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Iterative algorithms based on the viscosity approximation method for equilibrium and constrained convex minimization problem

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Abstract
The gradient-projection algorithm (GPA) plays an important role in solving constrained convex minimization problems. Based on the viscosity approximation method, we combine the GPA and averaged mapping approach to propose implicit and explicit composite iterative algorithms for finding a common solution of an equilibrium and a constrained convex minimization problem for the first time in this paper. Under suitable conditions, strong convergence theorems are obtained.
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1 Introduction

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. Let C be a nonempty closed convex subset of H. Let $T: C \to C$ be a nonexpansive mapping, namely $\|Tx - Ty\| \le \|x - y\|$, for all $x, y \in C$. The set of fixed points of T is denoted by $F(T)$.

Let ϕ be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. Consider the equilibrium problem (EP) which is to find $z \in C$ such that

$$\phi(z,y) \ge 0, \quad \forall y \in C. \tag{1.1}$$

We denoted the set of solutions of EP by $\mathrm{EP}(\phi)$. Given a mapping $F: C \to H$, let $\phi(x,y) = \langle Fx, y - x\rangle$ for all $x, y \in C$, then $z \in \mathrm{EP}(\phi)$ if and only if $\langle Fz, y - z\rangle \ge 0$ for all $y \in C$, that is, z is a solution of the variational inequality. Numerous problems in physics, optimizations, and economics reduce to find a solution of (1.1). Some methods have been proposed to solve the equilibrium problem; see, for instance, [1–3] and the references therein.

Composite iterative algorithms were proposed by many authors for finding a common solution of an equilibrium problem and a fixed point problem (see [4–18]).

On the other hand, consider the constrained convex minimization problem as follows:

$$\text{minimize}\{g(x) : x \in C\}, \tag{1.2}$$

where $g: C \to \mathbb{R}$ is a real-valued convex function. It is well known that the gradient-projection algorithm (GPA) plays an important role in solving constrained convex minimization problems. If g is (*Fréchet*) differentiable, then the GPA generates a sequence $\{x_n\}$ using the following recursive formula:

$$x_{n+1} = P_C\big(x_n - \lambda \nabla g(x_n)\big), \quad \forall n \geq 0, \tag{1.3}$$

or more generally,

$$x_{n+1} = P_C\big(x_n - \lambda_n \nabla g(x_n)\big), \quad \forall n \geq 0, \tag{1.4}$$

where in both (1.3) and (1.4) the initial guess x_0 is taken from C arbitrarily, and the parameters, λ or λ_n, are positive real numbers satisfying certain conditions. The convergence of the algorithms (1.3) and (1.4) depends on the behavior of the gradient ∇g. As a matter of fact, it is known that if ∇g is α-strongly monotone and L-Lipschitzian with constants $\alpha, L \geq 0$, then the operator

$$W := P_C(I - \lambda \nabla g) \tag{1.5}$$

is a contraction; hence the sequence $\{x_n\}$ defined by the algorithm (1.3) converges in norm to the unique minimizer of (1.2). However, if the gradient ∇g fails to be strongly monotone, the operator W defined by (1.5) would fail to be contractive; consequently, the sequence $\{x_n\}$ generated by the algorithm (1.3) may fail to converge strongly (see [19]). If ∇g is Lipschitzian, then the algorithms (1.3) and (1.4) can still converge in the weak topology under certain conditions.

Recently, Xu [19] proposed an explicit operator-oriented approach to the algorithm (1.4); that is, an averaged mapping approach. He gave his averaged mapping approach to the GPA (1.4) and the relaxed gradient-projection algorithm. Moreover, he constructed a counterexample which shows that the algorithm (1.3) does not converge in norm in an infinite-dimensional space and also presented two modifications of GPA which are shown to have strong convergence [20, 21].

In 2011, Ceng *et al.* [22] proposed the following explicit iterative scheme:

$$x_{n+1} = P_C\big[s_n \gamma V x_n + (I - s_n \mu F) T_n x_n\big], \quad n \geq 0,$$

where $s_n = \frac{2-\lambda_n L}{4}$ and $P_C(I - \lambda_n \nabla g) = s_n I + (1 - s_n) T_n$ for each $n \geq 0$. He proved that the sequences $\{x_n\}$ converge strongly to a minimizer of the constrained convex minimization problem, which also solves a certain variational inequality.

In 2000, Moudafi [2] introduced the viscosity approximation method for nonexpansive mappings, extended in [23]. Let f be a contraction on H, starting with an arbitrary initial $x_0 \in H$, define a sequence $\{x_n\}$ recursively by

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) T x_n, \quad n \geq 0, \tag{1.6}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$. Xu [24] proved that if $\{\alpha_n\}$ satisfies certain conditions, the sequence $\{x_n\}$ generated by (1.6) converges strongly to the unique solution $x^* \in F(T)$ of the variational inequality

$$\big\langle (I - f)x^*, x - x^*\big\rangle \geq 0, \quad \forall x \in F(T).$$

The purpose of the paper is to study the iterative method for finding the common solution of an equilibrium problem and a constrained convex minimization problem. Based on the viscosity approximation method, we combine the GPA and averaged mapping approach to propose implicit and explicit composite iterative method for finding the common element of the set of solutions of an equilibrium problem and the solution set of a constrained convex minimization problem. We also prove some strong convergence theorems.

2 Preliminaries

Throughout this paper, we always assume that C is a nonempty closed convex subset of a Hilbert space H. We use '$\rightharpoonup$' for weak convergence and '$\to$' for strong convergence.

It is widely known that H satisfies Opial's condition [25]; that is, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

holds for every $y \in H$ with $y \neq x$.

In order to solve the equilibrium problem for a bifunction $\phi : C \times C \to \mathbb{R}$, let us assume that ϕ satisfies the following conditions:

(A1) $\phi(x,x) = 0$, for all $x \in C$;

(A2) ϕ is monotone, that is, $\phi(x,y) + \phi(y,x) \leq 0$ for all $x, y \in C$;

(A3) for all $x, y, z \in C$, $\lim_{t\downarrow 0} \phi(tz + (1-t)x, y) \leq \phi(x,y)$;

(A4) for each fixed $x \in C$, the function $y \mapsto \phi(x,y)$ is convex and lower semicontinuous.

Let us recall the following lemmas which will be useful for our paper.

