \leq \limsup_{n_i \to \infty} \phi(x_{n_i}, x_0) \leq \phi\big(p^*, x_0\big).$$

This implies that $\lim_{n_i \to \infty} \phi(x_{n_i}, x_0) = \phi(p^*, x_0)$, and so $\|x_{n_i}\| \to \|p^*\|$. Since $x_{n_i} \rightharpoonup p^*$, and by virtue of the Kadec-Klee property of E, we obtain

$$\lim_{n_i \to \infty} x_{n_i} = p^*, \quad \text{as } n_i \to \infty.$$

The sequence $\{\phi(x_n, x_0)\}$ is convergent, and $\lim_{n_i \to \infty} \phi(x_{n_i}, x_0) = \phi(p^*, x_0)$, which implies that $\lim_{n \to \infty} \phi(x_n, x_0) = \phi(p^*, x_0)$. If there exists some subsequence $\{x_{n_j}\} \subset \{x_n\}$ such that $x_{n_j} \rightharpoonup q$, then from Lemma 1.3(2) we have

$$\phi\big(p^*, q\big) = \lim_{n_i, n_j \to \infty} \phi(x_{n_i}, x_{n_j})$$
$$= \lim_{n_i, n_j \to \infty} \phi\big(x_{n_i}, \Pi_{C_{n_j}}(x_0)\big)$$

$$\leq \lim_{n_i, n_j \to \infty} \left[\phi(x_{n_i}, x_0) - \phi\left(\Pi_{C_{n_j}}(x_0), x_0\right) \right]$$

$$= \lim_{n_i, n_j \to \infty} \left[\phi(x_{n_i}, x_0) - \phi(x_{n_j}, x_0) \right]$$

$$= \phi(p^*, x_0) - \phi(p^*, x_0) = 0.$$

This implies that $p^* = q$, and so

$$\lim_{n \to \infty} x_n = p^*. \tag{3.9}$$

Step 5. We will show that $\lim_{n \to \infty} \|Jx_n - Jy_n\| = 0$ and $\lim_{n \to \infty} \|Jx_n - Jz_n\| = 0$ as $n \to \infty$. Since $x_{n+1} \in C_{n+1} \subset C_n$, by the definition of C_{n+1}, we have

$$\phi(x_{n+1}, x_n) = \phi\left(x_{n+1}, \Pi_{C_n}(x_0)\right)$$

$$\leq \phi(x_{n+1}, x_0) - \phi\left(\Pi_{C_n}(x_0), x_0\right)$$

$$= \phi(x_{n+1}, x_0) - \phi(x_n, x_0). \tag{3.10}$$

Since $\lim_{n \to \infty} \phi(x_n, x_0)$ exists, and we are taking $n \to \infty$ in (3.10), then $\phi(x_{n+1}, x_n) \to 0$.
It follows from Lemma 2.3 that

$$\lim_{n \to \infty} \|x_{n+1} - x_n\| = 0. \tag{3.11}$$

By the definition of C_{n+1} and $x_{n+1} \in C_{n+1}$, we get

$$\phi(x_{n+1}, y_n) \leq \phi(x_{n+1}, x_n) + \xi_n, \quad \text{and}$$
$$\phi(x_{n+1}, z_n) \leq \phi(x_{n+1}, x_n) + \theta_n. \tag{3.12}$$

From $\phi(x_{n+1}, x_n) \to 0$, $\xi_n \to 0$, and $\theta_n \to 0$, as $n \to \infty$, we obtain

$$\phi(x_{n+1}, y_n) \to 0, \quad \text{and} \quad \phi(x_{n+1}, z_n) \to 0, \quad \text{as } n \to \infty. \tag{3.13}$$

Since $\lim_{n \to \infty} x_n = p^*$, by virtue of Lemma 2.4, we get

$$\lim_{n \to \infty} y_n = p^*, \quad \text{and} \quad \lim_{n \to \infty} z_n = p^*. \tag{3.14}$$

It follows from (3.9) and (3.14) that

$$\lim_{n \to \infty} \|x_n - y_n\| = 0, \quad \text{and} \quad \lim_{n \to \infty} \|x_n - z_n\| = 0. \tag{3.15}$$

Since J is uniformly continuous on each bounded subset of E, then

$$\lim_{n \to \infty} \|Jx_n - Jy_n\| = 0, \quad \text{and} \quad \lim_{n \to \infty} \|Jx_n - Jz_n\| = 0. \tag{3.16}$$

Step 6. We will show that $p^* \in \Omega$, where $\Omega := \left(\bigcap_{i=1}^{\infty} \text{Fix}(T_i)\right) \cap \left(\bigcap_{i=1}^{\infty} \text{Fix}(S_i)\right)$.
(6.1) First, we will show that $p^* \in \bigcap_{i=1}^{\infty} \text{Fix}(T_i)$.

