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Common fixed points of ordered g-quasicontractions and weak contractions in ordered metric spaces

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Abstract

We introduce ordered quasicontractions and *g*-quasicontractions in partially ordered metric spaces and prove the respective coincidence point and (common) fixed point results. An example shows that the new concepts are distinct from the existing ones. We also prove fixed point theorems for mappings satisfying so-called weak contractive conditions in the setting of partially ordered metric space. Hence, generalizations of several known results are obtained.

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1 Introduction

It is well known that the Banach contraction principle has been generalized in various directions. A lot of authors have used generalized contractive conditions of the type

$$d(fx, fy) \leq \lambda M(x, y),$$

where (X, d) is a metric space, $f, g : X \to X$, $\lambda \in [0,1)$ and $M(x, y)$ is the maximum of one of the sets

$$\{d(gx, gy)\}, \qquad \{d(gx, gy), d(gx, fx), d(gy, fy)\},$$
$$\left\{d(gx, gy), d(gx, fx), d(gy, fy), \frac{1}{2}(d(fx, gy) + d(gx, fy))\right\},$$

and alike. To obtain results about coincidence points or common fixed points of mappings f and g, in the case when $fX \subset gX$, usually the technique of so-called Jungck sequence $y_n = fx_n = gx_{n+1}$ was used (see, [1,2]).

In the case when

$$M(x, y) = \max \{d(gx, gy), d(gx, fx), d(gy, fy), d(gx, fy), d(gy, fx)\},$$

the mapping f is called a *g*-quasicontraction (or simply quasicontraction if $g = i_X$), see [3,4]. In this case different techniques were used to obtain common fixed point results in the cited and subsequent articles.

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The existence of fixed points in partially ordered metric spaces was first investigated by Ran and Reurings [5], and then by Nieto and Lopez [6,7]. Further results in this direction were proved, e.g., in [8-16].

In Section 2 of this article, we introduce ordered quasicontractions and *g*-quasicontractions in partially ordered metric spaces and prove the respective (common) fixed point results. An example shows that the new concepts are distinct from the existing ones, so our results are extensions of known ones.

Recently, several authors (see, e.g., [13,17-23]) have begun to use more general conditions (so-called weak contractive conditions) of the type

$$\psi\left(d(fx, fy)\right) \leq \psi(M(x, y)) - \varphi(M(x, y)),$$

where ψ and ϕ are so-called control functions (see definition in Section 3). The usage of Jungck sequence is here also possible, but becomes more involved. Also, some applications were obtained, in particular when dealing with differential, matrix and integral equations (see, e.g., [5-7,10]).

In Section 3 of the present article, we consider weak contractive conditions in the setting of partially ordered metric space (it is then restricted to the case when *gx* and *gy* are comparable) and prove respective common fixed point theorems. These results can be considered as generalizations of theorems from [9-11] since control functions are more general than comparison functions used in these articles.

2 Common fixed points of ordered *g*-quasicontractions

Consider a partially ordered set $(X, \preccurlyeq)$ and two self-maps $f, g : X \rightarrow X$ such that $fX \subset gX$. Following [9] we shall say that the mapping *f* is *g-nondecreasing* (resp., *g-nonincreasing*) if $gx \preccurlyeq gy \Rightarrow fx \preccurlyeq fy$ (resp., $gx \preccurlyeq gy \Rightarrow fx \succcurlyeq fy$) holds for each $x, y \in X$.

For arbitrary $x_0 \in X$ one can construct a so-called Jungck sequence $\{y_n\}$ in the following way: denote $y_0 = fx_0 \in fX \subset gX$; there exists $x_1 \in X$ such that $gx_1 = y_0 = fx_0$; now $y_1 = fx_1 \in fX \subset gX$ and there exists $x_2 \in X$ such that $gx_2 = y_1 = fx_1$ and the procedure can be continued. The following properties of this sequence can be easily deduced.

Lemma 1 1° *If f is g-nondecreasing and* $gx_0 \preccurlyeq fx_0$ *(resp.,* $gx_0 \succcurlyeq fx_0$*), then the sequence* $\{y_n\}$ *is nondecreasing (resp., nonincreasing) w.r.t.* $\preccurlyeq$.

2° *If f is g-nonincreasing and* fx_0 *is comparable with* gx_0*, then the sequence* $\{y_n\}$ *is not monotonous, but arbitrary two of its adjacent terms are comparable.*

Proof 1° It follows from $gx_0 \preccurlyeq fx_0 = gx_1$ that $fx_0 \preccurlyeq fx_1$, i.e., $y_0 \preccurlyeq y_1$ and it is easy to proceed by induction. The other case follows similarly.

