Fixed Point Theory and Applications
a SpringerOpen Journal

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Mixed g-monotone property and quadruple fixed point theorems in partially ordered G-metric spaces using $(\phi - \psi)$ contractions

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Abstract

In this paper, we prove some quadruple coincidence and quadruple common fixed point theorems for $F : X^4 \to X$ and $g : X \to X$ satisfying $(\phi - \psi)$ contractions in partially ordered G-metric spaces. We illustrate our results based on an example of the main theorems. Also, we deduce quadruple coincidence points for mappings satisfying the contraction condition of integral type.

MSC: 54H25; 47H10; 54E50

Keywords: quadruple fixed point; ordered sets; generalized metric spaces; mixed g-monotone property

1 Introduction and preliminaries

In 1992, Dhage introduced a new class of generalized metric spaces called D-metric spaces (see [1]). In a subsequent series of papers, Dhage attempted to develop topological structures in such spaces (see [2–4]). He claimed that D-metrics provide a generalization of ordinary metric functions and went on to present several fixed point results. In [5], in collaboration with Sims we demonstrate that most of the claims concerning the fundamental topological structure of D-metric space are incorrect, we also introduce a valid generalized metric space structure, which we call G-metric spaces. Some other papers dealing with G-metric spaces are those in [6–17]. Recently, there has been growing interest in establishing fixed point theorems in partially ordered complete G-metric spaces with the contractive condition which holds for all points that are related by partial ordering ([18, 19] and [20]).

In [21], coupled fixed point results in partially ordered metric spaces were established. After the publication of this work, several coupled fixed point and coincidence point results have appeared in recent literatures (see, for instance, [19, 22–37] and [38]).

Recently, Vasile Berinde and Marin Borcut [39] extended and generalized the results of [21] to the case of a contractive operator $F : X \times X \times X \to X$, where X is a complete ordered metric space. They introduced the concept of a tripled fixed point and the mixed monotone property of a mapping $F : X \times X \times X \to X$. For more details on tripled fixed point results, we refer the reader to [39] and [40].

Very recently, the notion of a fixed point of order $N \geq 3$ was introduced in [30], and later in [41] Erdal Karapinar and Nguyen Van Luong introduced the concept of a quadruple fixed point and the mixed monotone property of a mapping $F : X \times X \times X \times X \to X$ and they presented some new fixed point results. Then, a quadruple fixed point is developed and related fixed points are obtained (see [41–47]).

In this paper, we prove some quadruple coincidence and quadruple common fixed point theorems for $F : X^4 \to X$ and $g : X \to X$ satisfying $(\phi - \psi)$ contractions in partially ordered G-metric spaces. We illustrate our results based on an example of the main theorems. Also, we deduce quadruple coincidence points for mappings satisfying the contraction condition of integral type, we shall recall some mathematical preliminaries.

Definition 1.1 [48] Let X be a nonempty set, and let $G : X \times X \times X \to \mathbf{R}^+$ be a function satisfying the following properties:

(G1) $G(x, y, z) = 0$ if $x = y = z$;
(G2) $0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$;
(G3) $G(x, x, y) \le G(x, y, z)$ for all $x, y, z \in X$ with $z \neq y$;
(G4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$ (symmetry in all three variables); and
(G5) $G(x, y, z) \le G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$ (rectangle inequality).

Then the function G is called a *generalized metric* or, more specifically, a *G-metric* on X, and the pair (X, G) is called a *G-metric space*.

Example 1.1 [48] Let (X, d) be a usual metric space, and define G_s and G_m on $X \times X \times X$ to $\mathbf{R}^+$ by

$$G_s(x, y, z) = d(x, y) + d(y, z) + d(x, z), \quad \text{and}$$
$$G_m(x, y, z) = \max\{d(x, y), d(y, z), d(x, z)\}$$

for all $x, y, z \in X$. Then (X, G_s) and (X, G_m) are G-metric spaces.

Definition 1.2 [48] Let (X, G) be a G-metric space, and let (x_n) be a sequence of points of X. A point $x \in X$ is said to be the *limit* of the sequence (x_n) if $\lim_{n,m\to\infty} G(x, x_n, x_m) = 0$, and one says that the sequence (x_n) is *G-convergent* to x.

Thus, if $x_n \longrightarrow 0$ in a G-metric space (X, G), then for any $\epsilon > 0$, there exists $N \in \mathbf{N}$ such that $G(x, x_n, x_m) < \epsilon$ for all $n, m \ge N$ (we mean by $\mathbf{N}$ the natural numbers).

Proposition 1.1 [48] *Let (X, G) be a G-metric space. Then the following are equivalent.*

(1) *(x_n) is G-convergent to x.*
(3) *$G(x_n, x_n, x) \to 0$, as $n \to \infty$.*
(4) *$G(x_n, x, x) \to 0$, as $n \to \infty$.*
(5) *$G(x_m, x_n, x) \to 0$, as $m, n \to \infty$.*

Definition 1.3 [48] Let (X, G) be a G-metric space. A sequence (x_n) is called G-Cauchy if, given $\epsilon > 0$, there is $N \in \mathbf{N}$ such that $G(x_n, x_m, x_l) < \epsilon$ for all $n, m, l \ge N$. That is, $G(x_n, x_m, x_l) \longrightarrow 0$ as $n, m, l \longrightarrow \infty$.

Proposition 1.2 [48] *In a G-metric space, (X, G), the following are equivalent.*

(1) *The sequence (x_n) is G-Cauchy.*
(2) *For every $\epsilon > 0$, there exists $N \in \mathbf{N}$ such that $G(x_n, x_m, x_m) < \epsilon$ for all $n, m \ge N$.*

Proposition 1.3 [48] *Let (X,G) and (X',G') be two G-metric spaces. Then a function $f: X \longrightarrow X'$ is G-continuous at a point $x \in X$ if and only if it is G-sequentially continuous at x; that is, whenever (x_n) is G-convergent to x, we have $(f(x_n))$ is G-convergent to $f(x)$.*

Definition 1.4 [48] A G-metric space (X,G) is called a symmetric G-metric space if $G(x,y,y) = G(y,x,x)$ for all $x,y \in X$.

It is clear that any G-metric space where G derives from an underlying metric via G_s or G_m in Example 1.1 is symmetric.

Proposition 1.4 [48] *Let (X,G) be a G-metric space, then the function $G(x,y,z)$ is jointly continuous in all three of its variables.*

Proposition 1.5 [48] *Every G-metric space (X,G) induces a metric space (X,d_G) defined by*

$$d_G(x,y) = G(x,y,y) + G(y,x,x), \quad \forall x,y \in X.$$

Note that if (X,G) is symmetric, then

$$d_G(x,y) = 2G(x,y,y), \quad \forall x,y \in X. \tag{1.1}$$

However, if (X,G) is not symmetric, then it holds by the G-metric properties that

$$\frac{3}{2}G(x,y,y) \le d_G(x,y) \le 3G(x,y,y), \quad \forall x,y \in X. \tag{1.2}$$

Definition 1.5 [48] A G-metric space (X,G) is said to be *G-complete* (or complete G-metric) if every G-Cauchy sequence in (X,G) is G-convergent in (X,G).

Definition 1.6 Let (X,G) be a G-metric space. A mapping $F: X \times X \times X \times X \to X$ is said to be continuous if for any G-convergent sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, and $\{w_n\}$ converging to x, y, z, and w respectively, $\{F(x_n,y_n,z_n,w_n)\}$ is G-convergent to $F(x,y,z,w)$.

Proposition 1.6 [48] *A G-metric space (X,G) is G-complete if and only if (X,d_G) is a complete metric space.*

Following Erdal [41], we introduce the following definitions.

Definition 1.7 [41] Let X be a nonempty set and $F: X \times X \times X \times X \to X$ be a given mapping. An element $(x,y,z,w) \in X \times X \times X \times X$ is called a quadruple fixed point of F if

$$\begin{aligned} &F(x,y,z,w) = x, \qquad F(y,z,w,x) = y, \\ &F(z,w,x,y) = z, \quad \text{and} \quad F(w,x,y,z) = w. \end{aligned}$$

Definition 1.8 [41] Let $(X,\le)$ be a partially ordered set and $F: X \times X \times X \times X \to X$ be a mapping. We say that F has the mixed monotone property if $F(x,y,z,w)$ is monotone non-decreasing in x and z and is monotone non-increasing in y and w; that is, for any $x, y, z, w \in X$,

$$x_1, x_2 \in X, \qquad x_1 \le x_2 \quad \text{implies} \quad F(x_1, y, z, w) \le F(x_2, y, z, w),$$
$$y_1, y_2 \in X, \qquad y_1 \le y_2 \quad \text{implies} \quad F(x, y_2, z, w) \le F(x, y_1, z, w),$$
$$z_1, z_2 \in X, \qquad z_1 \le z_2 \quad \text{implies} \quad F(x, y, z_1, w) \le F(x, y, z_2, w),$$

and

$$w_1, w_2 \in X, \qquad w_1 \le w_2 \quad \text{implies} \quad F(x, y, z, w_2) \le F(x, y, z, w_1).$$

Definition 1.9 [41] Let X be a non-empty set. Then we say that the mappings $F : X^4 \to X$ and $g : X \to X$ are commutative if for all $x, y, z, w \in X$,

$$g\big(F(x, y, z, w)\big) = F(gx, gy, gz, gw).$$

Definition 1.10 [47] Let $(X, \le)$ be a partially ordered set. Let $F : X^4 \to X$ and $g : X \to X$. The mapping F is said to have the mixed g-monotone property if for any $x, y, z, w \in X$,

$$x_1, x_2 \in X, \qquad gx_1 \le gx_2 \quad \Longrightarrow \quad F(x_1, y, z, w) \le F(x_2, y, z, w),$$
$$y_1, y_2 \in X, \qquad gy_1 \le gy_2 \quad \Longrightarrow \quad F(x, y_1, z, w) \ge F(x, y_2, z, w),$$
$$z_1, z_2 \in X, \qquad gz_1 \le gz_2 \quad \Longrightarrow \quad F(x, y, z_1, w) \le F(x, y, z_2, w), \quad \text{and}$$
$$w_1, w_2 \in X, \qquad gw_1 \le gw_2 \quad \Longrightarrow \quad F(x, y, z, w_1) \ge F(x, y, z, w_2).$$

Definition 1.11 [47] Let $F : X^4 \to X$ and $g : X \to X$. An element (x, y, z, w) is called a quadruple coincidence point of F and g if

$$F(x, y, z, w) = gx, \qquad F(y, z, w, x) = gy, \qquad F(z, w, x, y) = gz, \quad \text{and} \quad F(w, x, y, z) = gw.$$

(gx, gy, gz, gw) is said to be a quadruple point of coincidence of F and g.