For any $i \geq 1$ and for any $p \in \Omega$, it follows from (3.3), (3.7), (3.15), and (3.16) that

$$
\begin{aligned}
\beta_{n,0}\beta_{n,i}g\left(\left\|Jx_n - JT_i^n x_n\right\|\right) &\leq \phi(p, x_n) - \phi(p, y_n) + \xi_n \\
&= \|x_n\|^2 - \|y_n\|^2 - 2\langle p, Jx_n - Jy_n\rangle + \xi_n \\
&= \left(\|x_n\| + \|y_n\|\right)\left(\|x_n\| - \|y_n\|\right) - 2\langle p, Jx_n - Jy_n\rangle + \xi_n \\
&\leq \left(\|x_n\| + \|y_n\|\right)\left(\|x_n - y_n\|\right) - 2\|p\|\|Jx_n - Jy_n\| + \xi_n \\
&\to 0, \quad \text{as } n \to \infty.
\end{aligned}
$$

By the condition $\liminf_{n\to\infty} \beta_{n,0}\beta_{n,i} > 0, \forall i \geq 1$, we obtain

$$
g\left(\left\|Jx_n - JT_i^n x_n\right\|\right) \to 0, \quad \text{as } n \to \infty.
$$

It follows from the property of g that

$$
\left\|Jx_n - JT_i^n x_n\right\| \to 0, \quad \text{as } n \to \infty. \tag{3.17}
$$

Since $x_n \to p^*$ and J is uniformly continuous on each bounded subset of E, it yields that $Jx_n \to Jp^*$.

Hence, from (3.17) we get

$$
JT_i^n x_n \to Jp^*, \quad \text{as } n \to \infty, \forall i \geq 1. \tag{3.18}
$$

Since $J^{-1}: E^* \to E$ is norm-to-*weak**-continuous, we also have

$$
T_i^n x_n \rightharpoonup p^*, \quad \text{as } n \to \infty, \forall i \geq 1. \tag{3.19}
$$

Again, since for any $i \geq 1$,

$$
\left|\left\|T_i^n x_n\right\| - \left\|p^*\right\|\right| = \left|\left\|J\left(T_i^n x_n\right)\right\| - \left\|Jp^*\right\|\right| \leq \left\|J\left(T_i^n x_n\right) - Jp^*\right\| \to 0, \quad \text{as } n \to \infty,
$$

from (3.19) and the Kadec-Klee property of E, it follows that

$$
\lim_{n\to\infty} T_i^n x_n = p^*. \tag{3.20}
$$

On the other hand, by the assumption that for each $i \geq 1$, T_i is uniformly L_i-Lipschitz continuous, we get

$$
\begin{aligned}
\left\|T_i^{n+1}x_n - T_i^n x_n\right\| &\leq \left\|T_i^{n+1}x_n - T_i^{n+1}x_{n+1}\right\| + \left\|T_i^{n+1}x_{n+1} - x_{n+1}\right\| \\
&\quad + \|x_{n+1} - x_n\| + \left\|x_n - T_i^n x_n\right\| \\
&\leq (L_i + 1)\|x_{n+1} - x_n\| + \left\|T_i^{n+1}x_{n+1} - x_{n+1}\right\| + \left\|x_n - T_i^n x_n\right\|.
\end{aligned}
$$

It follows from (3.9), (3.11), and (3.20) that

$$
\lim_{n\to\infty} \left\|T_i^{n+1}x_n - T_i^n x_n\right\| = 0, \quad \text{and} \quad \lim_{n\to\infty} T_i^{n+1}x_n = p^*,
$$

and so

$$\lim_{n \to \infty} T_i^{n+1} x_n = \lim_{n \to \infty} T_i T_i^n x_n = \lim_{n \to \infty} T_i p^* = p^*.$$

In view of (3.20) and the closeness of T_i, it yields that $T_i p^* = p^*$ for all $i \geq 1$. This implies that

$$p^* \in \bigcap_{i=1}^{\infty} \text{Fix}(T_i). \tag{3.21}$$

(6.2) Next, we will show that $p^* \in \bigcap_{i=1}^{\infty} \text{Fix}(S_i)$.

For any $i \geq 1$ and for any $p \in \Omega$, it follows from (3.5), (3.8), (3.15), and (3.16) that

$$
\begin{aligned}
\alpha_{n,0} \alpha_{n,i} g\left(\left\| Jy_n - JS_i^n y_n \right\|\right) &\leq \phi(p, x_n) - \phi(p, z_n) + \theta_n \\
&= \|x_n\|^2 - \|z_n\|^2 - 2\langle p, Jx_n - Jz_n \rangle + \theta_n \\
&= \left(\|x_n\| + \|z_n\|\right)\left(\|x_n\| - \|z_n\|\right) - 2\langle p, Jx_n - Jz_n \rangle + \theta_n \\
&\leq \left(\|x_n\| + \|z_n\|\right)\left(\|x_n - z_n\|\right) - 2\|p\|\|Jx_n - Jz_n\| + \theta_n \\
&\to 0, \quad \text{as } n \to \infty.
\end{aligned}
$$

By the condition $\liminf_{n \to \infty} \alpha_{n,0} \alpha_{n,i} > 0$, $\forall i \geq 1$, we obtain

$$g\left(\left\| Jy_n - JS_i^n y_n \right\|\right) \to 0, \quad \text{as } n \to \infty.$$

It follows from the property of g that

$$\left\| Jy_n - JS_i^n y_n \right\| \to 0, \quad \text{as } n \to \infty. \tag{3.22}$$

Since $y_n \to p^*$ and J is uniformly continuous on each bounded subset of E, it yields that $Jy_n \to Jp^*$.