Lemma 2.1 [26] *Let ϕ be a bifunction from $C \times C$ into $\mathbb{R}$ satisfying* (A1), (A2), (A3), *and* (A4), *then for any $r > 0$ and $x \in H$, there exists $z \in C$ such that*

$$\phi(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C.$$

Further, if

$$Q_r x = \left\{ z \in C : \phi(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C \right\},$$

then the following hold:

(1) *Q_r is single-valued*;

(2) *Q_r is firmly nonexpansive*; *that is,*

$$\|Q_r x - Q_r y\|^2 \leq \langle Q_r x - Q_r y, x - y\rangle, \quad \forall x, y \in H;$$

(3) $F(Q_r) = \mathrm{EP}(\phi)$;

(4) $\mathrm{EP}(\phi)$ *is closed and convex.*

Definition 2.1 A mapping $T : H \to H$ is said to be firmly nonexpansive if and only if $2T - I$ is nonexpansive, or equivalently,

$$\langle x - y, Tx - Ty\rangle \geq \|Tx - Ty\|^2, \quad x, y \in H.$$

Alternatively, T is firmly nonexpansive if and only if T can be expressed as

$$T = \frac{1}{2}(I + S),$$

where $S : H \to H$ is nonexpansive. Obviously, projections are firmly nonexpansive.

Definition 2.2 A mapping $T : H \to H$ is said to be an *averaged mapping* if it can be written as the average of the identity I and a nonexpansive mapping; that is,

$$T = (1 - \alpha)I + \alpha S, \tag{2.1}$$

where $\alpha \in (0,1)$ and $S : H \to H$ is nonexpansive. More precisely, when (2.1) holds, we say that T is α-averaged.

Clearly, a firmly nonexpansive mapping is a $\frac{1}{2}$-averaged map.

Proposition 2.1 [27] *For given operators $S, T, V : H \to H$:*

(i) *If $T = (1 - \alpha)S + \alpha V$ for some $\alpha \in (0,1)$ and if U is averaged and V is nonexpansive, then T is averaged.*

(ii) *T is firmly nonexpansive if and only if the complement I-T is firmly nonexpansive.*

(iii) *If $T = (1 - \alpha)S + \alpha V$ for some $\alpha \in (0,1)$, U is firmly nonexpansive and V is nonexpansive, then T is averaged.*

(iv) *The composite of finitely many averaged mappings is averaged. That is, if each of the mappings $\{T_i\}_{i=1}^N$ is averaged, then so is the composite $T_1 \cdots T_N$. In particular, if T_1 is α_1-averaged, and T_2 is α_2-averaged, where $\alpha_1, \alpha_2 \in (0,1)$, then the composite T_1T_2 is α-averaged, where $\alpha = \alpha_1 + \alpha_2 - \alpha_1\alpha_2$.*

Recall that the metric projection from H onto C is the mapping $P_C : H \to C$ which assigns, to each point $x \in H$, the unique point $P_C x \in C$ satisfying the property

$$\|x - P_C x\| = \inf_{y \in C} \|x - y\| =: d(x, C).$$

Lemma 2.2 *For a given $x \in H$:*

(a) *$z = P_C x$ if and only if $\langle x - z, y - z\rangle \le 0$, $\forall y \in C$.*

(b) *$z = P_C x$ if and only if $\|x - z\|^2 \le \|x - y\|^2 - \|y - z\|^2$, $\forall y \in C$.*

(c) *$\langle P_C x - P_C y, x - y\rangle \ge \|P_C x - P_C y\|^2$, $\forall x, y \in H$.*

Consequently, P_C is nonexpansive and monotone.

Lemma 2.3 *The following inequality holds in an inner product space X:*

$$\|x + y\|^2 \le \|x\|^2 + 2\langle y, x + y\rangle, \quad \forall x, y \in X.$$

The so-called demiclosedness principle for nonexpansive mappings will be used.

Lemma 2.4 (Demiclosedness principle [28]) *Let $T : C \to C$ be a nonexpansive mapping with* $\mathrm{Fix}(T) \neq \emptyset$. *If $\{x_n\}$ is a sequence in C that converges weakly to x and if $\{(I - T)x_n\}$ converges strongly to y, then $(I - T)x = y$. In particular, if $y = 0$, then $x \in$* $\mathrm{Fix}(T)$.

Next, we introduce monotonicity of a nonlinear operator.

Definition 2.3 A nonlinear operator G whose domain $D(G) \subseteq H$ and range $R(G) \subseteq H$ is said to be:

(a) monotone if

$$\langle x - y, Gx - Gy \rangle \geq 0, \quad \forall x, y \in D(G),$$

(b) β-strongly monotone if there exists $\beta > 0$ such that

$$\langle x - y, Gx - Gy \rangle \geq \beta \|x - y\|^2, \quad \forall x, y \in D(G),$$

(c) ν-inverse strongly monotone (for short, ν-ism) if there exists $\nu > 0$ such that

$$\langle x - y, Gx - Gy \rangle \geq \nu \|Gx - Gy\|^2, \quad \forall x, y \in D(G).$$

It can be easily seen that if G is nonexpansive, then $I - G$ is monotone; and the projection map P_C is a 1-ism.

The inverse strongly monotone (also referred to as co-coercive) operators have been widely used to solve practical problems in various fields, for instance, in traffic assignment problems; see, for example, [29, 30] and reference therein.

The following proposition summarizes some results on the relationship between averaged mappings and inverse strongly monotone operators.

Proposition 2.2 [27] *Let $T : H \to H$ be an operator from H to itself.*

(a) *T is nonexpansive if and only if the complement $I - T$ is $\frac{1}{2}$-ism.*

(b) *If T is ν-ism, then for $\gamma > 0$, γT is $\frac{\nu}{\gamma}$-ism.*

(c) *T is averaged if and only if the complement $I - T$ is ν-ism for some $\nu > \frac{1}{2}$. Indeed, for $\alpha \in (0,1)$, T is α-averaged if and only if $I - T$ is $\frac{1}{2\alpha}$-ism.*

Lemma 2.5 [24] *Let $\{a_n\}$ be a sequence of nonnegative numbers satisfying the condition*

$$a_{n+1} \leq (1 - \gamma_n) a_n + \gamma_n \delta_n, \quad \forall n \geq 0,$$

where $\{\gamma_n\}$, $\{\delta_n\}$ are sequences of real numbers such that:

(i) $\{\gamma_n\} \subset (0,1)$ *and* $\sum_{n=0}^{\infty} \gamma_n = \infty$,

(ii) $\limsup_{n\to\infty} \delta \leq 0$ *or* $\sum_{n=0}^{\infty} \gamma_n |\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

3 Main results

In this paper, we always assume that $g : C \to \mathbb{R}$ is a real-valued convex function and ∇g is an L-Lipschitzian mapping with $L \geq 0$. Since the Lipschitz continuity of ∇g implies that it is indeed inverse strongly monotone, its complement can be an averaged mapping. Consequently, the GPA can be rewritten as the composite of a projection and an averaged mapping, which is again an averaged mapping. This shows that an averaged mapping plays an important role in the gradient-projection algorithm.