Definition 1.12 [47] Let $F : X^4 \to X$ and $g : X \to X$. An element (x, y, z, w) is called a quadruple common fixed point of F and g if

$$F(x, y, z, w) = gx = x, \qquad F(y, z, w, x) = gy = y,$$
$$F(z, w, x, y) = gz = z, \quad \text{and} \quad F(w, x, y, z) = gw = w.$$

Let Φ be the set of all functions $\phi : [0, +\infty) \to [0, +\infty)$ which satisfy
(1) ϕ is continuous and non-decreasing;
(2) $\phi(t) = 0$ iff $t = 0$;
(3) $\phi(t + s) \le \phi(t) + \phi(s)$, $\forall t, s \in [0, \infty)$.
And let Ψ denote all functions $\psi : [0, +\infty) \to [0, +\infty)$ which satisfy
(1) $\lim_{t \to r} \psi(t) > 0$ for all $r > 0$, and
(2) $\lim_{t \to +0} \psi(t) = 0$.

For example [26], the functions $\phi_1(t) = kt$, $k > 0$, $\phi_2(t) = \frac{t}{1+t}$ are in Φ and $\psi_1(t) = kt$, $k > 0$, $\psi_2(t) = \frac{ln(2k+1)}{2}$ are in Ψ.

Remark 1 $\Phi \subseteq \Psi$.

Remark 2 For all $t \in [0, +\infty)$, we have $\frac{1}{2}\phi(t) \leq \phi(\frac{t}{2})$.

2 Main results

Theorem 2.1 *Let $(X, \leq)$ be a partially ordered set and (X, G) be a G-metric space. Let $F : X \times X \times X \times X \to X$ and $g : X \to X$ be such that F has the mixed g-monotone property. Assume that there exist $\phi \in \Phi$ and $\psi \in \Psi$ such that*

$$\begin{aligned}
&\phi\big(G\big(F(x,y,z,w), F(u,v,s,t), F(a,b,c,d)\big)\big) \\
&\quad \leq \frac{1}{4}\phi\big(G(gx,gu,gs) + G(gy,gv,gb) + G(gz,gs,gc) + G(gw,gt,gd)\big) \\
&\qquad - \psi\left(\frac{G(gx,gu,gs) + G(gy,gv,gb) + G(gz,gs,gc), G(gw,gt,gd)}{4}\right) \qquad (2.1)
\end{aligned}$$

for all $x, y, z, w, u, v, s, t, a, b, c, d \in X$ with $gx \geq gu \geq ga$, $gy \leq gv \leq gb$, $gz \geq gs \geq gc$, and $gw \leq gt \leq gd$. Suppose $F(X^4) \subseteq g(X)$, g is continuous and commutes with F. If there exist $x_0, y_0, z_0, w_0 \in X$ such that

$$\begin{aligned}
&gx_0 \leq F(x_0, y_0, z_0, w_0), \qquad gy_0 \geq F(y_0, z_0, w_0, x_0), \\
&gz_0 \leq F(z_0, w_0, x_0, y_0), \quad \text{and} \quad gw_0 \geq F(w_0, x_0, y_0, z_0),
\end{aligned}$$

suppose either

(a) *(X, G) is a complete G-metric space and F is continuous or*

(b) *$(g(X), G)$ is complete and $(X, G, \leq)$ has the following property:*

  (i) *if a non-decreasing sequence $x_n \to a$, then $x_n \leq x$ for all n,*

  (ii) *if a non-increasing sequence $y_n \to y$, then $y \leq y_n$ for all n,*

  then there exist $x, y, z, w \in X$ such that

$$F(x,y,z,w) = gx, \qquad F(y,z,w,x) = gy, \qquad F(z,w,x,y) = gz, \quad \text{and} \quad F(w,x,y,z) = gw,$$

  that is, F and g have a quadruple coincidence point.

Proof Let $x_0, y_0, z_0, w_0 \in X$ be such that

$$\begin{aligned}
&gx_0 \leq F(x_0, y_0, z_0, w_0), \qquad gy_0 \geq F(y_0, z_0, w_0, x_0), \\
&gz_0 \leq F(z_0, w_0, x_0, y_0), \quad \text{and} \quad gw_0 \geq F(w_0, x_0, y_0, z_0).
\end{aligned}$$

Since $F(X^4) \subset g(X)$, then we can choose $x_1, y_1, z_1, w_1 \in X$ such that

$$\begin{aligned}
&gx_1 = F(x_0, y_0, z_0, w_0), \qquad gy_1 = F(y_0, z_0, w_0, x_0), \\
&gz_1 = F(z_0, w_0, x_0, y_0), \quad \text{and} \quad gw_1 = F(w_0, x_0, y_0, z_0).
\end{aligned} \qquad (2.2)$$

Taking into account $F(X^4) \subset g(X)$, by continuing this process, we can construct sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, and $\{w_n\}$ in X such that

$$\begin{aligned} &gx_{n+1} = F(x_n, y_n, z_n, w_n), \qquad gy_{n+1} = F(y_n, z_n, w_n, x_n), \\ &gz_{n+1} = F(z_n, w_n, x_n, y_n), \quad \text{and} \quad gw_{n+1} = F(w_n, x_n, y_n, z_n). \end{aligned} \tag{2.3}$$

We shall show that

$$\begin{aligned} &gx_n \leq gx_{n+1}, \qquad gy_{n+1} \leq gy_n, \\ &gz_n \leq gz_{n+1}, \quad \text{and} \quad gw_{n+1} \leq gw_n \quad \text{for } n = 0, 1, 2, \ldots. \end{aligned} \tag{2.4}$$

For this purpose, we use the mathematical induction. Since $gx_0 \leq F(x_0, y_0, z_0, w_0)$, $gy_0 \geq F(y_0, z_0, w_0, x_0)$, $gz_0 \leq F(z_0, w_0, x_0, y_0)$, and $gw_0 \geq F(w_0, x_0, y_0, z_0)$, then by (2.2), we get

$$gx_0 \leq gx_1, \qquad gy_1 \leq gy_0, \qquad gz_0 \leq gz_1, \quad \text{and} \quad gw_1 \leq gw_0,$$

that is, (2.4) holds for $n = 0$.

We presume that (2.4) holds for some $n > 0$. As F has the mixed g-monotone property and $gx_n \leq gx_{n+1}$, $gy_{n+1} \leq gy_n$, $gz_n \leq gz_{n+1}$, and $gw_{n+1} \leq gw_n$, we obtain

$$\begin{aligned} gx_{n+1} &= F(x_n, y_n, z_n, w_n) \leq F(x_{n+1}, y_n, z_n, w_n) \\ &\leq F(x_{n+1}, y_n, z_{n+1}, w_n) \leq F(x_{n+1}, y_{n+1}, z_{n+1}, w_n) \\ &\leq F(x_{n+1}, y_{n+1}, z_{n+1}, w_{n+1}) = gx_{n+2}, \\ gy_{n+2} &= F(y_{n+1}, z_{n+1}, w_{n+1}, x_{n+1}) \leq F(y_{n+1}, z_n, x_{n+1}, w_{n+1}) \\ &\leq F(y_n, z_n, x_{n+1}, w_{n+1}) \leq F(y_n, z_n, x_n, w_{n+1}) \\ &\leq F(y_n, z_n, x_n, w_n) = gy_{n+1}, \\ gz_{n+1} &= F(z_n, y_n, x_n, w_n) \leq F(z_{n+1}, y_n, x_n, w_n) \\ &\leq F(z_{n+1}, y_{n+1}, x_n, w_n) \leq F(z_{n+1}, y_{n+1}, x_{n+1}, w_n) \\ &\leq F(z_{n+1}, y_{n+1}, x_{n+1}, w_{n+1}) = gz_{n+2} \end{aligned}$$

and

$$\begin{aligned} gw_{n+2} &= F(w_{n+1}, x_{n+1}, y_{n+1}, z_{n+1}) \leq F(w_{n+1}, x_n, y_{n+1}, z_{n+1}) \\ &\leq F(w_n, x_n, y_{n+1}, z_{n+1}) \leq F(w_n, x_n, y_n, z_{n+1}) \\ &\leq F(w_n, x_n, y_n, z_n) = gw_{n+1}. \end{aligned}$$

Thus, (2.4) holds for any $n \in \mathbb{N}$. Assume for some $n \in \mathbb{N}$,

$$gx_n = gx_{n+1}, \qquad gy_n = gy_{n+1}, \qquad gz_n = gz_{n+1}, \quad \text{and} \quad gw_n = gw_{n+1}$$

then by (2.3), we have $gx_n = F(x_n, y_n, z_n, w_n)$, $gy_n = F(y_n, z_n, w_n, x_n)$, $gz_n = F(z_n, w_n, x_n, y_n)$, and $gw_n = F(w_n, x_n, y_n, z_n) \Rightarrow (x_n, y_n, z_n, w_n)$ is a quadruple coincidence point of F and g.