Hence, from (3.22) we have

$$JS_i^n y_n \to Jp^*, \quad \text{as } n \to \infty, \forall i \geq 1. \tag{3.23}$$

Since $J^{-1} : E^* \to E$ is norm-to-*weak**-continuous, we also have

$$S_i^n y_n \rightharpoonup p^* \quad \text{as } n \to \infty, \forall i \geq 1. \tag{3.24}$$

Again, since for any $i \geq 1$,

$$\left| \left\| S_i^n y_n \right\| - \left\| p^* \right\| \right| = \left| \left\| J\left(S_i^n y_n\right) \right\| - \left\| Jp^* \right\| \right| \leq \left\| J\left(S_i^n y_n\right) - Jp^* \right\| \to 0, \quad \text{as } n \to \infty,$$

from (3.24) and the Kadec-Klee property of E, it follows that

$$\lim_{n \to \infty} S_i^n y_n = p^*. \tag{3.25}$$

On the other hand, by the assumption that for each $i \geq 1$, S_i is uniformly ℓ_i-Lipschitz continuous, we have

$$
\begin{aligned}
\left\| S_i^{n+1} y_n - S_i^n y_n \right\| &\leq \left\| S_i^{n+1} y_n - S_i^{n+1} y_{n+1} \right\| + \left\| S_i^{n+1} y_{n+1} - x_{n+1} \right\| + \| x_{n+1} - x_n \| \\
&\quad + \| x_n - y_n \| + \left\| y_n - S_i^n y_n \right\| \\
&\leq \ell_i \| y_n - y_{n+1} \| + \left\| S_i^{n+1} y_{n+1} - x_{n+1} \right\| + \| x_{n+1} - x_n \| \\
&\quad + \| x_n - y_n \| + \left\| y_n - S_i^n y_n \right\|.
\end{aligned}
$$

From $x_n \to p^*$, $y_n \to p^*$ and $S_i^n y_n \to p^*$, as $n \to \infty$, we obtain

$$
\| y_n - y_{n+1} \| \to 0, \qquad \left\| S_i^{n+1} y_{n+1} - x_{n+1} \right\| \to 0, \quad \text{and} \quad \left\| y_n - S_i^n y_n \right\| \to 0, \quad \text{as } n \to \infty.
$$

And it follows from (3.11) and (3.15) that

$$
\lim_{n \to \infty} \left\| S_i^{n+1} y_n - S_i^n y_n \right\| = 0, \quad \text{and} \quad \lim_{n \to \infty} S_i^{n+1} y_n = p^*,
$$

and so

$$
\lim_{n \to \infty} S_i^{n+1} y_n = \lim_{n \to \infty} S_i S_i^n y_n = \lim_{n \to \infty} S_i p^* = p^*.
$$

In view of (3.25) and the closeness of S_i, it yields that $S_i p^* = p^*$ for all $i \geq 1$. This implies that

$$
p^* \in \bigcap_{i=1}^{\infty} \operatorname{Fix}(S_i). \tag{3.26}
$$

From (3.21) and (3.26), we can conclude that $p^* \in \Omega := (\bigcap_{i=1}^{\infty} \operatorname{Fix}(T_i)) \cap (\bigcap_{i=1}^{\infty} \operatorname{Fix}(S_i))$.

Step 7. Finally, we will show that $x_n \to p^* = \Pi_\Omega(x_0)$.

Let $w = \Pi_\Omega(x_0)$. Since $w \in \Omega \subset C_n$ and $x_n = \Pi_{C_n}(x_0)$, we have

$$
\phi(x_n, x_0) \leq \phi(w, x_0), \quad \forall n \geq 0.
$$

This implies that

$$
\phi(p^*, x_0) = \lim_{n \to \infty} \phi(x_n, x_0) \leq \phi(w, x_0). \tag{3.27}
$$

In view of the definition of $\Pi_\Omega(x_0)$, from (3.27), we have $p^* = w$. Therefore, $x_n \to p^* = \Pi_\Omega(x_0)$.

This completes the proof of Theorem 3.1. $\qquad \square$

Theorem 3.2 *Let E be a uniformly smooth and strictly convex real Banach space with the Kadec-Klee property, and let C be a nonempty, closed, and convex subset of E. We assume the following*:

(B1) *Let $\{T_i\}_{i=1}^{\infty} : C \to C$ be a countable family of closed, uniformly L_i-Lipschitz continuous, and uniformly quasi-ϕ-asymptotically nonexpansive mappings with a real sequence $\{k_n\} \subset [1, \infty)$, $k_n \to 1$.*

(B2) *Let $\{S_i\}_{i=1}^{\infty} : C \to C$ be a countable family of closed, uniformly ℓ_i-Lipschitz continuous, and uniformly quasi-ϕ-asymptotically nonexpansive mappings with a real sequence $\{l_n\} \subset [1, \infty)$, $l_n \to 1$.*

Suppose that $\Omega := (\bigcap_{i=1}^{\infty} \text{Fix}(T_i)) \cap (\bigcap_{i=1}^{\infty} \text{Fix}(S_i))$ is a nonempty and bounded in C. Let $\{x_n\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty}$, and $\{z_n\}_{n=1}^{\infty}$ be sequences generated by

$$
\begin{cases}
x_0 \in C \quad \text{chosen arbitrarily, } C_0 = C, \\
z_n = J^{-1}(\alpha_{n,0} J x_n + \sum_{i=1}^{\infty} \alpha_{n,i} J S_i^n y_n), \\
y_n = J^{-1}(\beta_{n,0} J x_n + \sum_{i=1}^{\infty} \beta_{n,i} J T_i^n x_n), \\
C_{n+1} = \{v \in C_n : \phi(v, y_n) \le \phi(v, x_n) + \xi_n \text{ and } \phi(v, z_n) \le \phi(v, x_n) + \theta_n\}, \\
x_{n+1} = \Pi_{C_{n+1}}(x_0), \quad \forall n \ge 0,
\end{cases}
\tag{3.28}
$$

where $\xi_n = \sup_{p \in \Omega}(k_n - 1)(\phi(p, x_n))$ and $\theta_n = \sup_{p \in \Omega}(l_n - 1)(\phi(p, x_n))$.