Note that ∇g is L-Lipschitzian. This implies that ∇g is $(1/L)$-ism, which then implies that $\lambda\nabla g$ is $(1/\lambda L)$-ism. So, by Proposition 2.2, $I - \lambda\nabla g$ is $(\lambda L/2)$-averaged. Now since the projection P_C is $(1/2)$-averaged, we see from Proposition 2.1 that the composite $P_C(I - \lambda\nabla g)$ is $((2 + \lambda L)/4)$-averaged for $0 < \lambda < 2/L$. Hence, we have that for each n, $P_C(I - \lambda_n\nabla g)$ is $((2 + \lambda_n L)/4)$-averaged. Therefore, we can write

$$P_C(I - \lambda_n\nabla g) = \frac{2 - \lambda_n L}{4} I + \frac{2 + \lambda_n L}{4} T_n = s_n I + (1 - s_n)T_n,$$

where T_n is nonexpansive and $s_n = \frac{2-\lambda_n L}{4}$.

Let $f : C \to C$ be a contraction with the constant $\rho \in (0,1)$. Suppose that the minimization problem (1.2) is consistent, and let U denote its solution set. Let $\{Q_{\beta_n}\}$ be a sequence of mappings defined as in Lemma 2.1. Consider the following mapping G_n on C defined by

$$G_n x = \alpha_n f(x) + (1 - \alpha_n)T_n Q_{\beta_n} x, \quad x \in C, n \in \mathbb{N},$$

where $\alpha_n \in (0,1)$. By Lemma 2.1, we have

$$\|G_n x - G_n y\| \le \big(1 - \alpha_n(1 - \rho)\big)\|x - y\|.$$

Since $0 < 1 - \alpha_n(1 - \rho) < 1$, it follows that G_n is a contraction. Therefore, by the Banach contraction principle, G_n has a unique fixed point $x_n^f \in C$ such that

$$x_n^f = \alpha_n f\big(x_n^f\big) + (1 - \alpha_n)T_n Q_{\beta_n} x_n^f.$$

For simplicity, we will write x_n for x_n^f provided no confusion occurs. Next, we prove the convergence of $\{x_n\}$, while we claim the existence of the $q \in U \cap \mathrm{EP}(\phi)$, which solves the variational inequality

$$\big\langle (I - f)q, p - q\big\rangle \ge 0, \quad \forall p \in U \cap \mathrm{EP}(\phi). \tag{3.1}$$

Equivalently, $q = P_{U\cap \mathrm{EP}(\phi)} f(q)$.

Theorem 3.1 *Let C be a nonempty closed convex subset of a real Hilbert space H and ϕ be a bifunction from $C \times C$ into $\mathbb{R}$ satisfying* (A1), (A2), (A3), *and* (A4). *Let $g : C \to \mathbb{R}$ be a real-valued convex function, and assume that ∇g is an L-Lipschitzian mapping with $L \ge 0$ and $f : C \to C$ is a contraction with the constant $\rho \in (0,1)$. Assume that $U \cap \mathrm{EP}(\phi) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} \phi(u_n, y) + \frac{1}{\beta_n}\langle y - u_n, u_n - x_n\rangle \ge 0, & \forall y \in C, \\ x_n = \alpha_n f(x_n) + (1 - \alpha_n)T_n u_n, & \forall n \in \mathbb{N}, \end{cases}$$

where $u_n = Q_{\beta_n} x_n$, $P_C(I - \lambda_n\nabla g) = s_n I + (1 - s_n)T_n$, $s_n = \frac{2-\lambda_n L}{4}$ and $\{\lambda_n\} \subset (0, \frac{2}{L})$. Let $\{\beta_n\}$ and $\{\alpha_n\}$ satisfy the following conditions:

(i) $\{\beta_n\} \subset (0,\infty)$, $\liminf_{n\to\infty} \beta_n > 0$;
(ii) $\{\alpha_n\} \subset (0,1)$, $\lim_{n\to\infty} \alpha_n = 0$.

Then $\{x_n\}$ *converges strongly, as* $s_n \to 0$ ($\Leftrightarrow \lambda_n \to \frac{2}{L}$), *to a point* $q \in U \cap \mathrm{EP}(\phi)$ *which solves the variational inequality* (3.1).

Proof First, we claim that $\{x_n\}$ is bounded. Indeed, pick any $p \in U \cap \mathrm{EP}(\phi)$, since $u_n = Q_{\beta_n} x_n$ and $p = Q_{\beta_n} p$, then we know that for any $n \in \mathbb{N}$,

$$\|u_n - p\| = \|Q_{\beta_n} x_n - Q_{\beta_n} p\| \le \|x_n - p\|. \tag{3.2}$$

Thus, we derive that (noting $T_n p = p$ and T_n is nonexpansive)

$$\begin{aligned}
\|x_n - p\| &= \|\alpha_n f(x_n) + (1-\alpha_n) T_n u_n - p\| \\
&\le \|\alpha_n f(x_n) - \alpha_n f(p)\| + \|\alpha_n f(p) - \alpha_n p\| + (1-\alpha_n)\|T_n u_n - T_n p\| \\
&\le \bigl[1 - \alpha_n(1-\rho)\bigr]\|x_n - p\| + \alpha_n \|(I-f)p\|.
\end{aligned}$$

Then we have

$$\|x_n - p\| \le \frac{1}{1-\rho}\|(I-f)p\|,$$

and hence $\{x_n\}$ is bounded. From (3.2), we also derive that $\{u_n\}$ is bounded.