2° Let, e.g., $gx_0 \preccurlyeq fx_0 = gx_1$. Then, by assumption, $fx_0 \succcurlyeq fx_1$, i.e., $y_0 \succcurlyeq y_1$. Now $gx_1 \succcurlyeq gx_2$ implies $fx_1 \preccurlyeq fx_2$, i.e., $y_1 \preccurlyeq y_2$ and the conclusion again follows.

Putting $g = i_X$ (identity map) in the previous lemma, we obtain

Corollary 1 1° *If f is nondecreasing and* $x_0 \preccurlyeq fx_0$ *(resp.* $x_0 \succcurlyeq fx_0$*), then the Picard sequence* $\{f^n x_0\}$ *is nondecreasing (nonincreasing) w.r.t.* $\preccurlyeq$.

2° *If f is nonincreasing and* fx_0 *is comparable with* x_0*, then the sequence* $\{f^n x_0\}$ *is not monotonous, but arbitrary two of its adjacent terms are comparable.*

Quasicontractions and *g*-quasicontractions in metric spaces were first studied in [3,4]. We shall call the mapping *f* an *ordered g-quasicontraction* if there exists $\lambda \in (0,1)$ such that for each $x, y \in X$ satisfying $gy \preccurlyeq gx$, the inequality

$$d(fx, fy) \leq \lambda \cdot M(x, y)$$

holds, where

$$M(x, y) = \max \left\{ d(gx, gy), d(gx, fx), d(gy, fy), d(gx, fy), d(gy, fx) \right\}. \tag{2.1}$$

Theorem 1 *Let* $(X, d, \preccurlyeq)$ *be a partially ordered metric space and let* $f, g : X \to X$ *be two self-maps on* X *satisfying the following conditions:*

(i) $fX \subset gX$;

(ii) gX *is complete;*

(iii) f *is* g-*nondecreasing;*

(iv) f *is an ordered* g-*quasicontraction;*

(v) there exists $x_0 \in X$ *such that* $gx_0 \preccurlyeq fx_0$;

(vi) if $\{gx_n\}$ *is a nondecreasing sequence that converges to some* $gz \in gX$ *then* $gx_n \preccurlyeq gz$ *for each* $n \in \mathbb{N}$ *and* $gz \preccurlyeq g(gz)$.

Then f *and* g *have a coincidence point, i.e., there exists* $z \in X$ *such that* $fz = gz$. *If, in addition, (vii)* f *and* g *are weakly compatible ([1,2]), i.e.,* $fx = gx$ *implies* $f\,gx = g\,fx$, *for each* $x \in X$, *then they have a common fixed point.*

Proof Starting with the point x_0 from condition (*v*), construct a Jungck sequence $y_n = fx_n = gx_{n+1}$. By Lemma 1, using condition (*iii*) we conclude that $\{y_n\}$ is nondecreasing. It can be proved by the Das-Naik's method (as in [8]) that $\{y_n\}$ is a Cauchy sequence. Since gX is complete (condition (*ii*)), there exists $z \in X$ such that $y_n \to w = gz \in gX$. We shall prove that $fz = w$.

By condition (*vi*), $gx_n \preccurlyeq gz$ holds. Hence, putting in the contractive condition (*iv*) $x = x_n$, $y = z$, we obtain that

$$d\left(fx_n, fz\right) \leq \lambda \max \left\{ d(gx_n, gz), d\left(gx_n, fx_n\right), d\left(gz, fz\right), d\left(gx_n, fz\right), d\left(fx_n, gz\right) \right\},$$

and passing to the limit when $n \to \infty$ (and using that $d(gx_n, fz) \leq d(gx_n, gz) + d(gz, fz)$) we deduce that

$$d\left(fz, gz\right) \leq \lambda \max \left\{ 0, 0, d\left(fz, gz\right), 0 + d\left(gz, fz\right), 0 \right\} = \lambda d\left(fz, gz\right).$$

It follows that $fz = gz = w$ and f and g have a coincidence point.

In the case when condition (*vii*) holds, we obtain that $fw = fgz = gfz = gw$ and it remains to prove that $fw = w$. By condition (*vi*) we have that $gz \preccurlyeq ggz = gw$ and then $d(fw, w) = d(fw, fz) \leq \lambda M(w, z)$, where

$$
\begin{aligned}
M(w, z) &= \max \left\{ d(gw, gz), d(gw, fw), d(gz, fz), d(gw, fz), d(gz, fw) \right\} \\
&= \max \left\{ d(fw, w), 0, 0, d(fw, w), d(w, fw) \right\} = d(fw, w).
\end{aligned}
$$

It follows that $fw = w$ and w is a common fixed point for f and g.