Let $\{\alpha_{n,i}\}$ and $\{\beta_{n,i}\}$ be coefficient sequences in $[0, 1]$ satisfying the following conditions:

(1) $\sum_{i=0}^{\infty} \alpha_{n,i} = 1$ *and* $\sum_{i=0}^{\infty} \beta_{n,i} = 1$;

(2) $\liminf_{n \to \infty} \alpha_{n,0} \alpha_{n,i} > 0$ *and* $\liminf_{n \to \infty} \beta_{n,0} \beta_{n,i} > 0$, $\forall i \ge 1$.

Then the sequence $\{x_n\}_{n=1}^{\infty}$ converges strongly to some point p^*, where $p^* = \Pi_{\Omega}(x_0)$.

Proof Since $\{T_i\}_{i=1}^{\infty}$, $\{S_i\}_{i=1}^{\infty}$ are countable families of closed and uniformly quasi-ϕ-asymptotically nonexpansive mappings, by virtue of Remark 1.7(2), $\{T_i\}_{i=1}^{\infty}$, $\{S_i\}_{i=1}^{\infty}$ are countable families of closed and uniformly total quasi-ϕ-asymptotically nonexpansive mappings with nonnegative sequences $\nu_n = k_n - 1$, $\mu_n = 0$, and $\kappa_n = l_n - 1$, $\omega_n = 0$, respectively, and a strictly increasing and continuous function $\zeta(t) = \rho(t) = t$, $t \ge 0$. Hence, $\xi_n = \sup_{p \in \Omega}(k_n - 1)(\phi(p, x_n)) \to 0$ and $\theta_n = \sup_{p \in \Omega}(l_n - 1)(\phi(p, x_n)) \to 0$ (as $n \to \infty$). Therefore, all the conditions in Theorem 3.1 are satisfied. The conclusion of Theorem 3.2 can be obtained from Theorem 3.1 immediately. $\qquad\square$

Theorem 3.3 *Let E be a uniformly smooth and strictly convex real Banach space with the Kadec-Klee property, and let C be a nonempty, closed, and convex subset of E. We assume the following:*

(C1) *Let $\{T_i\}_{i=1}^{\infty} : C \to C$ be a countable family of closed and quasi-ϕ-nonexpansive mappings.*

(C2) *Let $\{S_i\}_{i=1}^{\infty} : C \to C$ be a countable family of closed and quasi-ϕ-nonexpansive mappings.*

Suppose that $\Omega := (\bigcap_{i=1}^{\infty} \text{Fix}(T_i)) \cap (\bigcap_{i=1}^{\infty} \text{Fix}(S_i))$ is nonempty and bounded in C. Let $\{x_n\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty}$, and $\{z_n\}_{n=1}^{\infty}$ be sequences generated by

$$
\begin{cases}
x_0 \in C \quad \text{chosen arbitrarily, } C_0 = C, \\
z_n = J^{-1}(\alpha_{n,0} J x_n + \sum_{i=1}^{\infty} \alpha_{n,i} J S_i y_n), \\
y_n = J^{-1}(\beta_{n,0} J x_n + \sum_{i=1}^{\infty} \beta_{n,i} J T_i x_n), \\
C_{n+1} = \{v \in C_n : \phi(v, y_n) \le \phi(v, x_n) \text{ and } \phi(v, z_n) \le \phi(v, x_n)\}, \\
x_{n+1} = \Pi_{C_{n+1}}(x_0), \quad \forall n \ge 0,
\end{cases}
\tag{3.29}
$$

and let $\{\alpha_{n,i}\}$ and $\{\beta_{n,i}\}$ be coefficient sequences in $[0, 1]$ satisfying the following conditions:

(1) $\sum_{i=0}^{\infty} \alpha_{n,i} = 1$, *and* $\sum_{i=0}^{\infty} \beta_{n,i} = 1$;

(2) $\liminf_{n \to \infty} \alpha_{n,0} \alpha_{n,i} > 0$, *and* $\liminf_{n \to \infty} \beta_{n,0} \beta_{n,i} > 0$, $\forall i \geq 1$.

Then the sequence $\{x_n\}_{n=1}^{\infty}$ *converges strongly to some point* p^*, *where* $p^* = \Pi_{\Omega}(x_0)$.

Proof Since $\{T_i\}_{i=1}^{\infty}$, $\{S_i\}_{i=1}^{\infty}$ are countable families of closed and quasi-ϕ-nonexpansive mappings, by Remark 1.7(3), $\{T_i\}_{i=1}^{\infty}$, $\{S_i\}_{i=1}^{\infty}$ are countable families of closed and quasi-ϕ-asymptotically nonexpansive mappings with nonnegative sequences $k_n = 1$ and $l_n = 1$, respectively. Hence, $\xi_n = \sup_{p \in \Omega}(k_n - 1)(\phi(p, x_n)) = 0$ and $\theta_n = \sup_{p \in \Omega}(l_n - 1)(\phi(p, x_n)) = 0$. Therefore, all the conditions in Theorem 3.2 as 'Ω is bounded in C' and 'for each $i \geq 1$, $\{T_i\}_{i=1}^{\infty}$, $\{S_i\}_{i=1}^{\infty}$ are uniformly L_i, ℓ_i-Lipschitz continuous' are of no use here. Thus, all the conditions in Theorem 3.2 are satisfied. The conclusion of Theorem 3.3 can be obtained from Theorem 3.2 immediately. □