From now on, assume for any $n \in \mathbb{N}$ that at least

$$gx_n \neq gx_{n+1} \quad \text{or} \quad gy_n \neq gy_{n+1} \quad \text{or} \quad gz_n \neq gz_{n+1} \quad \text{or} \quad gw_n \neq gw_{n+1}. \tag{2.5}$$

Since $gx_n \leq gx_{n+1}$, $gy_{n+1} \leq gy_n$, $gz_n \leq gz_{n+1}$, and $gw_{n+1} \leq gw_n$, then from (2.1) and (2.3), we have

$$\begin{aligned}
&\phi\bigl(G(gx_{n+1},gx_{n+1},gx_n)\bigr)\\
&\quad := \phi\bigl(G\bigl(F(x_n,y_n,z_n,w_n),F(x_n,y_n,z_n,w_n),F(x_{n-1},y_{n-1},z_{n-1}w_{n-1})\bigr)\bigr)\\
&\quad \leq \frac{1}{4}\phi\bigl(G(gx_n,gx_n,gx_{n-1})+G(gy_n,gy_n,gy_{n-1})\\
&\qquad + G(gz_n,gz_n,gz_{n-1})+G(gw_n,gw_n,gw_{n-1})\bigr)\\
&\qquad - \psi\bigl(\bigl(G(gx_n,gx_n,gx_{n-1})+G(gy_n,gy_n,gy_{n-1})\\
&\qquad + G(gz_n,gz_n,gz_{n-1})+G(gw_n,gw_n,gw_{n-1})\bigr)/4\bigr),
\end{aligned}\tag{2.6}$$

$$\begin{aligned}
&\phi\bigl(G(gy_n,gy_{n+1},gy_{n+1})\bigr)\\
&\quad := \phi\bigl(G\bigl(F(y_{n-1},z_{n-1},w_{n-1},x_{n-1}),F(y_n,z_n,w_n,x_n),F(y_n,z_n,w_n,x_n)\bigr)\bigr)\\
&\quad \leq \frac{1}{4}\phi\bigl(G(gy_{n-1},gy_n,gy_n)+G(gz_{n-1},gz_n,gz_n)\\
&\qquad + G(gw_{n-1},gw_n,gw_n)+G(gx_{n-1},gx_n,gx_n)\bigr)\\
&\qquad - \psi\bigl(\bigl(G(gy_{n-1},gy_n,gy_n)+G(gz_{n-1},gz_n,gz_n)\\
&\qquad + G(gw_{n-1},gw_n,gw_n)+G(gx_{n-1},gx_n,gx_n)\bigr)/4\bigr),
\end{aligned}\tag{2.7}$$

$$\begin{aligned}
&\phi\bigl(G(gz_{n+1},gz_{n+1},gz_n)\bigr)\\
&\quad := \phi\bigl(G\bigl(F(z_n,w_n,x_n,y_n),F(z_n,w_n,x_n,y_n),F(z_{n-1},w_{n-1},x_{n-1},y_{n-1})\bigr)\bigr)\\
&\quad \leq \frac{1}{4}\phi\bigl(G(gz_n,gz_n,gz_{n-1})+G(gw_n,gw_n,gw_{n-1})\\
&\qquad + G(gx_n,gx_n,gx_{n-1})+G(gy_n,gy_n,gy_{n-1})\bigr)\\
&\qquad - \psi\bigl(\bigl(G(gz_n,gz_n,gz_{n-1})+G(gw_n,gw_n,gw_{n-1})\\
&\qquad + G(gx_n,gx_n,gx_{n-1})+G(gy_n,gy_n,gy_{n-1})\bigr)/4\bigr),
\end{aligned}\tag{2.8}$$

and

$$\begin{aligned}
&\phi\bigl(G(gw_n,gw_{n+1},gw_{n+1})\bigr)\\
&\quad := \phi\bigl(G\bigl(F(w_{n-1},x_{n-1},y_{n-1},z_{n-1}),F(w_n,x_n,y_n,z_n),F(w_n,x_n,y_n,z_n)\bigr)\bigr)\\
&\quad \leq \frac{1}{4}\phi\bigl(G(gw_{n-1},gw_n,gw_n)+G(gx_{n-1},gx_n,gx_n)\\
&\qquad + G(gy_{n-1},gy_n,gy_n)+G(gz_{n-1},gz_n,gz_n)\bigr)\\
&\qquad - \psi\bigl(\bigl(G(gw_{n-1},gw_n,gw_n)+G(gx_{n-1},gx_n,gx_n)\\
&\qquad + G(gy_{n-1},gy_n,gy_n)+G(gz_{n-1},gz_n,gz_n)\bigr)/4\bigr).
\end{aligned}\tag{2.9}$$

From (2.6), (2.7), (2.8), and (2.9) it follows that

$$\begin{aligned}
&\phi\bigl(G(gx_{n+1},gx_{n+1},gx_n)\bigr) + \phi\bigl(G(gy_n,gy_{n+1},gy_{n+1})\bigr) \\
&\qquad + \phi\bigl(G(gz_{n+1},gz_{n+1},gz_n)\bigr) + \phi\bigl(G(gw_n,gw_{n+1},gw_{n+1})\bigr) \\
&\quad \le \phi\bigl(G(gw_{n-1},gw_n,gw_n) + G(gx_{n-1},gx_n,gx_n) + G(gy_{n-1},gy_n,gy_n) + G(gz_{n-1},gz_n,gz_n)\bigr) \\
&\qquad - 4\psi\bigl(\bigl(G(gw_{n-1},gw_n,gw_n) + G(gx_{n-1},gx_n,gx_n) \\
&\qquad + G(gy_{n-1},gy_n,gy_n) + G(gz_{n-1},gz_n,gz_n)\bigr)/4\bigr),
\end{aligned} \tag{2.10}$$

but from the property (3) of ϕ, we get $\phi(A+B+C+D) \le \phi(A)+\phi(B)+\phi(C)+\phi(D)$; therefore,

$$\begin{aligned}
&\phi\bigl(G(gx_{n+1},gx_{n+1},gx_n) + G(gy_n,gy_{n+1},gy_{n+1}) \\
&\qquad + G(gz_{n+1},gz_{n+1},gz_n) + G(gw_n,gw_{n+1},gw_{n+1})\bigr) \\
&\quad \le \phi\bigl(G(gw_{n-1},gw_n,gw_n) + G(gx_{n-1},gx_n,gx_n) + G(gy_{n-1},gy_n,gy_n) + G(gz_{n-1},gz_n,gz_n)\bigr) \\
&\qquad - 4\psi\bigl(\bigl(G(gw_{n-1},gw_n,gw_n) + G(gx_{n-1},gx_n,gx_n) \\
&\qquad + G(gy_{n-1},gy_n,gy_n) + G(gz_{n-1},gz_n,gz_n)\bigr)/4\bigr),
\end{aligned} \tag{2.11}$$

which implies that

$$\begin{aligned}
&\phi\bigl(G(gx_{n+1},gx_{n+1},gx_n) + G(gy_n,gy_{n+1},gy_{n+1}) \\
&\qquad + G(gz_{n+1},gz_{n+1},gz_n) + G(gw_n,gw_{n+1},gw_{n+1})\bigr) \\
&\quad \le \phi\bigl(G(gw_{n-1},gw_n,gw_n) + G(gx_{n-1},gx_n,gx_n) \\
&\qquad + G(gy_{n-1},gy_n,gy_n) + G(gz_{n-1},gz_n,gz_n)\bigr).
\end{aligned} \tag{2.12}$$

Using the fact that ϕ is non-decreasing, we get

$$\begin{aligned}
&G(gx_{n+1},gx_{n+1},gx_n) + G(gy_n,gy_{n+1},gy_{n+1}) + G(gz_{n+1},gz_{n+1},gz_n) + G(gw_n,gw_{n+1},gw_{n+1}) \\
&\quad \le G(gw_{n-1},gw_n,gw_n) + G(gx_{n-1},gx_n,gx_n) \\
&\qquad + G(gy_{n-1},gy_n,gy_n) + G(gz_{n-1},gz_n,gz_n).
\end{aligned} \tag{2.13}$$

Let $\delta_n = G(gx_{n+1},gx_{n+1},gx_n) + G(gy_n,gy_{n+1},gy_{n+1}) + G(gz_{n+1},gz_{n+1},gz_n) + G(gw_n,gw_{n+1},gw_{n+1})$.

Then the sequence (δ_n) is decreasing; therefore, there is some $\delta \ge 0$ such that

$$\begin{aligned}
\lim_{n\to\infty} \delta_n &= \lim\bigl(G(gx_{n+1},gx_{n+1},gx_n) + G(gy_n,gy_{n+1},gy_{n+1}) \\
&\qquad + G(gz_{n+1},gz_{n+1},gz_n) + G(gw_n,gw_{n+1},gw_{n+1})\bigr) = \delta.
\end{aligned} \tag{2.14}$$

We will show that $\delta = 0$. Suppose to the contrary that $\delta > 0$, taking the limit as $n \to \infty$ of both sides of (1.3) and using the fact that ϕ is continuous and $\lim_{t\to r} \psi(t) > 0$ for $r > 0$, we have

$$\phi(\delta) = \lim_{n\to\infty} \phi(\delta_n) \leq \lim_{n\to\infty} \left[\phi(\delta_{n-1}) - 4\psi\left(\frac{\delta_{n-1}}{4}\right) \right] = \phi(\delta) - 4 \lim_{n\to\infty} \psi\left(\frac{\delta_{n-1}}{4}\right) < \phi(\delta),$$

a contradiction. Thus, $\delta = 0$, that is,

$$\begin{aligned} \lim_{n\to\infty} \delta_n = \lim\bigl(&G(gx_{n+1}, gx_{n+1}, gx_n) + G(gy_n, gy_{n+1}, gy_{n+1}) \\ &+ G(gz_{n+1}, gz_{n+1}, gz_n) + G(gw_n, gw_{n+1}, gw_{n+1})\bigr) = 0. \end{aligned} \tag{2.15}$$

We will show that the sequences (gx_n), (gy_n), (gz_n), and (gw_n) are G-Cauchy sequences in (X, G).