Remark 1 If condition (*v*) in the previous theorem is replaced by $fx_0 \preccurlyeq gx_0$, the respective Jungck sequence is nonincreasing and the conclusion of the theorem holds.

Theorem 2 *Let the conditions of Theorem 1 be satisfied, except that (iii), (v), and (vi) are, respectively, replaced by:*

(iii') f *is* g-*nonincreasing;*

(v') there exists $x_0 \in X$ *such that* fx_0 *and* gx_0 *are comparable;*

(vi') if $\{gx_n\}$ is a sequence in gX which has comparable adjacent terms and that converges to some $gz \in gX$, then there exists a subsequence $\{gx_{n_k}\}$ of $\{gx_n\}$ having all the terms comparable with gz and gz is comparable with ggz.

Then all the conclusions of Theorem 1 hold.

Proof Regardless whether $fx_0 \preccurlyeq gx_0$ or $gx_0 \preccurlyeq fx_0$ (condition(v')), Lemma 1 implies that the adjacent terms of the Jungck sequence $\{y_n\}$ are comparable. This is again sufficient to imply that $\{y_n\}$ is a Cauchy sequence. Hence, it converges to some $gz \in gX$.

By (vi'), there exists a subsequence $y_{n_k} = fx_{n_k} = gx_{n_k+1}$, $k \in \mathbb{N}$, having all the terms comparable with gz. Hence, we can apply the contractive condition to obtain $d\left(fz, fx_{n_k}\right) \leq \lambda M\left(z, x_{n_k}\right)$ where

$$
\begin{aligned}
M\left(z, x_{n_k}\right) &= \max\left\{d\left(gz, gx_{n_k}\right), d(gz, fz), d\left(gx_{n_k}, fx_{n_k}\right), d\left(gx_{n_k}, fz\right), d\left(gz, fx_{n_k}\right)\right\} \\
&\rightarrow \max\left\{0, d\left(fz, gz\right), 0, d(gz, fz), 0\right\} = d(fz, gz) \quad (k \rightarrow \infty),
\end{aligned}
$$

and since $d\left(fz, gz\right) \leq d\left(fz, fx_{n_k}\right) + d\left(fx_{n_k}, gz\right)$ and $fx_{n_k} \rightarrow gz$, $k \rightarrow \infty$, it follows that $fz = gz = w$. The rest of conclusions follow in the same way as in Theorem 1.

Remark 2 If the five-term set in condition (2.1) is replaced by any of the following sets

$$
\{d(gx, gy)\}, \qquad \{d(gx, gy), d(gx, fx), d(gy, fy)\},
$$

$$
\left\{d\left(gx, gy\right), \frac{1}{2}\left(d(gx, fx) + d(gy, fy)\right), \frac{1}{2}\left(d(gx, fy) + d(gy, fx)\right)\right\},
$$

$$
\left\{d\left(gx, gy\right), d(gx, fx), d(gy, fy), \frac{1}{2}\left(d(gx, fy) + d(gy, fx)\right)\right\},
$$

then conclusions of both Theorems 1 and 2 remain valid. Thus, "ordered" versions of several generalized contraction theorems proved recently are obtained. In particular, the case of the set $\{d(gx, gy)\}$ gives a generalization of the results of Ran and Reurings [5], and Nieto and Lopez [6,7]. Note also that in most of these cases the proof that the Jungck sequence is a Cauchy sequence is directly obtained from the famous Jungck relation

$$
d\left(y_{n+1}, y_n\right) \leq \lambda d\left(y_n, y_{n-1}\right).
$$

Remark 3 Taking $g = i_X$ we obtain "ordered" variants of various fixed point theorems involving one function f. In particular, the following "ordered" version of the known result of Ćirić on quasicontractions [3] is obtained.