Theorem 3.4 *Let E be a uniformly smooth and strictly convex real Banach space with the Kadec-Klee property, and let C be a nonempty, closed, and convex subset of E. We assume the following*:

(D1) *Let* $\{T_i\}_{i=1}^{\infty} : C \to C$ *be a countable family of weak relatively nonexpansive mappings.*

(D2) *Let* $\{S_i\}_{i=1}^{\infty} : C \to C$ *be a countable family of weak relatively nonexpansive mappings.*

Suppose that $\Omega := (\bigcap_{i=1}^{\infty} \text{Fix}(T_i)) \cap (\bigcap_{i=1}^{\infty} \text{Fix}(S_i))$ *is nonempty and bounded in C. Let* $\{x_n\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty}$, *and* $\{z_n\}_{n=1}^{\infty}$ *be sequences generated by*

$$
\begin{cases}
x_0 \in C \quad \text{chosen arbitrarily, } C_0 = C, \\
z_n = J^{-1}(\alpha_{n,0}Jx_n + \sum_{i=1}^{\infty} \alpha_{n,i}JS_iy_n), \\
y_n = J^{-1}(\beta_{n,0}Jx_n + \sum_{i=1}^{\infty} \beta_{n,i}JT_ix_n), \\
C_{n+1} = \{v \in C_n : \phi(v, y_n) \leq \phi(v, x_n) \text{ and } \phi(v, z_n) \leq \phi(v, x_n)\}, \\
x_{n+1} = \Pi_{C_{n+1}}(x_0), \quad \forall n \geq 0,
\end{cases}
\tag{3.30}
$$

and let $\{\alpha_{n,i}\}$ and $\{\beta_{n,i}\}$ be coefficient sequences in $[0,1]$ satisfying the following conditions:

(1) $\sum_{i=0}^{\infty} \alpha_{n,i} = 1$ *and* $\sum_{i=0}^{\infty} \beta_{n,i} = 1$;

(2) $\liminf_{n \to \infty} \alpha_{n,0} \alpha_{n,i} > 0$ *and* $\liminf_{n \to \infty} \beta_{n,0} \beta_{n,i} > 0$, $\forall i \geq 1$.

Then the sequence $\{x_n\}_{n=1}^{\infty}$ *converges strongly to some point* p^*, *where* $p^* = \Pi_{\Omega}(x_0)$.

Proof Since $\{T_i\}_{i=1}^{\infty}$, $\{S_i\}_{i=1}^{\infty}$ are countable families of weak relatively nonexpansive mappings, from Remark 1.7(4), $\{T_i\}_{i=1}^{\infty}$, $\{S_i\}_{i=1}^{\infty}$ are countable families of quasi-ϕ-nonexpansive mappings. Therefore, all the conditions in Theorem 3.3 are satisfied. The conclusion of Theorem 3.4 can be obtained from Theorem 3.3 immediately. □

Theorem 3.5 *Let E be a uniformly smooth and strictly convex real Banach space with the Kadec-Klee property, and let C be a nonempty, closed, and convex subset of E. We assume the following*:

(E1) *Let* $\{T_i\}_{i=1}^{\infty} : C \to C$ *be a countable family of relatively nonexpansive mappings.*

(E2) *Let* $\{S_i\}_{i=1}^{\infty} : C \to C$ *be a countable family of relatively nonexpansive mappings.*

Suppose that $\Omega := (\bigcap_{i=1}^{\infty} \text{Fix}(T_i)) \cap (\bigcap_{i=1}^{\infty} \text{Fix}(S_i))$ *is nonempty and bounded in* C. *Let* $\{x_n\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty}$, *and* $\{z_n\}_{n=1}^{\infty}$ *be sequences generated by*

$$
\begin{cases}
x_0 \in C \quad \text{chosen arbitrarily}, C_0 = C, \\
z_n = J^{-1}(\alpha_{n,0}Jx_n + \sum_{i=1}^{\infty} \alpha_{n,i}JS_iy_n), \\
y_n = J^{-1}(\beta_{n,0}Jx_n + \sum_{i=1}^{\infty} \beta_{n,i}JT_ix_n), \\
C_{n+1} = \{v \in C_n : \phi(v,y_n) \le \phi(v,x_n) \text{ and } \phi(v,z_n) \le \phi(v,x_n)\}, \\
x_{n+1} = \Pi_{C_{n+1}}(x_0), \quad \forall n \ge 0,
\end{cases}
\tag{3.31}
$$

and let $\{\alpha_{n,i}\}$ *and* $\{\beta_{n,i}\}$ *be coefficient sequences in* $[0,1]$ *satisfying the following conditions*:
  (1) $\sum_{i=0}^{\infty} \alpha_{n,i} = 1$ *and* $\sum_{i=0}^{\infty} \beta_{n,i} = 1$;
  (2) $\liminf_{n\to\infty} \alpha_{n,0}\alpha_{n,i} > 0$ *and* $\liminf_{n\to\infty} \beta_{n,0}\beta_{n,i} > 0, \forall i \ge 1$.
Then the sequence $\{x_n\}_{n=1}^{\infty}$ *converges strongly to some point* p^*, *where* $p^* = \Pi_{\Omega}(x_0)$.