Next, we claim that $\|x_n - u_n\| \to 0$. Indeed, for any $p \in U \cap \mathrm{EP}(\phi)$, by Lemma 2.1, we have

$$\begin{aligned}
\|u_n - p\|^2 &= \|Q_{\beta_n} x_n - Q_{\beta_n} p\|^2 \\
&\le \langle x_n - p, u_n - p\rangle \\
&= \frac{1}{2}\bigl(\|x_n - p\|^2 + \|u_n - p\|^2 - \|u_n - x_n\|^2\bigr).
\end{aligned}$$

This implies that

$$\|u_n - p\|^2 \le \|x_n - p\|^2 - \|u_n - x_n\|^2. \tag{3.3}$$

Then from (3.3), we derive that

$$\begin{aligned}
\|x_n - p\|^2 &= \|\alpha_n f(x_n) + (1-\alpha_n) T_n u_n - p\|^2 \\
&= \|\alpha_n f(x_n) - \alpha_n p + (1-\alpha_n) T_n u_n - (1-\alpha_n) T_n p\|^2 \\
&\le (1-\alpha_n)^2 \|u_n - p\|^2 + 2\alpha_n \bigl\langle f(x_n) - p, x_n - p\bigr\rangle \\
&\le \|x_n - p\|^2 - \|u_n - x_n\|^2 + 2\alpha_n\bigl[\rho\|x_n - p\| + \|(I-f)p\|\bigr]\|x_n - p\|.
\end{aligned}$$

Since $\alpha_n \to 0$, it follows that

$$\lim_{n\to\infty} \|x_n - u_n\| = 0.$$

Then we show that $\|x_n - T_n x_n\| \to 0$. Indeed,

$$\begin{aligned}\|x_n - T_n x_n\| &= \|x_n - T_n u_n + T_n u_n - T_n x_n\| \\ &\le \|x_n - T_n u_n\| + \|T_n u_n - T_n x_n\| \\ &\le \alpha_n \big\|f(x_n) - T_n u_n\big\| + \|u_n - x_n\|.\end{aligned}$$

Since $\alpha_n \to 0$ and $\|x_n - u_n\| \to 0$, we obtain that

$$\|x_n - T_n x_n\| \to 0.$$

Thus,

$$\begin{aligned}\|u_n - T_n u_n\| &= \|u_n - x_n + x_n - T_n x_n + T_n x_n - T_n u_n\| \\ &\le \|u_n - x_n\| + \|x_n - T_n x_n\| + \|T_n x_n - T_n u_n\| \\ &\le \|u_n - x_n\| + \|x_n - T_n x_n\| + \|x_n - u_n\|,\end{aligned}$$

and

$$\|x_n - T_n u_n\| \le \|x_n - u_n\| + \|u_n - T_n u_n\|,$$

we have

$$\|u_n - T_n u_n\| \to 0 \quad \text{and} \quad \|x_n - T_n u_n\| \to 0.$$

Observe that

$$\begin{aligned}\big\|P_C(I - \lambda_n \nabla g)u_n - u_n\big\| &= \big\|s_n u_n + (1 - s_n) T_n u_n - u_n\big\| \\ &= (1 - s_n)\|T_n u_n - u_n\| \\ &\le \|T_n u_n - u_n\|,\end{aligned}$$

where $s_n = \frac{2-\lambda_n L}{4} \in (0, \frac{1}{2})$. Hence, we have

$$\begin{aligned}&\left\|P_C\left(I - \frac{2}{L}\nabla g\right)u_n - u_n\right\| \\ &\quad\le \left\|P_C\left(I - \frac{2}{L}\nabla g\right)u_n - P_C(I - \lambda_n \nabla g)u_n\right\| + \big\|P_C(I - \lambda_n \nabla g)u_n - u_n\big\| \\ &\quad\le \left\|\left(I - \frac{2}{L}\nabla g\right)u_n - (I - \lambda_n \nabla g)u_n\right\| + \big\|P_C(I - \lambda_n \nabla g)u_n - u_n\big\| \\ &\quad\le \left(\frac{2}{L} - \lambda_n\right)\big\|\nabla g(u_n)\big\| + \|T_n u_n - u_n\|.\end{aligned}$$

From the boundedness of $\{u_n\}$, $s_n \to 0$ ($\Leftrightarrow \lambda_n \to \frac{2}{L}$) and $\|u_n - T_n u_n\| \to 0$, we conclude that

$$\lim_{n\to\infty}\left\|u_n - P_C\left(I - \frac{2}{L}\nabla g\right)u_n\right\| = 0.$$

Since ∇g is L-Lipschitzian, ∇g is $\frac{1}{L}$-ism. Consequently, $P_C(I - \frac{2}{L}\nabla g)$ is a nonexpansive self-mapping on C. As a matter of fact, we have for each $x, y \in C$

$$\begin{aligned}
&\left\| P_C\left(I - \frac{2}{L}\nabla g\right)x - P_C\left(I - \frac{2}{L}\nabla g\right)y \right\|^2 \\
&\quad\leq \left\| \left(I - \frac{2}{L}\nabla g\right)x - \left(I - \frac{2}{L}\nabla g\right)y \right\|^2 \\
&\quad= \left\| x - y - \frac{2}{L}\big(\nabla g(x) - \nabla g(y)\big) \right\|^2 \\
&\quad= \|x - y\|^2 - \frac{4}{L}\langle x - y, \nabla g(x) - \nabla g(y)\rangle + \frac{4}{L^2}\big\|\nabla g(x) - \nabla g(y)\big\|^2 \\
&\quad\leq \|x - y\|^2 - \frac{4}{L^2}\big\|\nabla g(x) - \nabla g(y)\big\|^2 + \frac{4}{L^2}\big\|\nabla g(x) - \nabla g(y)\big\|^2 \\
&\quad= \|x - y\|^2.
\end{aligned}$$

Consider a subsequence $\{u_{n_i}\}$ of $\{u_n\}$. Since $\{u_{n_i}\}$ is bounded, there exists a subsequence $\{u_{n_{i_j}}\}$ of $\{u_{n_i}\}$ which converges weakly to q. Next, we show that $q \in U \cap \mathrm{EP}(\phi)$. Without loss of generality, we can assume that $u_{n_i} \rightharpoonup q$. Then, by Lemma 2.4, we obtain

$$q = P_C\left(I - \frac{2}{L}\nabla g\right)q.$$

This shows that $q \in U$.