Corollary 2 (a) *Let* $(X, d, \preccurlyeq)$ *be a partially ordered complete metric space and let* f : $X \rightarrow X$ *be a nondecreasing self-map such that for some* $\lambda \in (0,1)$,

$$
d\left(fx, fy\right) \leq \lambda \max\left\{d(x, y), d(x, fx), d(y, fy), d(x, fy), d(y, fx)\right\} \tag{2.2}
$$

holds for all $x, y \in X$ *such that* $y \preccurlyeq x$. *Suppose also that either*
(i) f is continuous or
(ii) for each nondecreasing sequence $\{x_n\}$ *converging to some* $u \in X$, $x_n \preccurlyeq u$ *holds for each* $n \in \mathbb{N}$.
If there exists $x_0 \in X$ *such that* $x_0 \preccurlyeq fx_0$, *then* f *has at least one fixed point.*

(b) *The same holds if f is nonincreasing, there exists x_0 comparable with fx_0 and (ii) is replaced by*

(ii') *if a sequence $\{x_n\}$ converging to some $u \in X$ has every two adjacent terms comparable, then there exists a subsequence $\{x_{n_k}\}$ having each term comparable with x.*

Proof As an illustration, we include a direct proof of case (b) when condition (*ii'*) is fulfilled.

Let u be the limit of the Picard sequence $\{f^n x_0\}$ and let $\{f^{n_k} x_0\}$ be a subsequence having all the terms comparable with u. Then we can apply contractivity condition to obtain

$$d\left(fu, f^{n_k+1}x_0\right) \leq \lambda \max \left\{ d\left(u, f^{n_k}x_0\right), d(u, fu), d\left(f^{n_k}x_0, f^{n_k+1}x_0\right), \right.$$
$$\left. d\left(u, f^{n_k+1}x_0\right), d\left(fu, f^{n_k}x_0\right) \right\}.$$

Using that $d(fu, u) \leq d\left(fu, f^{n_k+1}x_0\right) + d\left(f^{n_k+1}x_0, u\right)$ and passing to the limit when $k \to \infty$ we get

$$d\left(fu, u\right) \leq \lambda \max \left\{0, d\left(u, fu\right), 0, 0, d\left(u, fu\right)\right\} = \lambda d\left(u, fu\right),$$

and it follows that $fu = u$.

Note also that instead of the completeness of X, its f-orbitally completeness is sufficient to obtain the conclusion of the corollary.

It is clear that the following diagram, where arrows stand for implications, is valid:

$$\begin{array}{ccc} \text{contraction} & \to & \text{quasicontraction} \\ \downarrow & & \downarrow \\ \text{ordered contraction} & \to & \text{ordered quasicontraction} \end{array}$$
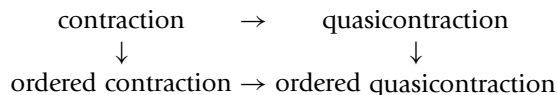
Reverse implications do not hold. An example of a quasicontraction which is not a contraction was given in [3]. We shall present an example of an ordered quasicontraction which is not a quasicontraction.

Example 1 Let $X = [0,4]$, equipped with the usual metric and ordered by

$$x \preceq y \Leftrightarrow (x = y) \quad \text{or} \quad (x, y \in [1, 4] \text{ and } x \leq y).$$

Consider the function $f: X \to X$, $fx = \begin{cases} 2x, & 0 \leq x < 1, \\ \frac{1}{3}(x + 5) & 1 \leq x \leq 4. \end{cases}$ Then

$$M(x, y) = \max_{0 \leq x < 1} \left\{ d(x, y), d(x, fx), d(y, fy), d(x, fy), d(y, fy) \right\}$$
$$= \max_{0 \leq x < 1} \left\{ |x - y|, |x|, |y|, |x - 2y|, |y - 2x| \right\},$$

and we have to check the condition $d(fx, fy) = 2|x - y| \leq \lambda M(x, y)$. Taking in each of the possible five inequalities that ($x \to 1-$ and $y = 0$) or ($x = 0$ and $y \to 1-$) we obtain that λ cannot be in the interval $[0,1)$. Hence, f is not a quasicontraction. But it is an ordered contraction, and so an ordered quasicontraction, since $d(fx, fy) \leq \lambda d(x, y)$ holds for $\lambda \in \left[\frac{1}{3}, 1\right)$ and all $x, y \in X$ such that $y \preceq x$.

2.1 Uniqueness of the fixed point

The following simple example shows that conditions of theorems in the previous section are not sufficient for the uniqueness of fixed points (resp., common fixed points).

Example 2 Let $X = \{(1, 0), (0,1)\}$, let $(a, b) \preccurlyeq (c, d)$ if and only if $a \le c$ and $b \le d$, and let d_1 be the Euclidean metric. The function $f((x, y)) = (x, y)$ is continuous. The only comparable pairs of points in X are (x, x) for $x \in X$ and then (2.2) reduces to $d_1(fx, fx) = 0$, and is trivially fulfilled. However, f has two fixed points $(1,0)$ and $(0,1)$.