Proof Since $\{T_i\}_{i=1}^{\infty}$, $\{S_i\}_{i=1}^{\infty}$ are countable families of relatively nonexpansive mappings, it follows from Remark 1.7(5) that $\{T_i\}_{i=1}^{\infty}$, $\{S_i\}_{i=1}^{\infty}$ are countable families of weak relatively nonexpansive mappings. Therefore, all the conditions in Theorem 3.4 are satisfied. The conclusion of Theorem 3.5 can be obtained from Theorem 3.4 immediately. □

Remark 3.6 Theorems 3.1-3.5 generalize, improve, and extend the corresponding results in [5–8, 11–18], and [19] in the following aspects:

(a) For the framework of spaces, we extend the space from a uniformly smooth and uniformly convex Banach space to a uniformly smooth and strictly convex real Banach space with the Kadec-Klee property. (Note that each uniformly convex Banach space must have the Kadec-Klee property.)

(b) For the mappings, we extend the mappings from a nonexpansive mapping, a relatively nonexpansive mapping, a weakly relatively nonexpansive mapping, a quasi-ϕ-nonexpansive mapping or a quasi-ϕ-asymptotically nonexpansive mapping to a total quasi-ϕ-asymptotically nonexpansive mapping.

(c) We extend a countable family of closed and uniformly L_i-Lipschitz continuous and uniformly total quasi-ϕ-asymptotically nonexpansive mappings to two countable families of closed and uniformly L_i-Lipschitz continuous and uniformly total quasi-ϕ-asymptotically nonexpansive mappings.

4 Deduced theorem
If we take $i = 1$ in Theorem 3.1, then we obtain the following result.

Theorem 4.1 *Let E be a uniformly smooth and strictly convex real Banach space with the Kadec-Klee property, and let C be a nonempty, closed, and convex subset of E. We assume the following*:

(F1) *Let $T : C \to C$ be a closed, uniformly L-Lipschitz continuous, and uniformly total quasi-ϕ-asymptotically nonexpansive mapping with nonnegative real sequences $\{\nu_n\}$, $\{\mu_n\}$, and a strictly increasing function $\zeta : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\nu_n \to 0$, $\mu_n \to 0$ (as $n \to \infty$), $\zeta(0) = 0$, and*

$$
\phi(p, T^n x) \le \phi(p,x) + \nu_n \zeta(\phi(p,x)) + \mu_n, \quad \forall x \in C, p \in \text{Fix}(T), \text{ and } \forall n \ge 1.
$$

(F2) *Let* $S : C \to C$ *be a closed, uniformly ℓ-Lipschitz continuous, and uniformly total quasi-ϕ-asymptotically nonexpansive mapping with nonnegative real sequences* $\{\kappa_n\}$, $\{\omega_n\}$, *and a strictly increasing function* $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ *such that* $\kappa_n \to 0$, $\omega_n \to 0$ *(as $n \to \infty$), $\rho(0) = 0$, and*

$$\phi(p, S^n x) \le \phi(p, x) + \kappa_n \rho(\phi(p, x)) + \omega_n, \quad \forall x \in C, p \in \mathrm{Fix}(S), \text{ and } \forall n \ge 1.$$

Suppose that $\Omega := \mathrm{Fix}(T) \cap \mathrm{Fix}(S)$ *is nonempty and bounded in C. Let* $\{x_n\}_{n=1}^\infty$, $\{y_n\}_{n=1}^\infty$, *and* $\{z_n\}_{n=1}^\infty$ *be sequences generated by*

$$\begin{cases} x_0 \in C \quad \text{chosen arbitrarily, } C_0 = C, \\ z_n = J^{-1}(\alpha_n J x_n + (1 - \alpha_n) J S^n y_n), \\ y_n = J^{-1}(\beta_n J x_n + (1 - \beta_n) J T^n x_n), \\ C_{n+1} = \{v \in C_n : \phi(v, y_n) \le \phi(v, x_n) + \xi_n \text{ and } \phi(v, z_n) \le \phi(v, x_n) + \theta_n\}, \\ x_{n+1} = \Pi_{C_{n+1}}(x_0), \quad \forall n \ge 0, \end{cases} \quad (4.1)$$

where $\xi_n = v_n \sup_{p \in \Omega} \zeta(\phi(p, x_n)) + \mu_n$ *and* $\theta_n = \kappa_n \sup_{p \in \Omega} \rho(\phi(p, x_n)) + \omega_n$.

Let $\{\alpha_n\}$ *and* $\{\beta_n\}$ *be coefficient sequences in $[0, 1]$ satisfying the following conditions*:

(1) $0 \le \alpha_n \le \alpha < 1$ *for some* $\alpha \in (0, 1)$;

(2) $0 \le \beta_n \le \beta < 1$ *for some* $\beta \in (0, 1)$.

Then the sequence $\{x_n\}_{n=1}^\infty$ *converges strongly to some point p^*, where $p^* = \Pi_\Omega(x_0)$.*

If we set $S_i^n = I$ (identity mapping) for all $i = 1, 2, 3, \dots$ in Theorem 3.1, then we obtain the following result.