Next, we show that $q \in \mathrm{EP}(\phi)$. Since $u_n = Q_{\beta_n}x_n$, for any $y \in C$, we obtain

$$\phi(u_n, y) + \frac{1}{\beta_n}\langle y - u_n, u_n - x_n\rangle \geq 0.$$

From (A2), we have

$$\frac{1}{\beta_n}\langle y - u_n, u_n - x_n\rangle \geq \phi(y, u_n).$$

Replacing n by n_i, we have

$$\left\langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{\beta_{n_i}}\right\rangle \geq \phi(y, u_{n_i}).$$

Since $\frac{u_{n_i} - x_{n_i}}{\beta_{n_i}} \to 0$ and $u_{n_i} \rightharpoonup q$, it follows from (A4) that $0 \geq \phi(y, q)$ for all $y \in C$. Let

$$z_t = ty + (1 - t)q, \quad \forall t \in (0, 1], y \in C,$$

then we have $z_t \in C$ and hence $\phi(z_t, q) \leq 0$. Thus, from (A1) and (A4), we have

$$\begin{aligned}
0 &= \phi(z_t, z_t) \\
&\leq t\phi(z_t, y) + (1 - t)\phi(z_t, q) \\
&\leq t\phi(z_t, y),
\end{aligned}$$

and hence $0 \le \phi(z_t, y)$. From (A3), we have $0 \le \phi(q, y)$ for all $y \in C$ and hence $q \in \mathrm{EP}(\phi)$. Therefore, $q \in \mathrm{EP}(\phi) \cap U$.

On the other hand, we note that

$$\begin{aligned} x_n - q &= \alpha_n f(x_n) + (1-\alpha_n)T_n u_n - q \\ &= \alpha_n f(x_n) - \alpha_n f(q) + \alpha_n f(q) - \alpha_n q + (1-\alpha_n)(T_n u_n - q). \end{aligned}$$

Hence, we obtain

$$\begin{aligned} \|x_n - q\|^2 &= \alpha_n \langle (f-I)q, x_n - q\rangle \\ &\quad + \langle \alpha_n \big(f(x_n) - f(q)\big) + (1-\alpha_n)(T_n u_n - T_n q), x_n - q\rangle \\ &\le \alpha_n \langle (f-I)q, x_n - q\rangle + \big(1 - \alpha_n(1-\rho)\big)\|x_n - q\|^2. \end{aligned}$$

It follows that

$$\|x_n - q\|^2 \le \frac{1}{1-\rho}\langle (f-I)q, x_n - q\rangle.$$

In particular,

$$\|x_{n_i} - q\|^2 \le \frac{1}{1-\rho}\langle (f-I)q, x_{n_i} - q\rangle. \tag{3.4}$$

Since $x_{n_i} \rightharpoonup q$, it follows from (3.4) that $x_{n_i} \to q$ as $i \to \infty$.

Next, we show that q solves the variational inequality (3.1). Observe that

$$x_n = \alpha_n f(x_n) + (1-\alpha_n)T_n u_n = \alpha_n f(x_n) + (1-\alpha_n)T_n Q_{\beta_n} x_n.$$

Hence, we conclude that

$$(I-f)x_n = -\frac{1}{\alpha_n}(I - T_n Q_{\beta_n})x_n - T_n Q_{\beta_n} x_n + x_n.$$

Since T_n is nonexpansive, we have that $I - T_n Q_{\beta_n}$ is monotone. Note that for any given $z \in U \cap \mathrm{EP}(\phi)$,

$$\begin{aligned} &\langle (I-f)x_n, x_n - z\rangle \\ &\quad = -\frac{1}{\alpha_n}\langle (I - T_n Q_{\beta_n})x_n - (I - T_n Q_{\beta_n})z, x_n - z\rangle - \langle T_n u_n - x_n, x_n - z\rangle \\ &\quad \le \|T_n u_n - x_n\|\|x_n - z\|. \end{aligned}$$

Now, replacing n with n_i in the above inequality, and letting $i \to \infty$, we have

$$\langle (I-f)q, q - z\rangle = \lim_{i\to\infty}\langle (I-f)x_{n_i}, x_{n_i} - z\rangle \le 0.$$

From the arbitrariness of $z \in U \cap \mathrm{EP}(\phi)$, it follows that $q \in U \cap \mathrm{EP}(\phi)$ is a solution of the variational inequality (3.1). Further, by the uniqueness of solution of the variational inequality (3.1), we conclude that $x_n \to q$ as $n \to \infty$. The variational inequality (3.1) can be written as

$$\langle f(q) - q, q - z \rangle \geq 0, \quad \forall z \in U \cap \mathrm{EP}(\phi).$$

So, in terms of Lemma 2.2, it is equivalent to the following equality:

$$P_{U\cap \mathrm{EP}(\phi)} f(q) = q.$$

This completes the proof. □

Theorem 3.2 *Let C be a nonempty closed convex subset of a real Hilbert space H and ϕ be a bifunction from $C \times C$ into $\mathbb{R}$ satisfying* (A1), (A2), (A3), *and* (A4). *Let $g : C \to \mathbb{R}$ be a real-valued convex function, and assume that ∇g is an L-Lipschitzian mapping with $L \geq 0$ and $f : C \to C$ is a contraction with the constant $\rho \in (0,1)$. Assume that $U \cap \mathrm{EP}(\phi) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $x_1 \in C$ and*

$$\begin{cases} \phi(u_n, y) + \frac{1}{\beta_n}\langle y - u_n, u_n - x_n\rangle \geq 0, & \forall y \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) T_n u_n, & \forall n \in \mathbb{N}, \end{cases}$$

where $u_n = Q_{\beta_n} x_n$, $P_C(I - \lambda_n \nabla g) = s_n I + (1 - s_n) T_n$, $s_n = \frac{2-\lambda_n L}{4}$ and $\{\lambda_n\} \subset (0, \frac{2}{L})$. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{s_n\}$ satisfy the following conditions:

(i) $\{\beta_n\} \subset (0, \infty)$, $\liminf \beta_n > 0$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$;
(ii) $\{\alpha_n\} \subset (0,1)$, $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;
(iii) $\{s_n\} \subset (0, \frac{1}{2})$, $\lim_{n\to\infty} s_n = 0$ ($\Leftrightarrow \lim_{n\to\infty} \lambda_n = \frac{2}{L}$), $\sum_{n=1}^{\infty} |s_{n+1} - s_n| < \infty$.

Then $\{x_n\}$ converges strongly to a point $q \in U \cap \mathrm{EP}(\phi)$ which solves the variational inequality (3.1).