We shall give a sufficient condition for the uniqueness of the fixed point in the case of an ordered quasicontraction.

Theorem 3 *Let all the conditions of Corollary 2 be satisfied with $\lambda \in (0, 1/2)$ in the contractivity condition (2.2) and let for all $x, y \in X$ there exists $z \in X$ comparable with both of them. Then:*

1. f has a unique fixed point u;

2. for arbitrary x_0 which can be a starting point of the Picard sequence $\{f^n x_0\}$ this sequence converges to u.

Proof Let u and v be two fixed points of f. If these two points are comparable, then

$$d(u, v) = d(fu, fv)$$
$$\le \lambda \max\{d(u, v), d(u, fu), d(v, fv), d(u, fv), d(v, fu)\} = \lambda d(u, v),$$

implying $u = v$.

Suppose that u and v are incomparable and that $w \in X$ is comparable with both of them. We shall prove that $d(u, f^n w) \to 0$ and $d(v, f^n w) \to 0$ when $n \to \infty$, which, taking into account that $d(u, v) \le d(u, f^n w) + d(f^n w, v)$, will imply that $d(u, v) = 0$ and $u = v$. Indeed,

$$d\left(u, f^n w\right) = d\left(ff^{n-1}u, ff^{n-1}w\right)$$
$$\le \lambda \max\left\{d\left(f^{n-1}u, f^{n-1}w\right), d\left(f^{n-1}u, f^n u\right), d\left(f^{n-1}w, f^n w\right),\right.$$
$$\left.d\left(f^{n-1}u, f^n w\right), d\left(f^{n-1}w, f^n u\right)\right\}$$
$$\le \lambda \max\left\{d\left(u, f^{n-1}w\right), 0, d\left(f^{n-1}w, u\right) + d\left(u, f^n w\right)\right.$$
$$\left.d\left(u, f^n w\right), d\left(f^{n-1}w, u\right)\right\}$$
$$= \lambda(d(u, f^{n-1}w) + d(u, f^n w)).$$

It follows that

$$d(u, f^n w) \le \frac{\lambda}{1 - \lambda} d(u, f^{n-1}w) = \mu d(u, f^{n-1}w).$$

Since $0 < \lambda < 1/2$, it is $\mu < 1$ and it follows that $d(u, f^n w) \le \mu^n d(u, w) \to 0$, when $n \to \infty$. Similarly, $d(v, f^n w) \to 0$ and the proof is complete in the case, when f is nondecreasing. The proof is similar when f is nonincreasing.

It is an open question whether the previous theorem is true for $\lambda \in [1/2, 1)$. Also, an open problem is to find sufficient conditions for the uniqueness of the common fixed point in the case of an ordered g-quasicontraction.

3 Weak ordered contractions

Functions $\psi, \phi : [0, \infty) \to [0, \infty)$ will be called *control functions* if:

(i) ψ is a continuous nondecreasing function with $\psi(t) = 0$ if and only if $t = 0$,

(ii) ϕ is a lower semi-continuous function with $\phi(t) = 0$ if and only if $t = 0$.

Let (X, d) be a metric space and let $f,g : X \to X$. In the articles [20,21] (in the setting of metric spaces) and [12,13] (in the setting of ordered metric spaces) contractive conditions of the form

$$\psi\left(d(fx, gy)\right) \leq \psi\left(M(x, y)\right) - \varphi(M(x, y)),$$

where

$$M(x, y) = \max\left\{d(x, y), d(x, fx), d(y, gy), \frac{1}{2}\left(d(x, gy) + d(y, fx)\right)\right\},$$

were used to obtain common fixed points results. We shall use here the following condition

$$\psi\left(d(fx, fy)\right) \leq \psi(M(x, y)) - \varphi(M(x, y)), \tag{3.1}$$

where

$$M(x, y) = \max\left\{d(gx, gy), d(gx, fx), d(gy, fy), \frac{1}{2}\left(d(gx, fy) + d(gy, fx)\right)\right\}. \tag{3.2}$$

Note that the respective result in the setting of ordered cone metric spaces is proved in [14].

Assertions similar to the following lemma (see, e.g., [24]) were used (and proved) in the course of proofs of several fixed point results in various articles.