Theorem 4.2 *Let E be a uniformly smooth and strictly convex real Banach space with the Kadec-Klee property, and let C be a nonempty, closed, and convex subset of E. We assume the following*:

(G1) *Let* $\{T_i\}_{i=1}^\infty : C \to C$ *be a countable family of closed, uniformly L_i-Lipschitz continuous, and uniformly total quasi-ϕ-asymptotically nonexpansive mappings with nonnegative real sequences* $\{v_n\}$, $\{\mu_n\}$, *and a strictly increasing function* $\zeta : \mathbb{R}^+ \to \mathbb{R}^+$ *such that* $v_n \to 0$, $\mu_n \to 0$ *(as $n \to \infty$), $\mu_1 = 0$, and $\zeta(0) = 0$, and for each $i \ge 1$,*

$$\phi(p, T_i^n x) \le \phi(p, x) + v_n \zeta(\phi(p, x)) + \mu_n, \quad \forall x \in C, p \in \bigcap_{i=1}^\infty \mathrm{Fix}(T_i), \text{ and } \forall n \ge 1.$$

(G2) *Suppose that* $\Omega := \bigcap_{i=1}^\infty \mathrm{Fix}(T_i)$ *is nonempty and bounded in C. Let* $\{x_n\}_{n=1}^\infty$, $\{y_n\}_{n=1}^\infty$, *and* $\{z_n\}_{n=1}^\infty$ *be sequences generated by*

$$\begin{cases} x_0 \in C \quad \text{chosen arbitrarily, } C_0 = C, \\ z_n = J^{-1}(\alpha_n J x_n + (1 - \alpha_n) J y_n), \\ y_n = J^{-1}(\beta_{n,0} J x_n + \sum_{i=1}^\infty \beta_{n,i} J T_i^n x_n), \\ C_{n+1} = \{v \in C_n : \phi(v, y_n) \le \phi(v, x_n) + \xi_n\}, \\ x_{n+1} = \Pi_{C_{n+1}}(x_0), \quad \forall n \ge 0, \end{cases} \quad (4.2)$$

where $\xi_n = \nu_n \sup_{p \in \Omega} \zeta(\phi(p, x_n)) + \mu_n$. *Let* $\{\alpha_n\}$ *and* $\{\beta_{n,i}\}$ *be coefficient sequences in* $[0, 1]$
satisfying the following conditions:

   (1) $0 \leq \alpha_n \leq \alpha < 1$ *for some* $\alpha \in (0, 1)$;

   (2) $\sum_{i=0}^{\infty} \beta_{n,i} = 1$;

   (3) $\liminf_{n \to \infty} \beta_{n,0} \beta_{n,i} > 0, \forall i \geq 1$.

Then the sequence $\{x_n\}_{n=1}^{\infty}$ *converges strongly to some point* p^*, *where* $p^* = \Pi_\Omega(x_0)$.

Remark 4.3 Theorem 4.2 contains the result of Chang *et al.* [8].

5 Applications

Now, we apply Theorem 3.5 to prove a strong convergence theorem concerning two maximal monotone operators in Banach spaces.

Let E be a smooth, strictly convex, and reflexive real Banach space, and let $A : E \to E^*$ be a maximal monotone operator. For each $r > 0$, we can define a single value mapping $J_r : E \to D(A)$ by $J_r = (J + rA)^{-1}J$, and such a mapping J_r is called the *resolvent* of A. It is easy to prove that $A^{-1}(0) = F(J_r)$ for all $r > 0$. Using Theorem 3.5, we can obtain the following strong convergence theorem for maximal monotone operators.

Theorem 5.1 *Let E be a uniformly smooth and strictly convex real Banach space with the Kadec-Klee property, and let C be a nonempty, closed, and convex subset of E. We assume the following*:

(H1) *Let A, B be two maximal monotone operators from E to E^*, and let J_r^A, J_r^B be the resolvent of A and B, respectively, where $r > 0$.*

(H2) *Suppose that $\Omega := A^{-1}(0) \cap B^{-1}(0)$ is nonempty, and let $\{x_n\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty}$, and $\{z_n\}_{n=1}^{\infty}$ be sequences generated by*

$$\begin{cases} x_0 \in C \quad \text{chosen arbitrarily, } C_0 = C \\ z_n = J^{-1}(\alpha_{n,0} J x_n + \sum_{i=1}^{\infty} \alpha_{n,i} J J_{r_i}^B y_n) \\ y_n = J^{-1}(\beta_{n,0} J x_n + \sum_{i=1}^{\infty} \beta_{n,i} J J_{r_i}^A x_n) \\ C_{n+1} = \{v \in C_n : \phi(v, y_n) \leq \phi(v, x_n) \text{ and } \phi(v, z_n) \leq \phi(v, x_n)\} \\ x_{n+1} = \Pi_{C_{n+1}}(x_0), \quad \forall n \geq 0, \end{cases} \tag{5.1}$$

where $r_n > 0$ with $\liminf_{n \to \infty} r_n > 0$, and let $\{\alpha_{n,i}\}$ and $\{\beta_{n,i}\}$ be coefficient sequences in $[0, 1]$ satisfying the following conditions:

   (1) $\sum_{i=0}^{\infty} \alpha_{n,i} = 1$, *and* $\sum_{i=0}^{\infty} \beta_{n,i} = 1$;

   (2) $\liminf_{n \to \infty} \alpha_{n,0} \alpha_{n,i} > 0$, *and* $\liminf_{n \to \infty} \beta_{n,0} \beta_{n,i} > 0, \forall i \geq 1$.

Then the sequence $\{x_n\}_{n=1}^{\infty}$ converges strongly to some point p^*, where $p^* = \Pi_\Omega(x_0)$.