Suppose to the contrary that at least one of (gx_n), (gy_n), (gz_n), and (gw_n) is not a G-Cauchy sequence, so there exists $\epsilon > 0$ for which we can find sequences $(gx_{n(k)}); (gx_{m(k)})$ of (gx_n), $(gy_{n(k)}); (gy_{m(k)})$ of (gy_n), $(gz_{n(k)}); (gz_{m(k)})$ of (gz_n), and $(gw_{n(k)}); (gw_{m(k)})$ of (gw_n) with $n(k) > m(k) \geq k$ such that

$$\begin{aligned} &\bigl(G(gx_{n(k)}, gx_{n(k)}, gx_{m(k)}) + G(gy_{n(k)}, gy_{n(k}, gy_{m(k)}) \\ &\quad + G(gz_{n(k)}, gz_{n(k)}, gz_{m(k)}) + G(gw_{n(k)}, gw_{n(k)}, gw_{m(k)})\bigr) \geq \epsilon. \end{aligned} \tag{2.16}$$

Further, corresponding to $m(k)$, we may choose $n(k)$ such that it is the smallest integer satisfying (2.16) and $n(k) > m(k) \geq k$.

Thus,

$$\begin{aligned} &\bigl(G(gx_{n(k)-1}, gx_{n(k)-1}, gx_{m(k)}) + G(gy_{n(k)-1}, gy_{n(k)-1}, gy_{m(k)}) \\ &\quad + G(gz_{n(k)-1}, gz_{n(k)-1}, gz_{m(k)}) + G(gw_{n(k)-1}, gw_{n(k)-1}, gw_{m(k)})\bigr) < \epsilon. \end{aligned} \tag{2.17}$$

Using the rectangle inequality and having in mind (2.16) and (2.17), we get

$$\begin{aligned} \epsilon &\leq t_k \\ &= G(gx_{n(k)}, gx_{n(k)}, gx_{m(k)}) + G(gy_{n(k)}, gy_{n(k}, gy_{m(k)}) + G(gz_{n(k)}, gz_{n(k)}, gz_{m(k)}) \\ &\quad + G(gw_{n(k)}, gw_{n(k)}, gw_{m(k)}) \\ &\leq G(gx_{n(k)}, gx_{n(k)}, gx_{n(k)-1}) + G(gx_{n(k)-1}, gx_{n(k)-1}, gx_{m(k)}) + G(gy_{n(k)}, gy_{n(k)}, gy_{n(k)-1}) \\ &\quad + G(gy_{n(k)-1}, gy_{n(k)-1}, gy_{m(k)}) + G(gz_{n(k)}, gz_{n(k)}, gz_{n(k)-1}) + G(gz_{n(k)-1}, gz_{n(k)-1}, gz_{m(k)}) \\ &\quad + G(gw_{n(k)}, gw_{n(k)}, gw_{n(k)-1}) + G(gw_{n(k)-1}, gw_{n(k)-1}, gw_{m(k)}) \\ &< G(gx_{n(k)}, gx_{n(k)}, gx_{n(k)-1}) + G(gy_{n(k)}, gy_{n(k)}, gy_{n(k)-1}) + G(gz_{n(k)}, gz_{n(k)}, gz_{n(k)-1}) \\ &\quad + G(gw_{n(k)}, gw_{n(k)}, gw_{n(k)-1}) + \epsilon = \epsilon + \delta_{n(k)-1}. \end{aligned} \tag{2.18}$$

Letting $k \to \infty$ in (2.18) and using $(\delta = 0)$, we get

$$\begin{aligned} \lim t_k = \lim\bigl(&G(gx_{n(k)}, gx_{n(k)}, gx_{m(k)}) + G(gy_{n(k)}, gy_{n(k)}, gy_{m(k)}) \\ &+ G(gz_{n(k)}, gz_{n(k)}, gz_{m(k)}) + G(gw_{n(k)}, gw_{n(k)}, gw_{m(k)})\bigr) = \epsilon. \end{aligned} \tag{2.19}$$

Again by the rectangle inequality and using the fact that $G(x,y,y) \leq 2G(y,x,x)$, we have

$$
\begin{aligned}
\epsilon &\leq t_k \\
&= G(gx_{n(k)}, gx_{n(k)}, gx_{m(k)}) + G(gy_{n(k)}, gy_{n(k)}, gy_{m(k)}) + G(gz_{n(k)}, gz_{n(k)}, gz_{m(k)}) \\
&\quad + G(gw_{n(k)}, gw_{n(k)}, w_{m(k)}) \\
&\leq G(gx_{n(k)}, gx_{n(k)}, gx_{n(k)+1}) + G(gx_{n(k)+1}, gx_{n(k)+1}, gx_{m(k)+1}) + G(gx_{m(k)+1}, gx_{m(k)+1}, gx_{m(k)}) \\
&\quad + G(gy_{n(k)}, gy_{n(k)}, gy_{n(k)+1}) + G(gy_{n(k)+1}, gy_{n(k)+1}, gy_{m(k)+1}) \\
&\quad + G(gy_{m(k)+1}, gy_{m(k)+1}, gy_{m(k)}) + G(gz_{n(k)}, gz_{n(k)}, gz_{n(k)+1}) \\
&\quad + G(gz_{n(k)+1}, gz_{n(k)+1}, gz_{m(k)+1}) + G(gz_{m(k)+1}, gz_{m(k)+1}, gz_{m(k)}) \\
&\quad + G(gw_{n(k)}, gw_{n(k)}, gw_{n(k)+1}) + G(gw_{n(k)+1}, gw_{n(k)+1}, gw_{m(k)+1}) \\
&\quad + G(gw_{m(k)+1}, gw_{m(k)+1}, gw_{m(k)}) \\
&\leq 2\delta_{n(k)} + \delta_{m(k)} + G(gx_{n(k)+1}, gx_{n(k)+1}, gx_{m(k)+1}) \\
&\quad + G(gy_{n(k)+1}, gy_{n(k)+1}, gy_{m(k)+1}) + G(gz_{n(k)+1}, gz_{n(k)+1}, gz_{m(k)+1}) \\
&\quad + G(gw_{n(k)+1}, gw_{n(k)+1}, gw_{m(k)+1}).
\end{aligned}
\tag{2.20}
$$

Using the property of ϕ, we have

$$
\begin{aligned}
\phi(t_k) &= \phi\bigl[G(gx_{n(k)}, gx_{n(k)}, gx_{m(k)}) + G(gy_{n(k)}, gy_{n(k)}, gy_{m(k)}) + G(gz_{n(k)}, gz_{n(k)}, gz_{m(k)}) \\
&\qquad + G(gw_{n(k)}, gw_{n(k)}, w_{m(k)})\bigr] \\
&\leq \phi(2\delta_{n(k)} + \delta_{m(k)}) + \phi\bigl(G(gx_{n(k)+1}, gx_{n(k)+1}, gx_{m(k)+1})\bigr) \\
&\qquad + \phi\bigl(G(gy_{n(k)+1}, gy_{n(k)+1}, gy_{m(k)+1})\bigr) + \phi\bigl(G(gz_{n(k)+1}, gz_{n(k)+1}, gz_{m(k)+1})\bigr) \\
&\qquad + \phi\bigl(G(gw_{n(k)+1}, gw_{n(k)+1}, gw_{m(k)+1})\bigr).
\end{aligned}
\tag{2.21}
$$

Since $n(k) > m(k)$, then $gx_{n(k)} \geq gx_{m(k)}$, $gy_{n(k)} \leq gx_{m(k)}$, $gz_{n(k)} \geq gz_{m(k)}$, and $gw_{n(k)} \leq gw_{m(k)}$. Then from (2.1), we have

$$
\begin{aligned}
&\phi\bigl(G(gx_{n(k)+1}, gx_{n(k)+1}, gx_{m(k)+1})\bigr) \\
&\quad = \phi\bigl(G\bigl(F(x_{n(k)}, y_{n(k)}, z_{n(k)}, w_{n(k)}), F(x_{n(k)}, y_{n(k)}, z_{n(k)}, w_{n(k)}), F(x_{m(k)}, y_{m(k)}, z_{m(k)}, w_{m(k)})\bigr)\bigr) \\
&\quad \leq \frac{1}{4}\phi\bigl(G(gx_{n(k)}, gx_{n(k)}, gx_{m(k)}) + G(gy_{n(k)}, gy_{n(k)}, gy_{m(k)}) \\
&\qquad + G(gz_{n(k)}, gz_{n(k)}, gz_{m(k)}) + G(gw_{n(k)}, gw_{n(k)}, gw_{m(k)})\bigr) \\
&\qquad - \psi\bigl(\bigl(G(gx_{n(k)}, gx_{n(k)}, gx_{m(k)}) + G(gy_{n(k)}, gy_{n(k)}, gy_{m(k)}) \\
&\qquad + G(gz_{n(k)}, gz_{n(k)}, gz_{m(k)}) + G(gw_{n(k)}, gw_{n(k)}, gw_{m(k)})\bigr)/4\bigr),
\end{aligned}
\tag{2.22}
$$

and similarly,

$$
\begin{aligned}
&\phi\bigl(G(gy_{n(k)+1}, gy_{n(k)+1}, gy_{m(k)+1})\bigr) \\
&\quad = \phi\bigl(G\bigl(F(y_{n(k)}, z_{n(k)}, w_{n(k)}, x_{n(k)}), F(y_{n(k)}, z_{n(k)}, w_{n(k)}, x_{n(k)}), F(y_{m(k)}, z_{m(k)}, w_{m(k)}, x_{m(k)})\bigr)\bigr)
\end{aligned}
$$

$$\begin{aligned}
&\leq \frac{1}{4}\phi\big(G(gy_{n(k)}, gy_{m(k)}, gy_{n(k)}) + G(gz_{n(k)}, gz_{n(k)}, gz_{m(k)}) \\
&\quad + G(gw_{n(k)}, gw_{n(k)}, gw_{m(k)}) + G(gx_{n(k)}, gx_{n(k)}, gx_{m(k)})\big) \\
&\quad - \psi\big(\big(G(gy_{n(k)}, gy_{n(k)}, gy_{m(k)}) + G(gz_{n(k)}, gz_{n(k)}, gz_{m(k)}) \\
&\quad + G(gw_{n(k)}, gw_{n(k)}, gw_{m(k)}) + G(gx_{n(k)}, gx_{n(k)}, gx_{m(k)})\big)/4\big),
\end{aligned} \tag{2.23}$$