Lemma 2 [24]*Let (X, d) be a metric space and let $\{y_n\}$ be a sequence in X such that the sequence $\{d(y_{n+1}, y_n)\}$ is nonincreasing and*

$$\lim_{n \to \infty} d\left(y_{n+1}, y_n\right) = 0.$$

If $\{y_{2n}\}$ is not a Cauchy sequence, then there exist $\varepsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that the following four sequences tend to ε, when $k \to \infty$:

$$d\left(y_{2m_k}, y_{2n_k}\right), \quad d\left(y_{2m_k}, y_{2n_k+1}\right), \quad d\left(y_{2m_k-1}, y_{2n_k}\right), \quad d\left(y_{2m_k-1}, y_{2n_k+1}\right). \tag{3.3}$$

Theorem 4 *Let $(X, d, \preccurlyeq)$ be a partially ordered metric space and let f, g be two self-maps on X satisfying the following conditions:*

(i) $fX \subset gX$;

(ii) gX is complete;

(iii) f is g-nondecreasing;

(iv) f and g satisfy condition (3.1) for each $x, y \in X$ such that $gy \preccurlyeq gx$, where (ψ, ϕ) is a pair of control functions and $M(x,y)$ is defined by (3.2);

(v) there exists $x_0 \in X$ such that $gx_0 \preccurlyeq fx_0$;

(vi) if $\{gx_n\}$ is a nondecreasing sequence that converges to some $gz \in gX$ then $gx_n \preccurlyeq gz$ for each $n \in \mathbb{N}$ and $gz \preccurlyeq g(gz)$.

Then f and g have a coincidence point. If in addition,

(vii) f and g are weakly compatible,

then they have a common fixed point.

Proof As in the proof of Theorem 1, a nondecreasing Jungck sequence y_n with $y_n = fx_n = gx_{n+1}$ can be constructed. Consider the following two possibilities: 1° $y_{n_0} = y_{n_0+1}$ for some $n_0 \in \mathbb{N}$ and 2° $y_n \neq y_{n+1}$ for each $n \in \mathbb{N}$.

$1°$ We shall prove that in this case $y_n = y_{n_0}$ for each $n \geq n_0$. Using that $gx_{n_0+1} \preceq gx_{n_0+2}$ we obtain

$$\psi\left(d\left(y_{n_0+1}, y_{n_0+2}\right)\right) = \psi\left(d\left(fx_{n_0+1}, fx_{n_0+2}\right)\right)$$
$$\leq \psi\left(M\left(x_{n_0+1}, x_{n_0+2}\right)\right) - \varphi\left(M\left(x_{n_0+1}, x_{n_0+2}\right)\right),$$

where

$$M(x_{n_0+1}, x_{n_0+2}) = \max\left\{d(gx_{n_0+1}, gx_{n_0+2}), d(gx_{n_0+1}, fx_{n_0+1}), d(gx_{n_0+2}, fx_{n_0+2}),\right.$$
$$\left.\frac{1}{2}(d(gx_{n_0+1}, fx_{n_0+2}) + d(gx_{n_0+2}, fx_{n_0+1}))\right\}$$
$$= \max\left\{d(y_{n_0}, y_{n_0+1}), d(y_{n_0}, y_{n_0+1}), d(y_{n_0+1}, y_{n_0+2}),\right.$$
$$\left.\frac{1}{2}(d(y_{n_0}, y_{n_0+2}) + 0)\right\}$$
$$= \max\left\{0, 0, d(y_{n_0+1}, y_{n_0+2}), \frac{1}{2}d(y_{n_0+1}, y_{n_0+2})\right\}$$
$$= d(y_{n_0+1}, y_{n_0+2}).$$

Hence $\psi(d(y_{n_0+1}, y_{n_0+2})) \leq \psi(d(y_{n_0+1}, y_{n_0+2})) - \varphi(d(y_{n_0+1}, y_{n_0+2}))$ which, by the properties of control functions, implies that $d(y_{n_0+1}, y_{n_0+2}) = 0$ and $y_{n_0+1} = y_{n_0+2}$. By induction, $y_n = y_{n_0}$ for each $n \geq n_0$. It follows that $gx_{n_0+1} = fx_{n_0+1}$ and x_{n_0+1} is a coincidence point of f and g.