Proof First, we show that $\{x_n\}$ is bounded. Indeed, pick any $p \in U \cap \mathrm{EP}(\phi)$, since $u_n = Q_{\beta_n} x_n$ and $p = Q_{\beta_n} p$, then we know that for any $n \in \mathbb{N}$,

$$\|u_n - p\| = \|Q_{\beta_n} x_n - Q_{\beta_n} p\| \leq \|x_n - p\|. \tag{3.5}$$

Thus, we derive that (noting $T_n p = p$ and T_n is nonexpansive)

$$\begin{aligned} \|x_{n+1} - p\| &= \left\|\alpha_n f(x_n) + (1 - \alpha_n) T_n u_n - p\right\| \\ &\leq \alpha_n \rho \|x_n - p\| + (1 - \alpha_n)\|x_n - p\| + \alpha_n \left\|f(p) - p\right\| \\ &\leq \bigl(1 - \alpha_n(1 - \rho)\bigr)\|x_n - p\| + \alpha_n \left\|f(p) - p\right\|. \end{aligned}$$

By induction, we have

$$\|x_n - p\| \leq \max\left\{\|x_1 - p\|, \frac{1}{1-\rho}\left\|f(p) - p\right\|\right\},$$

and hence $\{x_n\}$ is bounded. From (3.5), we also derive that $\{u_n\}$ is bounded.

Next, we show that $\|x_{n+1} - x_n\| \to 0$. Indeed, since ∇g is $\frac{1}{L}$-ism, $P_C(I - \lambda_n \nabla g)$ is nonexpansive. It follows that for any given $p \in S$,

$$\begin{aligned}\|P_C(I - \lambda_n\nabla g)u_{n-1}\| &\leq \|P_C(I - \lambda_n\nabla g)u_{n-1} - p\| + \|p\| \\ &\leq \|P_C(I - \lambda_n\nabla g)u_{n-1} - P_C(I - \lambda_n\nabla g)p\| + \|p\| \\ &\leq \|u_{n-1} - p\| + \|p\| \\ &\leq \|u_{n-1}\| + 2\|p\|.\end{aligned}$$

This together with the boundedness of $\{u_n\}$ implies that $\{P_C(I - \lambda_n\nabla g)u_{n-1}\}$ is bounded.

Also, observe that

$$\begin{aligned}&\|T_n u_{n-1} - T_{n-1}u_{n-1}\| \\ &\quad= \left\| \frac{4P_C(I - \lambda_n\nabla g) - (2 - \lambda_n L)I}{2 + \lambda_n L} u_{n-1} - \frac{4P_C(I - \lambda_{n-1}\nabla g) - (2 - \lambda_{n-1} L)I}{2 + \lambda_{n-1} L} u_{n-1} \right\| \\ &\quad\leq \left\| \frac{4P_C(I - \lambda_n\nabla g)}{2 + \lambda_n L} u_{n-1} - \frac{4P_C(I - \lambda_{n-1}\nabla g)}{2 + \lambda_{n-1} L} u_{n-1} \right\| + \left\| \frac{2 - \lambda_{n-1}L}{2 + \lambda_{n-1}L} u_{n-1} - \frac{2 - \lambda_n L}{2 + \lambda_n L} u_{n-1} \right\| \\ &\quad= \left\| \frac{4(2 + \lambda_{n-1}L)P_C(I - \lambda_n\nabla g)u_{n-1} - 4(2 + \lambda_n L)P_C(I - \lambda_{n-1}\nabla g)u_{n-1}}{(2 + \lambda_n L)(2 + \lambda_{n-1}L)} \right\| \\ &\qquad+ \frac{4L|\lambda_n - \lambda_{n-1}|}{(2 + \lambda_{n-1}L)(2 + \lambda_n L)} \|u_{n-1}\| \\ &\quad\leq \left\| \frac{4L(\lambda_{n-1} - \lambda_n)P_C(I - \lambda_n\nabla g)u_{n-1}}{(2 + \lambda_n L)(2 + \lambda_{n-1}L)} \right. \\ &\qquad \left. + \frac{4(2 + \lambda_n L)(P_C(I - \lambda_n\nabla g) - P_C(I - \lambda_{n-1}\nabla g))u_{n-1}}{(2 + \lambda_n L)(2 + \lambda_{n-1}L)} \right\| \\ &\qquad+ \frac{4L|\lambda_n - \lambda_{n-1}|}{(2 + \lambda_{n-1}L)(2 + \lambda_n L)} \|u_{n-1}\| \\ &\quad\leq \frac{4L|\lambda_{n-1} - \lambda_n| \cdot \|P_C(I - \lambda_n\nabla g)u_{n-1}\|}{(2 + \lambda_n L)(2 + \lambda_{n-1}L)} \\ &\qquad+ \frac{4(2 + \lambda_n L)\|P_C(I - \lambda_n\nabla g)u_{n-1} - P_C(I - \lambda_{n-1}\nabla g)u_{n-1}\|}{(2 + \lambda_n L)(2 + \lambda_{n-1}L)} \\ &\qquad+ \frac{4L|\lambda_n - \lambda_{n-1}|}{(2 + \lambda_{n-1}L)(2 + \lambda_n L)} \|u_{n-1}\| \\ &\quad\leq |\lambda_{n-1} - \lambda_n| \cdot \bigl[L\|P_C(I - \lambda_n\nabla g)u_{n-1}\| + 4\|\nabla g(u_{n-1})\| + L\|u_{n-1}\|\bigr] \\ &\quad\leq M_1|\lambda_{n-1} - \lambda_n|\end{aligned}$$

for some appropriate constant $M_1 > 0$ such that

$$M_1 \geq L\|P_C(I - \lambda_n\nabla g)u_{n-1}\| + 4\|\nabla g(u_{n-1})\| + L\|u_{n-1}\|, \quad \forall n \geq 1.$$