Proof It is well known that for each $i \geq 1$, $J_{r_i}^A$ is a relatively nonexpansive mapping (see, for example, [5, 6, 19]). Therefore, for each $p \in F(J_{r_i}^A)$ and $w \in E$, we have

$$\phi(p, J_{r_i}^A w) \leq \phi(p, w).$$

Again, by the same method, we can prove that the set of strong asymptotically fixed points

$$\widetilde{F}(\{J_{r_i}^A\}_{i=1}^{\infty}) = \bigcap_{i=1}^{\infty} F(J_{r_i}^A) = A^{-1}(0).$$

This implies that $\{J_{r_i}^A\}_{i=1}^\infty$ is a countable family of weak relatively nonexpansive mappings with the common fixed point set $\bigcap_{n=1}^\infty F(J_{r_i}^A) = A^{-1}(0)$. By a similar way, we can prove that $\{J_{r_i}^B\}_{i=1}^\infty$ is a countable family of weak relatively nonexpansive mappings with the common fixed point set $\bigcap_{n=1}^\infty F(J_{r_i}^B) = B^{-1}(0)$. Hence, the conclusion of Theorem 5.1 can be obtained from Theorem 3.5 immediately. □

Remark 5.2 Theorem 5.1 improves and extends Theorem 4.1 of Chang *et al.* [10] from one maximal monotone operator to two maximal monotone operators.

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References
1. Cioranescu, I: Geometry of Banach spaces, duality mappings and nonlinear problem. In: Hazewinkel, M (ed.) Mathematics and Its Applications, vol. 62. Kluwer Academic, Dordrecht (1990)
2. Takahashi, W: Nonlinear Functional Analysis. Fixed Point Theory and Its Application. Yokohama-Publishers, Yokohama (2000)
3. Alber, YI: Metric and generalized projection operators in Banach space: properties and applications. In: Kartosator, AG (ed.) Theory and Applications of Nonlinear Operators of Accretive and Monotone Type. Lecture Notes in Pure and Applied Mathematics, vol. 178, pp. 15-50. Dekker, New York (1996)
4. Kamimura, S, Takahashi, W: Strong convergence of a proximal-type algorithm in a Banach space. SIAM J. Optim. **13**(3), 938-945 (2002)
5. Matsushita, S, Takahashi, W: A strong convergence theorem for relatively nonexpansive mappings in Banach space. J. Approx. Theory **134**, 257-266 (2005)
6. Plubtieng, S, Ungchittrakool, K: Hybrid iterative methods for convex feasibility problems and fixed point problems of relatively nonexpansive mappings in Banach space. Fixed Point Theory Appl. **2008**, Article ID 583082 (2008)
7. Su, YF, Xu, HK, Zhang, X: Strong convergence theorems for two countable families of weak relatively nonexpansive mappings and applications. Nonlinear Anal. **73**, 3890-3906 (2010)
8. Chang, SS, Lee, HWJ, Chan, CK, Yang, L: Approximation theorems for total quasi-ϕ-asymptotically nonexpansive mapping with applications. Appl. Math. Comput. **218**, 2921-2931 (2011)
9. Chang, SS, Lee, HWJ, Chan, CK, Zhang, WB: A modified Halpern-type iteration algorithm for totally quasi-ϕ-asymptotically nonexpansive mappings with applications. Appl. Math. Comput. **12**(5), 1483-1492 (2012). doi:10.1016/j.amc.2011.12.019
10. Chang, SS, Lee, HWJ, Chan, CK: A block hybrid method for solving generalized equilibrium problems and convex feasibility problem. Adv. Comput. Math. (2011). doi:10.1007/s10444-011-9249-5
11. Chang, SS, Kim, JK, Wang, XR: Modified block iterative algorithm for solving convex feasibility problems in Banach spaces. J. Inequal. Appl. **14**, Article ID 869684 (2010)
12. Aleyner, A, Reich, S: Block-iterative algorithm for solving convex feasibility problems in Hilbert and Banach space. J. Math. Anal. Appl. **343**(1), 427-435 (2008)
13. Li, HY, Su, YF: Strong convergence theorems by a new hybrid for equilibrium problems and variational inequality problems. Nonlinear Anal. **72**(2), 847-855 (2010)
14. Takahashi, W, Zembayashi, K: Strong and weak convergence theorems for equilibrium problems and relatively nonexpansive mappings in Banach spaces. Nonlinear Anal. **70**(1), 45-57 (2009)
15. Wang, ZM, Su, YF, Wang, DX, Dong, YC: A modified Halpern-type iteration algorithm for a family of hemi-relative nonexpansive mappings and a system of equilibrium problems in Banach spaces. J. Comput. Appl. Math. **235**, 2364-2371 (2011)
16. Chang, SS, Chan, CK, Lee, HWJ: Modified block iterative algorithm for quasi-ϕ-asymptotically nonexpansive mappings and equilibrium problems in Banach spaces. Appl. Math. Comput. **217**, 7520-7530 (2011)
17. Yao, YH, Liou, YC, Kang, SM: Strong convergence of an iterative algorithm on an infinite countable family of nonexpansive mappings. Appl. Math. Comput. **208**, 211-218 (2009)

18. Zegeye, H, Ofoedu, EU, Shahzad, N: Convergence theorems for equilibrium problem, variational inequality problem and countable infinite relatively quasi-nonexpansive mappings. Appl. Math. Comput. **216**, 3439-3449 (2010)
19. Wattanawitoon, K, Kumam, P: Strong convergence theorems by a new hybrid projection algorithm for fixed point problem and equilibrium problems of two relatively quasi-nonexpansive mappings. Nonlinear Anal. Hybrid Syst. **3**(1), 11-20 (2010)