Suppose that condition (*vii*) holds. Then also

$$fy_{n_0} = fgx_{n_0+1} = gfx_{n_0+1} = gy_{n_0},$$

and if we prove that $fy_{n_0} = y_{n_0}$, y_{n_0} will be a common fixed point for f and g. Using condition (*vi*) with $z = y_{n_0}$ we obtain that $gx_{n_0+1} \preceq gy_{n_0}$ and we can apply condition (*iv*) to obtain

$$\psi\left(d\left(fx_{n_0+1}, fy_{n_0}\right)\right) \leq \psi\left(M\left(x_{n_0+1}, y_{n_0}\right)\right) - \varphi\left(M\left(x_{n_0+1}, y_{n_0}\right)\right),$$

where

$$M\left(x_{n_0+1}, y_{n_0}\right) = \max\left\{d\left(gx_{n_0+1}, gy_{n_0}\right), d\left(gx_{n_0+1}, fx_{n_0+1}\right), d\left(gy_{n_0}, fy_{n_0}\right),\right.$$
$$\left.\frac{1}{2}\left(d\left(g_{n_0+1}, fy_{n_0}\right) + d\left(gy_{n_0}, fx_{n_0+1}\right)\right)\right\}$$
$$= \max\left\{d\left(y_{n_0}, fy_{n_0}\right), d\left(y_{n_0}, y_{n_0}\right), d\left(fy_{n_0}, fy_{n_0}\right),\right.$$
$$\left.\frac{1}{2}\left(d\left(y_{n_0}, fy_{n_0}\right) + d\left(fy_{n_0}, y_{n_0}\right)\right)\right\}$$
$$= d\left(y_{n_0}, fy_{n_0}\right).$$

Thus, $\psi(d(y_{n_0}, fy_{n_0})) \leq \psi(d(y_{n_0}, fy_{n_0})) - \varphi(d(y_{n_0}, fy_{n_0}))$ and again using properties of control functions it follows that $fy_{n_0} = y_{n_0}$. This completes the proof of the theorem in the first case.

$2°$ We shall prove that in this case $\{y_n\}$ is a Cauchy sequence. Since $gx_{n+1} \preceq gx_{n+2}$, we have that

$$\psi(d(y_{n+1}, y_{n+2})) = \psi(d(fx_{n+1}, fx_{n+2}))$$
$$\leq \psi(M(x_{n+1}, x_{n+2})) - \varphi(M(x_{n+1}, x_{n+2})),$$

where

$$M(x_{n+1}, x_{n+2}) = \max \left\{ d\left(gx_{n+1}, gx_{n+2}\right), d\left(gx_{n+1}, fx_{n+1}\right), d\left(gx_{n+2}, fx_{n+2}\right), \right.$$
$$\left. \frac{1}{2}\left(d\left(gx_{n+1}, fx_{n+2}\right) + d\left(gy_{n+2}, fx_{n+1}\right)\right)\right\}$$
$$= \max \left\{ d\left(y_n, y_{n+1}\right), d\left(y_n, y_{n+1}\right), d\left(y_{n+1}, y_{n+2}\right), \frac{1}{2}d\left(y_n, y_{n+2}\right) + 0 \right\}$$
$$= \max \left\{ d\left(y_n, y_{n+1}\right), d\left(y_{n+1}, y_{n+2}\right)\right\}$$

($d\left(y_n, y_{n+2}\right) \le d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2})$ was used to obtain the last equality). If $d\left(y_{n+1}, y_{n+2}\right) \ge d(y_n, y_{n+1})$, then it follows from

$$\psi\left(d\left(y_{n+1}, y_{n+2}\right)\right) \le \psi\left(d\left(y_{n+1}, y_{n+2}\right)\right) - \varphi\left(d\left(y_{n+1}, y_{n+2}\right)\right) \tag{3.4}$$

and the properties of control functions that $d(y_{n+1}, y_{n+2}) = 0$, contrary to the assumption that all terms of the sequence $\{y_n\}$ are distinct. Hence, $d(y_{n+1}, y_{n+2}) < d(y_n, y_{n+1})$, i.e., the sequence $\{d(y_n, y_{n+1})\}$ is decreasing. Therefore, it converges to some $d^* \le 0$ when $n \to \infty$. Then, $M(x_{n+1}, x_{n+2}) \to d^*$, $n \to \infty$.

From (3.4), passing to the (upper) limit when $n \to \infty$, it follows that

$$\psi(d*) \le \psi(d*) - \liminf_{n\to\infty} \varphi(d*) \le \psi(d*) - \varphi(d*),$$

i.e., $\phi(d^*) = 0$ and $d^* = 0$. We conclude that $d(y_n, y_{n+1}) \to 0$, $n \to \infty$. As a consequence, in order to prove that $\{y_n\}$ is a Cauchy sequence, it is enough to prove that $\{y_{2n}\}$ is a Cauchy sequence.