$$\begin{aligned}
&\phi\big(G(gz_{n(k)+1}, gz_{n(k)+1}, gz_{m(k)+1})\big) \\
&\quad = \phi\big(G\big(F(z_{n(k)}, w_{n(k)}, x_{n(k)}, y_{n(k)}), F(z_{n(k)}, w_{n(k)}, x_{n(k)}, y_{n(k)}), F(z_{m(k)}, w_{m(k)}, x_{m(k)}, y_{m(k)})\big)\big) \\
&\quad \leq \frac{1}{4}\phi\big(G(gz_{n(k)}, gz_{n(k)}, gz_{m(k)}) + G(gw_{n(k)}, gw_{n(k)}, gw_{m(k)}) \\
&\qquad + G(gx_{n(k)}, gx_{n(k)}, gx_{m(k)}), G(gy_{n(k)}, gy_{n(k)}, gy_{m(k)})\big) \\
&\qquad - \psi\big(\big(G(gz_{n(k)}, gz_{n(k)}, gz_{m(k)}) + G(gw_{n(k)}, gw_{n(k)}, gw_{m(k)}) \\
&\qquad + G(gx_{n(k)}, gx_{n(k)}, gx_{m(k)}) + G(gy_{n(k)}, gy_{n(k)}, gy_{m(k)})\big)/4\big)
\end{aligned} \tag{2.24}$$

and

$$\begin{aligned}
&\phi\big(G(gw_{n(k)+1}, gw_{n(k)+1}, gw_{m(k)+1})\big) \\
&\quad = \phi\big(G\big(F(w_{n(k)}, x_{n(k)}, y_{n(k)}, z_{n(k)}), F(w_{n(k)}, x_{n(k)}, y_{n(k)}, z_{n(k)}), F(w_{m(k)}, x_{m(k)}, y_{m(k)}, z_{m(k)})\big)\big) \\
&\quad \leq \frac{1}{4}\phi\big(G(gw_{n(k)}, gw_{n(k)}, gw_{m(k)}) + G(gx_{n(k)}, gx_{n(k)}, gx_{m(k)}) \\
&\qquad + G(gy_{n(k)}, gy_{n(k)}, gy_{m(k)}) + G(gz_{n(k)}, gz_{n(k)}, gz_{m(k)})\big) \\
&\qquad - \psi\big(\big(G(gw_{n(k)}, gw_{n(k)}, gw_{m(k)}) + G(gx_{n(k)}, gx_{n(k)}, gx_{m(k)}) \\
&\qquad + G(gy_{n(k)}, gy_{n(k)}, gy_{m(k)}) + G(gz_{n(k)}, gz_{n(k)}, gz_{m(k)})\big)/4\big).
\end{aligned} \tag{2.25}$$

Combining (2.22), (2.23), (2.24), and (2.25) in (2.21), we get

$$\phi(t_k) \leq \phi(2\delta_{n(k)} + \delta_{m(k)}) + \phi(t_k) - 4\psi\left(\frac{t_k}{4}\right).$$

Letting, $k \to \infty$ and using (2.15) and (2.19), we get

$$\phi(\epsilon) \leq \phi(0) + \phi(0) + \phi(\epsilon) - 4\lim_{k\to\infty}\psi\left(\frac{t_k}{4}\right) \tag{2.26}$$

$$= \phi(\epsilon) - 4\lim_{t_k\to\epsilon}\psi\left(\frac{t_k}{4}\right) < \phi(\epsilon), \tag{2.27}$$

a contradiction. This implies that (gx_n), (gy_n), (gz_n), and (gw_n) are G-Cauchy sequences in (X, G).

Now suppose that the assumption (a) holds.

Since X is a G-complete metric space, there exist $x, y, z, w \in X$ such that

$$\begin{aligned}
&\lim_{n\to\infty} g(x_n) = x, \qquad \lim_{n\to\infty} g(y_n) = y, \\
&\lim_{n\to\infty} g(z_n) = z, \qquad \lim_{n\to\infty} g(w_n) = w.
\end{aligned} \tag{2.28}$$

From (2.28) and the continuity of g, we have

$$\begin{aligned} &\lim_{n\to\infty} g\bigl(g(x_n)\bigr) = gx, \qquad \lim_{n\to\infty} g\bigl(g(y_n)\bigr) = gy, \\ &\lim_{n\to\infty} g\bigl(g(z_n)\bigr) = gz, \quad \text{and} \quad \lim_{n\to\infty} g\bigl(g(w_n)\bigr) = gw. \end{aligned}$$

From the commutativity of F and g, we have

$$g(gx_{n+1}) = g\bigl(F(x_n,y_n,z_n,w_n)\bigr) = F(gx_n,gy_n,gz_n,gw_n), \tag{2.29}$$

$$g(gy_{n+1}) = g\bigl(F(y_n,z_n,w_n,z_n)\bigr) = F(gy_n,gz_n,gw_n,gx_n), \tag{2.30}$$

$$g(gz_{n+1}) = g\bigl(F(z_n,w_n,x_n,y_n)\bigr) = F(gz_n,gw_n,gx_n,gy_n), \tag{2.31}$$

and

$$g(gw_{n+1}) = g\bigl(F(w_n,x_n,y_n,z_n)\bigr) = F(gw_n,gx_n,gy_n,gz_n). \tag{2.32}$$

We shall show that $gx = F(x,y,z,w)$, $gy = F(y,z,w,x)$, $gz = F(z,w,x,y)$, and $gw = F(w,x,y,z)$.

By letting $n\to\infty$ in (2.29) $\to$ (2.32) and using the continuity of F, we obtain

$$\begin{aligned} gx &= \lim_{n\to\infty} g(gx_{n+1}) = \lim_{n\to\infty} F(gx_n,gy_n,gz_n,gw_n) \\ &= F\Bigl(\lim_{n\to\infty} gx_n, \lim_{n\to\infty} gy_n, \lim_{n\to\infty} gz_n, \lim_{n\to\infty} gw_n\Bigr) = F(x,y,z,w). \end{aligned}$$

Similarly, $gy = F(y,z,w,x)$, $gz = F(z,w,x,y)$, and $gw = F(w,x,y,z)$.

Hence, (x,y,z,w) is a coincidence point of F and g.

Now suppose that the assumption (b) holds.

Since $\{gx_n\}$, $\{gy_n\}$, $\{gz_n\}$, and $\{gw_n\}$ are G-Cauchy sequences in the complete G-metric space $(g(X),G)$, then there exist $x,y,z,w\in X$ such that

$$gx_n \to gx, \qquad gy_n \to gy, \qquad gz_n \to gz, \quad \text{and} \quad gw_n \to gw. \tag{2.33}$$

Since $\{gx_n\}$, $\{gz_n\}$ are non-decreasing and $\{gy_n\}$, $\{gw_n\}$ are non-increasing and since $(X,G,\le)$ satisfies conditions (i) and (ii), we have

$$gx_n \le gx, \qquad gy_n \ge gy, \qquad gz_n \le gz, \qquad gw_n \ge gw \quad \text{for all } n.$$

If $gx_n = gx$, $gy_n = gy$, $gz_n = gz$, and $gw_n = gw$ for some $n\ge 0$, then $gx = gx_n \le gx_{n+1} \le gx = gx_n$, $gy \le gy_{n+1} \le gy_n = gy$, $gz = gz_n \le gz_{n+1} \le gz = gz_n$, and $gw \le gw_{n+1} \le gw_n = gw$, which implies that

$$gx_n = gx_{n+1} = F(x_n,y_n,z_n,w_n), \qquad gy_n = gy_{n+1} = F(y_n,z_n,w_n,x_n),$$

and

$$gz_n = gz_{n+1} = F(z_n,w_n,x_n,y_n), \qquad gw_n = gw_{n+1} = F(w_n,w_n,y_n,z_n),$$

that is, (x_n, y_n, z_n, w_n) is a quadruple coincidence point of F and g. Then, we suppose that $(gx_n, gy_n, gz_n, gw_n) \neq (gx, gy, gz, gw)$ for all $n \geq 0$. By (2.1), consider now

$$\begin{aligned}
&G\bigl(gx, F(x,y,z,w), F(x,y,z,w)\bigr) \\
&\quad \leq G(gx, gx_{n+1}, gx_{n+1}) + G\bigl(gx_{n+1}, F(x,y,z,w), F(x,y,z,w)\bigr) \\
&\quad = G(gx, gx_{n+1}, gx_{n+1}) + G\bigl(F(x_n, y_n, z_n, w_n), F(x,y,z,w), F(x,y,z,w)\bigr) \\
&\quad \leq G(gx, gx_{n+1}, gx_{n+1}) \\
&\qquad + \frac{1}{4}\phi\bigl(\bigl\{G(gx_n, gx, gx) + G(gy_n, gy, gy) + G(gz_n, gz, gz) + G(gw_n, gw, gw)\bigr\}\bigr) \\
&\qquad - \psi\biggl(\frac{\{G(gx_n, gx, gx) + G(gy_n, gy, gy) + G(gz_n, gz, gz) + G(gw_n, gw, gw)\}}{4}\biggr). \qquad (2.34)
\end{aligned}$$

Taking the limit as $n \to \infty$ in (2.34) and using the property of ϕ, hence we get that $G(gx, F(x,y,z,w), F(x,y,z,w)) = 0$. Thus, $gx = F(x,y,z,w)$. Analogously, one finds

$$F(y,z,w,x) = gy, \qquad F(z,w,x,y) = gz, \quad \text{and} \quad F(w,x,y,z) = gw.$$

Thus, we proved that F and g have a quadruple coincidence point. This completes the proof of Theorem 2.1. □