Thus, we get

$$\begin{aligned}&\|x_{n+1} - x_n\| \\ &\quad= \|\alpha_n f(x_n) + (1 - \alpha_n)T_n u_n - (\alpha_{n-1}f(x_{n-1}) + (1 - \alpha_{n-1})T_{n-1}u_{n-1})\|\end{aligned}$$

$$\begin{aligned}
&= \big\|\alpha_n f(x_n) - \alpha_n f(x_{n-1}) + \alpha_n f(x_{n-1}) - \alpha_{n-1} f(x_{n-1}) + (1-\alpha_n) T_n u_n \\
&\quad - (1-\alpha_n) T_n u_{n-1} + (1-\alpha_n) T_n u_{n-1} - (1-\alpha_n) T_{n-1} u_{n-1} \\
&\quad + (1-\alpha_n) T_{n-1} u_{n-1} - (1-\alpha_{n-1}) T_{n-1} u_{n-1}\big\| \\
&\leq \alpha_n \rho \|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}| \big\|f(x_{n-1})\big\| + (1-\alpha_n)\|u_n - u_{n-1}\| \\
&\quad + (1-\alpha_n)\|T_n u_{n-1} - T_{n-1} u_{n-1}\| + |\alpha_n - \alpha_{n-1}| \|T_{n-1} u_{n-1}\| \\
&= \alpha_n \rho \|x_n - x_{n-1}\| + (1-\alpha_n)\|u_n - u_{n-1}\| + (1-\alpha_n)\|T_n u_{n-1} - T_{n-1} u_{n-1}\| \\
&\quad + |\alpha_n - \alpha_{n-1}| \big(\big\|f(x_{n-1})\big\| + \|T_{n-1} u_{n-1}\|\big) \\
&\leq \alpha_n \rho \|x_n - x_{n-1}\| + (1-\alpha_n)\|u_n - u_{n-1}\| + M_1 |\lambda_n - \lambda_{n-1}| \\
&\quad + |\alpha_n - \alpha_{n-1}| \big(\big\|f(x_{n-1})\big\| + \|T_{n-1} u_{n-1}\|\big) \\
&= \alpha_n \rho \|x_n - x_{n-1}\| + (1-\alpha_n)\|u_n - u_{n-1}\| + |s_n - s_{n-1}| \frac{4M_1}{L} \\
&\quad + |\alpha_n - \alpha_{n-1}| \big(\big\|f(x_{n-1})\big\| + \|T_{n-1} u_{n-1}\|\big) \\
&\leq \alpha_n \rho \|x_n - x_{n-1}\| + (1-\alpha_n)\|u_n - u_{n-1}\| + M_2 \big(|\alpha_n - \alpha_{n-1}| + |s_n - s_{n-1}|\big)
\end{aligned} \tag{3.6}$$

for some appropriate constant $M_2 > 0$ such that

$$M_2 \geq \max\left\{ \big\|f(x_{n-1})\big\| + \|T_{n-1} u_{n-1}\|, \frac{4M_1}{L} \right\}, \quad \forall n \geq 1.$$

From $u_{n+1} = Q_{\beta_{n+1}} x_{n+1}$ and $u_n = Q_{\beta_n} x_n$, we note that

$$\phi(u_{n+1}, y) + \frac{1}{\beta_{n+1}} \langle y - u_{n+1}, u_{n+1} - x_{n+1} \rangle \geq 0, \quad \forall y \in C, \tag{3.7}$$

and

$$\phi(u_n, y) + \frac{1}{\beta_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C. \tag{3.8}$$

Putting $y = u_n$ in (3.7) and $y = u_{n+1}$ in (3.8), we have

$$\phi(u_{n+1}, u_n) + \frac{1}{\beta_{n+1}} \langle u_n - u_{n+1}, u_{n+1} - x_{n+1} \rangle \geq 0, \quad \forall y \in C,$$

and

$$\phi(u_n, u_{n+1}) + \frac{1}{\beta_n} \langle u_{n+1} - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C.$$

So, from (A2), we have

$$\left\langle u_{n+1} - u_n, \frac{u_n - x_n}{\beta_n} - \frac{u_{n+1} - x_{n+1}}{\beta_{n+1}} \right\rangle \geq 0,$$

and hence

$$\left\langle u_{n+1} - u_n, u_n - u_{n+1} + u_{n+1} - x_n - \frac{\beta_n}{\beta_{n+1}} (u_{n+1} - x_{n+1}) \right\rangle \geq 0.$$

Since $\lim_{n\to\infty}\beta_n > 0$, without loss of generality, let us assume that there exists a real number a such that $\beta_n > a > 0$ for all $n \in \mathbb{N}$. Thus, we have

$$\begin{aligned}\|u_{n+1}-u_n\|^2 &\le \left\langle u_{n+1}-u_n, x_{n+1}-x_n + \left(1-\frac{\beta_n}{\beta_{n+1}}\right)(u_{n+1}-x_{n+1})\right\rangle \\ &\le \|u_{n+1}-u_n\|\left\{\|x_{n+1}-x_n\| + \left|1-\frac{\beta_n}{\beta_{n+1}}\right|\|u_{n+1}-x_{n+1}\|\right\},\end{aligned}$$

thus,

$$\|u_{n+1}-u_n\| \le \|x_{n+1}-x_n\| + \frac{1}{a}|\beta_{n+1}-\beta_n|M_3, \tag{3.9}$$

where $M_3 = \sup\{\|u_n - x_n\| : n \in \mathbb{N}\}$.

From (3.6) and (3.9), we obtain

$$\begin{aligned}&\|x_{n+1}-x_n\| \\ &\quad\le \bigl(1-\alpha_n(1-\rho)\bigr)\|x_n-x_{n-1}\| + M_2\bigl(|\alpha_n-\alpha_{n-1}| + |s_n-s_{n-1}|\bigr) + |\beta_n-\beta_{n-1}|\frac{M_3}{a} \\ &\quad\le \bigl(1-\alpha_n(1-\rho)\bigr)\|x_n-x_{n-1}\| + M\bigl(|\alpha_n-\alpha_{n-1}| + |s_n-s_{n-1}| + |\beta_n-\beta_{n-1}|\bigr),\end{aligned}$$

where $M = \max[M_2, \frac{M_3}{a}]$. Hence, by Lemma 2.5, we have

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0. \tag{3.10}$$

Then, from (3.9) and (3.10), and $|\beta_{n+1}-\beta_n| \to 0$, we have

$$\lim_{n\to\infty}\|u_{n+1}-u_n\| = 0.$$

For any $p \in U \cap \mathrm{EP}(\phi)$, as in the proof of Theorem 3.1, we have