Suppose that $\{y_{2n}\}$ is not a Cauchy sequence. Then, by Lemma 2, there exist $\varepsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that the sequences (3.3) tend to ε when $k \to \infty$. Now, by definition of $M(x, y)$, we obtain that $M\left(x_{2m_k}, x_{2n_k+1}\right) \to \varepsilon$, $k \to \infty$. Indeed,

$$M\left(x_{2m_k}, x_{2n_k+1}\right)$$
$$= \max \left\{ d(gx_{2m_k}, gx_{2n_k+1}), d(gx_{2m_k}, fx_{2m_k}), d(gx_{2n_k+1}, fx_{2n_k+1}), \right.$$
$$\left. \frac{1}{2}(d(fx_{2m_k}, gx_{2n_k+1}) + d(fx_{2n_k+1}, gx_{2m_k}))\right\}$$
$$= \max \left\{ d(y_{2m_k-1}, y_{2n_k}), d(y_{2m_k-1}, y_{2m_k}), d(y_{2n_k}, y_{2n_k+1}), \right.$$
$$\left. \frac{1}{2}(d(y_{2m_k}, y_{2n_k}) + d(y_{2n_k+1}, y_{2m_k-1}))\right\}$$
$$\to \max \left\{ \varepsilon, 0, 0, \frac{1}{2}(\varepsilon + \varepsilon)\right\} = \varepsilon.$$

Now, putting $x = x_{2m_k}$ and $y = x_{2n_k+1}$ into the contractive condition (iv) (which is possible since $gx_{2n_k+1} = y_{2n_k}$ and $gx = gx_{2m_k} = y_{2m_k-1}$ are comparable), we get that

$$\psi\left(d\left(y_{2m_k}, y_{2n_k+1}\right)\right) = \psi\left(d\left(fx_{2m_k}, fx_{2n_k+1}\right)\right)$$
$$\le \psi\left(M\left(x_{2m_k}, x_{2n_k+1}\right)\right) - \varphi\left(M\left(x_{2m_k}, x_{2n_k+1}\right)\right).$$

Passing to the limit when $k \to \infty$ and using properties of control functions, we obtain that $\psi(\varepsilon) \le \psi(\varepsilon) - \phi(\varepsilon)$, which is in contradiction with $\varepsilon > 0$.

The proof that $\{y_n\}$ is a Cauchy sequence is complete. By the assumption (ii), there exists $z \in X$ such that $gx_n \to gz$, when $n \to \infty$. We shall prove that $fz = gz$.

Condition (*vi*) implies that $gx_n \preccurlyeq gz$ and we can apply contractive condition to obtain

$$\psi\left(d\left(fx_n, fz\right)\right) \le \psi\left(M\left(x_n, z\right)\right) - \varphi\left(M\left(x_n, z\right)\right), \tag{3.5}$$

where

$$M(x_n, z) = \max\left\{d(gx_n, gz), d\left(gx_n, fx_n\right), d(gz, fz), \frac{1}{2}\left(d(gx_n, fz) + d(gz, fx_n)\right)\right\}$$

$$\to \max\left\{0, 0, d(gz, fz), \frac{1}{2}d(gz, fz)\right\} = d(gz, fz).$$

Passing to the (upper) limit when $n \to \infty$ in (3.5) we obtain that

$$\psi\left(d\left(gz, fz\right)\right) \le \psi\left(d\left(gz, fz\right)\right) - \varphi\left(d\left(gz, fz\right)\right),$$

wherefrom $gz = fz$ follows. Hence, z is a coincidence point of f and g.

If condition (*vii*) is fulfilled, put $w = fz = gz$, and obtain that $fw = fgz = gfz = gw$. Using that $gz \preccurlyeq ggz = gw$ it can be proved that $fw = gw = w$ in a similar way as it was done in the case 1°. The theorem is proved completely.

Remark 4 If the four-term set in condition (3.2) is replaced by any of the following sets

$$\left\{d(gx, gy)\right\}, \qquad \left\{d(gx, gy), d(gx, fx), d(gy, fy)\right\},$$
$$\left\{d(gx, gy), \frac{1}{2}\left(d(gx, fx), d(gy, fy)\right), \frac{1}{2}(d(gx, fy) + d(gy, fx))\right\},$$

similar conclusions for the existence of common fixed points of mappings f and g can be obtained.

Remark 5 In his important article [22], Jachymski showed that some fixed point results based on weak contractive conditions involving functions ψ and ϕ can be reduced to their counterparts using just one function, say ϕ'. We note that our results do not fall into this category, since conditions of our Theorem 4 are not covered by the conditions of [22, Theorem 7].

Remark 6 Very recently, Shatanawi et al. [25] have obtained some results closely related to the ones given in this section. However, their assumptions on the given mappings are different.

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Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

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