Corollary 2.1 *Let $(X, \leq)$ be a partially ordered set and (X, G) be a G-metric space. Let $F : X \times X \times X \times X \to X$ and $g : X \to X$ be such that F has the mixed g-monotone property. Assume that there exists $\psi \in \Psi$ such that*

$$\begin{aligned}
&G\bigl(F(x,y,z,w), F(u,v,s,t), F(a,b,c,d)\bigr) \\
&\quad \leq \frac{1}{4}\bigl(G(gx, gu, gs) + G(gy, gv, gb) + G(gz, gs, gc) + G(gw, gt, gd)\bigr) \\
&\qquad - \psi\biggl(\frac{G(gx, gu, gs) + G(gy, gv, gb) + G(gz, gs, gc) + G(gw, gt, gd)}{4}\biggr) \qquad (2.35)
\end{aligned}$$

for all $x, y, z, w, u, v, s, t, a, b, c, d \in X$ with $gx \geq gu \geq ga$, $gy \leq gv \leq gb$, $gz \geq gs \geq gc$, and $gw \leq gt \leq gd$. Suppose $F(X^4) \subseteq g(X)$, g is continuous and commutes with F. If there exist $x_0, y_0, z_0, w_0 \in X$ such that

$$\begin{aligned}
&gx_0 \leq F(x_0, y_0, z_0, w_0), \qquad gy_0 \geq F(y_0, z_0, w_0, x_0), \\
&gz_0 \leq F(z_0, w_0, x_0, y_0), \quad \textit{and} \quad gw_0 \geq F(w_0, x_0, y_0, z_0),
\end{aligned}$$

suppose either

(a) *(X, G) is a complete G-metric space and F is continuous or*
(b) *$(g(X), G)$ is complete and $(X, G, \leq)$ has the following property:*
  (i) *if a non-decreasing sequence $x_n \to a$, then $x_n \leq x$ for all n,*
  (ii) *if a non-increasing sequence $y_n \to y$, then $y \leq y_n$ for all n,*
  then there exist $x, y, z, w \in X$ such that

$$F(x,y,z,w) = gx, \qquad F(y,z,w,x) = gy, \qquad F(z,w,x,y) = gz, \quad \textit{and} \quad F(w,x,y,z) = gw,$$

that is, F and g have a quadruple coincidence point.

Proof In Theorem 2.1 taking $\phi(t) = t$, we get Corollary 2.1. □

Corollary 2.2 *Let $(X, \leq)$ be a partially ordered set and (X, G) be a G-metric space. Let $F : X \times X \times X \times X \to X$ and $g : X \to X$ be such that F has the mixed g-monotone property such that*

$$\begin{aligned} &G\bigl(F(x,y,z,w), F(u,v,s,t), F(a,b,c,d)\bigr) \\ &\quad \leq \frac{k}{4}\bigl(G(gx,gu,gs) + G(gy,gv,gb) + G(gz,gs,gc) + G(gw,gt,gd)\bigr) \end{aligned}$$

for all $x, y, z, w, u, v, s, t, a, b, c, d \in X$ with $gx \geq gu \geq ga$, $gy \leq gv \leq gb$, $gz \geq gs \geq gc$, and $gw \leq gt \leq gd$. Suppose $F(X^4) \subseteq g(X)$, g is continuous and commutes with F. If there exist $x_0, y_0, z_0, w_0 \in X$ such that

$$\begin{aligned} &gx_0 \leq F(x_0, y_0, z_0, w_0), \qquad gy_0 \geq F(y_0, z_0, w_0, x_0), \\ &gz_0 \leq F(z_0, w_0, x_0, y_0), \quad \text{and} \quad gw_0 \geq F(w_0, x_0, y_0, z_0), \end{aligned}$$

suppose either

(a) *(X, G) is a complete G-metric space and F is continuous or*

(b) *$(X, G, \leq)$ has the following property:*

   (i) *if a non-decreasing sequence $x_n \to a$, then $x_n \leq x$ for all n,*

   (ii) *if a non-increasing sequence $y_n \to y$, then $y \leq y_n$ for all n,*

   then there exist $x, y, z, w \in X$ such that

$$F(x,y,z,w) = gx, \qquad F(y,z,w,x) = gy, \qquad F(z,w,x,y) = gz, \quad \text{and} \quad F(w,x,y,z) = gw,$$

that is, F and g have a quadruple coincidence point.

Proof In Corollary 2.1 taking $\psi(t) = \frac{1-k}{4}t$, we get Corollary 2.2. □

Now, we shall prove the existence and uniqueness of a quadruple common fixed point. For a product X^4 of a partially ordered set $(X, \leq)$, we define a partial ordering in the following way. For all $(x, y, z, w), (u, v, r, h) \in X^4$,

$$(x,y,z,w) \leq (u,v,r,h) \quad \Leftrightarrow \quad x \leq u, \qquad y \geq v, \qquad z \leq r, \quad \text{and} \quad w \geq l. \tag{2.36}$$

We say that (x, y, z, w) and (u, v, r, l) are comparable if

$$(x,y,z,w) \leq (u,v,r,l) \quad \text{or} \quad (u,v,r,l) \leq (x,y,z,w).$$

Also, we say that (x, y, z, w) is equal to (u, v, r, l) if and only if $x = u$, $y = v$, $z = r$, $w = l$.

Theorem 2.2 *In addition to the hypothesis of Theorem* 2.1, *suppose that for all (x, y, z, w), $(u, v, r, l) \in X^4$, there exists $(a, b, c, d) \in X \times X \times X \times X$ such that $(F(a, b, c, d), F(b, c, d, a), F(c, d, a, b), F(d, a, b, c))$ is comparable to $(F(x, y, z, w), F(y, z, w, x), F(z, w, x, y), F(w, x, y, z))$ and $(F(u, v, r, l), F(v, r, l, u), F(r, l, u, v), F(l, u, v, r))$. Then F and g have a unique quadruple common fixed point (x, y, z, w) such that $x = gx = F(x, y, z, w)$, $y = gy = F(y, z, w, x)$, $z = gz = F(z, w, x, y)$, and $w = gw = F(w, x, y, z)$.*

Proof The set of quadruple coincidence points of F and g is not empty due to Theorem 2.1. Assume now (x,y,z,w) and (u,v,r,l) are two quadruple coincidence points of F and g, that is,

$$\begin{aligned} &F(x,y,z,w)=gx, &&F(u,v,r,l)=gu,\\ &F(y,z,w,x)=gy, &&F(v,r,l,u)=gv,\\ &F(z,w,x,y)=gz, &&F(r,l,u,v)=gr,\\ &F(w,x,y,z)=gw, &&F(l,u,v,r)=gl. \end{aligned}$$

We shall show that (gx,gy,gz,gw) and (gu,gv,gr,gl) are equal. By assumption, there exists $(a,b,c,d)\in X\times X\times X\times X$ such that $(F(a,b,c,d),F(b,c,d,a),F(c,d,a,b),F(d,a,b,c))$ is comparable to $(F(x,y,z,w),F(y,z,w,x),F(z,w,x,y),F(w,x,y,z))$ $(F(u,v,r,l),F(v,r,l,u),F(r,l,u,v),F(l,u,v,r))$.

Define the sequences $\{ga_n\}$, $\{gb_n\}$, $\{gc_n\}$, and $\{gd_n\}$ such that

$$\begin{aligned} &a_0=a,\qquad b_0=b,\qquad c_0=c,\qquad d_0=d\quad \text{and for any } n\geq 1,\\ &ga_n=F(a_{n-1},b_{n-1},c_{n-1},d_{n-1}),\\ &gb_n=F(b_{n-1},c_{n-1},d_{n-1},a_{n-1}),\\ &gc_n=F(c_{n-1},d_{n-1},a_{n-1},b_{n-1}),\\ &gd_n=F(d_{n-1},a_{n-1},b_{n-1},c_{n-1}) \end{aligned}\tag{2.37}$$

for all n. Further, set $x_0=x$, $y_0=y$, $z_0=z$, $w_0=w$ and $u_0=u$, $v_0=v$, $r_0=r$, $l_0=l$ and in the same way define the sequences $\{gx_n\}$, $\{gy_n\}$, $\{gz_n\}$, $\{gw_n\}$ and $\{gu_n\}$, $\{gv_n\}$, $\{gr_n\}$, $\{gl_n\}$. Then it is easy to see that

$$\begin{aligned} &gx_n=F(x,y,z,w), &&gu_n=F(u,v,r,l),\\ &gy_n=F(y,z,w,x), &&gv_n=F(v,r,l,u),\\ &gz_n=F(z,w,x,y), &&gr_n=F(r,l,u,v),\\ &gw_n=F(w,x,y,z), &&gl_n=F(l,u,v,r) \end{aligned}\tag{2.38}$$

for all $n\geq 1$. Since $(F(x,y,z,w),F(y,z,w,x),F(z,w,x,y),F(w,x,y,z))=(gx_1,gy_1,gz_1,gw_1)=(gx,gy,gz,gw)$ is comparable to $(F(a,b,c,d),F(b,c,d,a),F(c,d,a,b),F(d,a,b,c))=(ga_1,gb_1,gc_1,gd_1)$, then it is easy to show $(gx,gy,gz,gw)\geq(ga_1,gb_1,gc_1,gd_1)$. Recursively, we get that

$$(gx,gy,gz,gw)\geq(ga_n,gb_n,gc_n,gd_n)\quad\text{for all } n. \tag{2.39}$$

By (2.39) and (2.1), we have

$$\begin{aligned} &\phi\bigl(G(gx,gx,ga_{n+1})\bigr)\\ &\quad=\phi\bigl(G\bigl(F(x,y,z,w),F(x,y,z,w),F(a_n,b_n,c_n,d_n)\bigr)\bigr)\\ &\quad\leq\frac{1}{4}\phi\bigl(\bigl\{G(gx,gx,ga_n)+G(gy,gy,gb_n)+G(gz,gz,gc_n)+G(gw,gw,gd_n)\bigr\}\bigr)\\ &\qquad-\psi\biggl(\frac{G(gx,gx,ga_n)+G(gy,gy,gb_n)+G(gz,gz,gc_n)+G(gw,gw,gd_n)}{4}\biggr), \end{aligned}\tag{2.40}$$

$$\begin{aligned}
&\phi\big(G(gb_{n+1},gy,gy)\big)\\
&\quad=\phi\big(G\big(F(b_n,c_n,d_n,a_n),F(y,z,w,x),F(y,z,w,x)\big)\big)\\
&\quad\le\frac{1}{4}\phi\big(G(gb_n,gy,gy)+G(gc_n,gz,gz)+G(gd_n,gw,gw)+G(ga_n,gx,gx)\big)\\
&\qquad-\psi\left(\frac{G(gb_n,gy,gy)+G(gc_n,gz,gz)+G(gd_n,gw,gw)+G(ga_n,gx,gx)}{4}\right),
\end{aligned}\tag{2.41}$$

$$\begin{aligned}
&\phi\big(G(gz,gz,gc_{n+1})\big)\\
&\quad=\phi\big(G\big(F(z,w,x,y),F(z,w,x,y),F(c_n,d_n,a_n,b_n)\big)\big)\\
&\quad\le\frac{1}{4}\phi\big(G(gc_n,gz,gz)+G(gd_n,gw,gw)+G(ga_n,gx,gx)+G(gb_n,gy,gy)\big)\\
&\qquad-\psi\left(\frac{G(gc_n,gz,gz)+G(gd_n,gw,gw)+G(ga_n,gx,gx)+G(gb_n,gy,gy)}{4}\right),
\end{aligned}\tag{2.42}$$

and

$$\begin{aligned}
&\phi\big(G(gd_{n+1},gw,gw)\big)\\
&\quad=\phi\big(G\big(F(d_n,a_n,b_n,c_n),F(w,x,y,z),F(w,x,y,z)\big)\big)\\
&\quad\le\frac{1}{4}\phi\big(G(gd_n,gw,gw)+G(ga_n,gx,gx)+G(gb_n,gy,gy)+G(gc_n,gz,gz)\big)\\
&\qquad-\psi\left(\frac{G(gd_n,gw,gw)+G(ga_n,gx,gx)+G(gb_n,gy,gy)+G(gc_n,gz,gz)}{4}\right).
\end{aligned}\tag{2.43}$$

It follows from (2.40)-(2.43) and the property of ϕ that

$$\begin{aligned}
&\phi\big(G(gx,gx,ga_{n+1})+G(gb_{n+1},gy,gy)+G(gz,gz,gc_{n+1})+G(gd_{n+1},gw,gw)\big)\\
&\quad\le\phi\big(G(gx,gx,ga_{n+1})\big)+\phi\big(G(gb_{n+1},gy,gy)\big)\\
&\qquad+\phi\big(G(gz,gz,gc_{n+1})\big)+\phi\big(G(gd_{n+1},gw,gw)\big)\\
&\quad\le\phi\big(G(ga_n,gx,gx)+G(gb_n,gy,gy)+G(gc_n,gz,gz)+G(gd_n,gw,gw)\big)\\
&\qquad-4\psi\left(\frac{G(ga_n,gx,gx)+G(gb_n,gy,gy)+G(gc_n,gz,gz)+G(gd_n,gw,gw)}{4}\right).
\end{aligned}\tag{2.44}$$

Thus,

$$\begin{aligned}
&\phi\big(G(gx,gx,ga_{n+1})+G(gb_{n+1},gy,gy)+G(gz,gz,gc_{n+1})+G(gd_{n+1},gw,gw)\big)\\
&\quad\le\phi\big(G(ga_n,gx,gx)+G(gb_n,gy,gy)+G(gc_n,gz,gz)+G(gd_n,gw,gw)\big)
\end{aligned}\tag{2.45}$$

and therefore,

$$\begin{aligned}
&G(gx,gx,ga_{n+1})+G(gb_{n+1},gy,gy)+G(gz,gz,gc_{n+1})+G(gd_{n+1},gw,gw)\\
&\quad\le G(ga_n,gx,gx)+G(gb_n,gy,gy)+G(gc_n,gz,gz)+G(gd_n,gw,gw).
\end{aligned}\tag{2.46}$$

Hence, the sequence $\{G(ga_n,gx,gx)+G(gb_n,gy,gy)+G(gc_n,gz,gz)+G(gd_n,gw,gw)\}$ is a decreasing sequence; therefore, there exists $\alpha>0$ such that

$$\lim_{n\to\infty}\big(G(ga_n,gx,gx)+G(gb_n,gy,gy)+G(gc_n,gz,gz)+G(gd_n,gw,gw)\big)=\alpha.$$

We shall show that $\alpha = 0$. Suppose to the contrary $\alpha > 0$. Taking the limit as $n \to \infty$ in (2.44), then we have

$$\begin{aligned}\phi(\alpha) &\le \phi(\alpha) - 4 \lim_{n\to\infty} \psi\big(\big(G(ga_n, gx, gx) + G(gb_n, gy, gy) \\ &\quad + G(gc_n, gz, gz) + G(gd_n, gw, gw)\big)/4\big) < \phi(\alpha),\end{aligned}$$

a contradiction. Thus, $\alpha = 0$, that is,

$$\lim_{n\to\infty} \big(G(ga_n, gx, gx) + G(gb_n, gy, gy) + G(gc_n, gz, gz) + G(gd_n, gw, gw)\big) = 0.$$

This yields that

$$\begin{aligned}&\lim_{n\to\infty} G(ga_n, gx, gx) = 0, \qquad \lim_{n\to\infty} G(gb_n, gy, gy) = 0, \\ &\lim_{n\to\infty} G(gc_n, gz, gz) = 0 \quad \text{and} \quad \lim_{n\to\infty} G(gd_n, gw, gw) = 0.\end{aligned} \tag{2.47}$$

Analogously, we show that

$$\begin{aligned}&\lim_{n\to\infty} G(ga_n, gu, gu) = 0, \qquad \lim_{n\to\infty} G(gb_n, gv, gv) = 0, \\ &\lim_{n\to\infty} G(gc_n, gr, gr) = 0 \quad \text{and} \quad \lim_{n\to\infty} G(gd_n, gl, gl) = 0.\end{aligned} \tag{2.48}$$

Combining (2.47) and (2.48) yields that (gx, gy, gz, gw) and (gu, gv, gr, gl) are equal.

Since $gx = F(x, y, z, w)$, $gy = F(y, z, w, x)$, $gz = F(z, w, x, y)$, and $gz = F(z, w, x, y)$, by commutativity of F and g, we have

$$\begin{aligned}gx' &= g(gx) = g\big(F(x, y, z, w)\big) = F(gx, gy, gz, gw), \\ gy' &= g(gy) = g\big(F(y, z, w, x)\big) = F(gy, gz, gw, gx), \\ gz' &= g(gz) = g\big(F(z, w, x, y)\big) = F(gz, gw, gx, gy),\end{aligned}$$

and

$$gw' = g(gw) = g\big(F(w, x, y, z)\big) = F(gw, gx, gy, gz),$$

where $gx = x'$, $gy = y'$, $gz = z'$, and $gw = w'$. Thus, (x', y', z', w') is a quadruple coincidence point of F and g. Consequently, (gx', gy', gz', gz') and (gx, gy, gz, gw) are equal. We deduce

$$gx' = gx = x', \qquad gy' = gy = y' \quad \text{and} \quad gz' = gz = z', \qquad gw' = gw = w'.$$

Therefore, (x', y', z', w') is a quadruple common fixed point of F and g. Its uniqueness follows easily from (2.1). □

Example 2.1 Let $X = \mathbb{R}$ with a usual ordering. Define $G : X \times X \times X \to X$ by $G(x, y, z) = \max\{|x - y|, |y - z|, |x - z|\}$. Let $g : X \to X$ and $F : X \times X \times X \times X \to X$ be defined by

$$g(x) = \frac{x}{2}, \qquad F(x, y, z, w) = \frac{x - y + z - w}{24} \quad \text{for all } x, y, z, w \in X.$$

Take $\phi : [0,+\infty) \to [0,+\infty)$ be given by $\phi(t) = \frac{4}{3}t$ for all $t \in [0,+\infty)$ and $\psi : [0,+\infty) \to [0,+\infty)$ by $\psi(t) = \frac{2}{3}t$ for all $t \in [0,+\infty)$. Then

a. $(X, G, \preceq)$ is a complete ordered G-metric space.

b. For $x, y, z, w, u, v, s, t, a, b, c, d \in X$ with $gx \geq gu \geq ga$, $gy \leq gv \leq gb$, $gz \geq gh \geq gc$, and $gw \leq gt \leq gd$, we have the condition (2.1) of Theorem 2.1 satisfied.

c. F and g have the mixed g-monotone property.

Proof To prove (b), given $x, y, z, w, u, v, s, t, a, b, c, d \in X$ with $gx \geq gu \geq ga$, $gy \leq gv \leq gb$, $gz \geq gs \geq gc$, and $gw \leq gt \leq gd$. Then

$$\left|F(x,y,z,w) - F(u,v,s,t)\right| = \frac{1}{24}\big((x-u) + (v-y) + (z-s) + (t-w)\big)$$

but

$$\begin{aligned}
&\frac{1}{18}(x-u) \leq \frac{1}{2}\max\{x-u, x-a, u-a\} = G(gx, gu, ga),\\
&\frac{1}{18}(v-y) \leq \frac{1}{2}\max\{v-y, b-v, b-y\} = G(gy, gv, gb),\\
&\frac{1}{18}(z-s) \leq \frac{1}{2}\max\{z-s, z-c, s-c\} = G(gz, gs, gc), \quad \text{and}\\
&\frac{1}{18}(l-w) \leq \frac{1}{2}\max\{t-w, d-t, d-w\} = G(gw, gt, gd); \quad \text{therefore,}\\
&\phi\big(\left|F(x,y,z,w) - F(u,v,s,t)\right|\big)\\
&\quad = \left(\frac{4}{3}\right)\frac{1}{24}\big((x-u) + (v-y) + (z-s) + (t-w)\big)\\
&\quad = \frac{1}{18}\big((x-u) + (v-y) + (z-s) + (t-w)\big)\\
&\quad \leq \frac{1}{6}\big\{G(gx, gu, ga) + G(gy, gv, gb) + G(gz, gs, gc) + G(gw, gt, gd)\big\}\\
&\quad = \left(\frac{1}{4}\right)\left(\frac{4}{3}\right)\big(G(gx, gu, ga) + G(gy, gv, gb) + G(gz, gs, gc) + G(gw, gt, gd)\big)\\
&\qquad - \frac{2}{3}\left(\frac{G(gx, gu, ga) + G(gy, gv, gb) + G(gz, gs, gc) + G(gw, gt, gd)}{4}\right)\\
&\quad = \frac{1}{4}\phi\big(G(gx, gu, ga) + G(gy, gv, gb) + G(gz, gs, gc) + G(gw, gt, gd)\big)\\
&\qquad - \psi\left(\frac{G(gx, gu, ga) + G(gy, gv, gb) + G(gz, gs, gc) + G(gw, gt, gd)}{4}\right).
\end{aligned} \tag{2.49}$$

Similarly,

$$\begin{aligned}
&\phi\big(\left|F(x,y,z,w) - F(a,b,c,d)\right|\big)\\
&\quad \leq \frac{1}{4}\phi\big(G(gx, gu, ga) + G(gy, gv, gb) + G(gz, gs, gc) + G(gw, gt, gd)\big)\\
&\qquad - \psi\left(\frac{G(gx, gu, ga) + G(gy, gv, gb) + G(gz, gs, gc) + G(gw, gt, gd)}{4}\right),
\end{aligned} \tag{2.50}$$

and

$$\begin{aligned}&\phi\bigl(\bigl|F(u,v,s,t)-F(a,b,c,d)\bigr|\bigr)\\&\quad\le\frac{1}{4}\phi\bigl(G(gx,gu,ga)+G(gy,gv,gb)+G(gz,gs,gc)+G(gw,gt,gd)\bigr)\\&\qquad-\psi\left(\frac{G(gx,gu,ga)+G(gy,gv,gb)+G(gz,gs,gc)+G(gw,gt,gd)}{4}\right).\end{aligned}\tag{2.51}$$

Therefore,

$$\begin{aligned}&\phi\left(\max\left\{\begin{array}{c}|F(x,y,z,w)-F(u,v,s,t)|\\|F(x,y,z,w)-F(a,b,c,d)|\\|F(u,v,s,t)-F(a,b,c,d)|\end{array}\right\}\right)\\&\quad=\max\left\{\begin{array}{c}\phi(|F(x,y,z,w)-F(u,v,s,t)|)\\\phi(|F(x,y,z,w)-F(a,b,c,d)|)\\\phi(|F(u,v,s,t)-F(a,b,c,d)|)\end{array}\right\}\\&\quad\le\frac{1}{4}\phi\bigl(G(gx,gu,ga)+G(gy,gv,gb)+G(gz,gs,gc)+G(gw,gt,gd)\bigr)\\&\qquad-\psi\left(\frac{G(gx,gu,ga)+G(gy,gv,gb)+G(gz,gs,gc)+G(gw,gt,gd)}{4}\right).\end{aligned}\tag{2.52}$$

Hence,

$$\begin{aligned}&\phi\bigl(G\bigl(F(x,y,z,w),F(u,v,s,t),F(a,b,c,d)\bigr)\bigr)\\&\quad\le\frac{1}{4}\phi\bigl(G(gx,gu,gs)+G(gy,gv,gb)+G(gz,gs,gc)+G(gw,gt,gd)\bigr)\\&\qquad-\psi\left(\frac{G(gx,gu,gs)+G(gy,gv,gb)+G(gz,gs,gc),G(gw,gt,gd)}{4}\right).\end{aligned}\tag{2.53}$$

To prove (c), let $x,y,z,w\in X$. To show that $F(x,y,z,w)$ is g-monotone non-decreasing in x, let $x_1,x_2\in X$ with $gx_1\le gx_2$. Then $x_1\le x_2$, and so $x_1-y+z-w\le x_2-y+z-w$. Hence, $F(x_1,y,z,w)\le F(x_2,y,z,w)$.

Therefore, $F(x,y,z,w)$ is g-monotone non-decreasing in x. Similarly, we may show that $F(x,y,z,w)$ is g-monotone non-decreasing in z.

To show that $F(x,y,z,w)$ is g-monotone non-increasing in y, let $y_1,y_2\in X$ with $gy_1\le gy_2$, then $y_1\le y_2$. Hence, $x-y_2+z-w\le x-y_1+z-w$, so $F(x,y_2,z,w)\le F(x,y_1,z,w)$.

Therefore, $F(x,y,z,w)$ is g-monotone non-increasing in y. Similarly, we may show that $F(x,y,z,w)$ is g-monotone non-increasing in w.

Let $x_o=y_o=z_o=w_o=0$. It is obvious that other hypotheses of Theorem 2.1 are satisfied. Thus, by Theorems 2.1 and 2.2, F and g have a unique quadruple common fixed point. Here, $(0,0,0,0)$ is the unique quadruple common fixed point of F and g. □

3 Application

In this part, we use previously obtained results to deduce some quadruple coincidence point results for mappings satisfying a contraction of integral type in a complete G-metric space. We first introduce some notations.

We denote by Γ the set of functions $\alpha : [0, +\infty) \to [0, +\infty)$ satisfying the following conditions:

(i) α is a Lebesgue integrable mapping on each compact subset of $[0, +\infty)$;

(ii) for all $\epsilon > 0$, we have

$$\int_0^{\epsilon} \alpha(s)\,ds > 0;$$

(iii) α is sub-additive on each $[a, b] \subset [0, +\infty)$, that is,

$$\int_0^{a+b} \alpha(t)\,dt \le \int_0^{a} \alpha(t)\,dt + \int_0^{b} \alpha(t)\,dt.$$

Let $N \in \mathbf{N}$ be fixed. Let $\{\alpha_i\}_{1\le i\le N}$ be a family of N functions that belong to Γ. For all $t \ge 0$, we denote $(I_i)_{i=1,\dots,N}$ as follows:

$$\begin{aligned}
&I_1(t) = \int_0^{t} \alpha_1(s)\,ds,\\
&I_2(t) = \int_0^{I_1(t)} \alpha_2(s)\,ds = \int_0^{\int_0^t \alpha_1(s)\,ds} \alpha_2(s)\,ds,\\
&\dots\\
&I_N(t) = \int_0^{I_{N-1}(t)} \alpha_N(s)\,ds.
\end{aligned}$$

We have the following result.

Theorem 3.1 *Let $(X, \le)$ be a partially ordered set and (X, G) be a complete G-metric space. Suppose $F : X^4 \to X$ and $g : X \to X$ are such that F is continuous and F has the mixed g-monotone property. Assume that there exist $\phi \in \Phi$ and $\psi \in \Psi$ such that*

$$\begin{aligned}
&I_N\bigl(\phi\bigl(G\bigl(F(x,y,z,w),F(u,v,s,t),F(a,b,c,d)\bigr)\bigr)\bigr)\\
&\quad\le \frac{1}{4} I_N\bigl(\phi\bigl(G(gx,gu,gs)+G(gy,gv,gb)+G(gz,gs,gc),Gg(w,gt,gd)\bigr)\bigr)\\
&\qquad - I_N\left(\psi\left(\frac{G(gx,gu,gs)+G(gy,gv,gb)+G(gz,gs,gc),G(gw,gt,gd)}{4}\right)\right)
\end{aligned} \tag{3.1}$$

for any $x, y, z, w, u, v, s, t, a, b, c, d \in X$, for which $gx \ge gu \ge ga$, $gy \le gv \le gb$, $gz \ge gs \ge gc$, and $gw \le gt \le gd$. Also, suppose $F(X^4) \subseteq g(X)$, g is continuous and commutes with F. If there exist $x_0, y_0, z_0, w_0 \in X$ such that $gx_0 \le F(x_0, y_0, z_0, w_0)$, $gy_0 \ge F(y_0, z_0, w_0, x_0)$, $gz_0 \le F(z_0, w_0, x_0, y_0)$, and $gw_0 \ge F(w_0, x_0, y_0, z_0)$, then there exist $x, y, z, w \in X$ such that

$$F(x,y,z,w) = gx, \qquad F(y,z,w,x) = gy, \qquad F(z,w,x,y) = gz, \quad \text{and} \quad F(w,x,y,z) = gw,$$

that is, F and g have a quadruple coincidence point.

Proof Take $\bar{\phi} = I_N \circ \phi$ and $\bar{\psi} = I_N \circ \psi$.

Note that the $\{\alpha_i\}_{1\leq i=1,\dots,N}$ are taken to be sub-additive on each $[a,b]\subset[0,+\infty)$ in order to get

$$\bar{\phi}(a+b)\leq\bar{\phi}(a)+\bar{\phi}(b).$$

Moreover, it is easy to show that $\bar{\phi}$ is continuous, non-decreasing and verifies $\bar{\phi}(t)=0$ iff $t=0$.

We get that $\bar{\phi}\in\Phi$. Also, we can find that $\bar{\psi}\in\Psi$. From (3.1), we have

$$\begin{aligned}
&\bar{\phi}\bigl(G\bigl(F(x,y,z,w),F(u,v,s,t),F(a,b,c,d)\bigr)\bigr)\\
&\quad\leq\frac{1}{4}\bar{\phi}\bigl(G(gx,gu,gs)+G(gy,gv,gb)+G(gz,gs,gc),Gg(w,gt,gd)\bigr)\\
&\qquad-\bar{\psi}\left(\frac{G(gx,gu,gs)+G(gy,gv,gb)+G(gz,gs,gc),G(gw,gt,gd)}{4}\right).
\end{aligned}\tag{3.2}$$

Now, applying Theorem 2.1, we obtain the desired result. □

Competing interests

The author declares that they have no competing interests.

Received: 18 May 2012 Accepted: 22 October 2012 Published: 6 November 2012

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doi:10.1186/1687-1812-2012-199
Cite this article as: Mustafa: **Mixed g-monotone property and quadruple fixed point theorems in partially ordered G-metric spaces using $(\phi - \psi)$ contractions.** *Fixed Point Theory and Applications* 2012 **2012**:199.