$$\|u_n - p\|^2 \le \|x_n - p\|^2 - \|u_n - x_n\|^2. \tag{3.11}$$

Then from (3.11), we derive that

$$\begin{aligned}\|x_{n+1}-p\|^2 &= \bigl\|\alpha_n f(x_n) - \alpha_n p + (1-\alpha_n)T_n u_n - (1-\alpha_n)T_n p\bigr\|^2 \\ &\le \alpha_n^2\bigl\|f(x_n)-p\bigr\|^2 + 2\alpha_n(1-\alpha_n)\bigl\|f(x_n)-p\bigr\|\|u_n-p\| \\ &\quad + (1-\alpha_n)^2\|u_n-p\|^2 \\ &\le \alpha_n\bigl(\bigl\|f(x_n)-p\bigr\|^2 + 2\bigl\|f(x_n)-p\bigr\|\|u_n-p\|\bigr) + \|u_n-p\|^2 \\ &\le \|x_n-p\|^2 - \|u_n-x_n\|^2 + \alpha_n\bigl(\bigl\|f(x_n)-p\bigr\|^2 \\ &\quad + 2\bigl\|f(x_n)-p\bigr\|\|u_n-p\|\bigr).\end{aligned}$$

Since $\alpha_n \to 0$ and $\|x_n - x_{n+1}\| \to 0$, we have

$$\lim_{n\to\infty}\|x_n - u_n\| = 0.$$

Next, we have

$$\begin{aligned}\|x_n - T_n x_n\| &= \left\|\alpha_n f(x_n) + (1-\alpha_n)T_n u_n - T_n x_n\right\| \\ &\le \alpha_n \left\|f(x_n) - T_n u_n\right\| + (1-\alpha_n)\|u_n - x_n\|.\end{aligned}$$

Then, $\|x_n - T_n x_n\| \to 0$, it follows that $\|u_n - T_n u_n\| \to 0$.

Now, we show that

$$\limsup_{n\to\infty}\langle x_n - q, -(I-f)q\rangle \le 0,$$

where $q = P_{U\cap \mathrm{EP}(\phi)} f(q)$ is a unique solution of the variational inequality (3.1). Indeed, take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle x_n - q, -(I-f)q\rangle = \lim_{k\to\infty}\langle x_{n_k} - q, -(I-f)q\rangle.$$

Since $\{x_n\}$ is bounded, without loss of generality, we may assume that $x_{n_k} \rightharpoonup \tilde{x}$. By the same argument as in the proof of Theorem 3.1, we have $\tilde{x} \in U \cap \mathrm{EP}(\phi)$.

Since $q = P_{U\cap \mathrm{EP}(\phi)} f(q)$, it follows that

$$\limsup_{n\to\infty}\langle (I-f)q, q - x_n\rangle = \langle (I-f)q, q - \tilde{x}\rangle \le 0. \tag{3.12}$$

From

$$\begin{aligned}x_{n+1} - q &= \alpha_n f(x_n) + (1-\alpha_n)T_n u_n - q \\ &= \alpha_n f(x_n) - \alpha_n f(q) + \alpha_n f(q) - \alpha_n q + (1-\alpha_n)T_n u_n - (1-\alpha_n)T_n q,\end{aligned}$$

we have

$$\begin{aligned}\|x_{n+1} - q\|^2 &= \left\|\alpha_n\big(f(x_n) - f(q)\big) + \alpha_n\big(f(q) - q\big) + (1-\alpha_n)(T_n u_n - T_n q)\right\|^2 \\ &\le (1-\alpha_n)^2\|T_n u_n - T_n q\|^2 + 2\alpha_n\langle f(x_n) - f(q) - (I-f)q, x_{n+1} - q\rangle.\end{aligned}$$

This implies that

$$\begin{aligned}\|x_{n+1} - q\|^2 &\le (1-\alpha_n)^2\|x_n - q\|^2 + 2\alpha_n\rho\|x_n - q\|\|x_{n+1} - q\| \\ &\quad + 2\alpha_n\langle -(I-f)q, x_{n+1} - q\rangle \\ &\le (1-\alpha_n)^2\|x_n - q\|^2 + \alpha_n\rho\big(\|x_n - q\|^2 + \|x_{n+1} - q\|^2\big) \\ &\quad + 2\alpha_n\langle -(I-f)q, x_{n+1} - q\rangle.\end{aligned}$$

Then, we have

$$\begin{aligned}\|x_{n+1} - q\|^2 &\le \frac{1 - 2\alpha_n + \alpha_n\rho}{1 - \alpha_n\rho}\|x_n - q\|^2 + \frac{\alpha_n^2}{1 - \alpha_n\rho}\|x_n - q\|^2 \\ &\quad + \frac{2\alpha_n}{1 - \alpha_n\rho}\langle -(I-f)q, x_{n+1} - q\rangle\end{aligned}$$

$$
\begin{aligned}
&\le \bigl(1-2(1-\rho)\alpha_n\bigr)\|x_n-q\|^2+\frac{\alpha_n^2}{1-\alpha_n\rho}\|x_n-q\|^2 \\
&\quad +\frac{2\alpha_n}{1-\alpha_n\rho}\bigl\langle -(I-f)q, x_{n+1}-q\bigr\rangle \\
&\le \bigl(1-2(1-\rho)\alpha_n\bigr)\|x_n-q\|^2+2(1-\rho)\alpha_n\biggl(\frac{\alpha_n}{2(1-\rho)(1-\alpha_n\rho)}M^* \\
&\quad +\frac{1}{(1-\rho)(1-\alpha_n\rho)}\bigl\langle -(I-f)q, x_{n+1}-q\bigr\rangle\biggr) \\
&= \bigl(1-2(1-\rho)\alpha_n\bigr)\|x_n-q\|^2+2(1-\rho)\alpha_n\delta_n,
\end{aligned}
$$

where $M^* = \sup\{\|x_n-q\|^2 : n \in \mathbb{N}\}$, and $\delta_n = \frac{\alpha_n}{2(1-\rho)(1-\alpha_n\rho)}M^* + \frac{1}{(1-\rho)(1-\alpha_n\rho)}\langle -(I-f)q, x_{n+1}-q\rangle$.

It is easy to see that $\lim_{n\to\infty} 2(1-\rho)\alpha_n = 0$, $\sum_{n=1}^{\infty} 2(1-\rho)\alpha_n = \infty$, and $\limsup_{n\to\infty}\delta_n \le 0$ by (3.12). Hence, by Lemma 2.5, the sequence $\{x_n\}$ converges strongly to q. This completes the proof. □

4 Conclusions

Methods for solving the equilibrium problem and the constrained convex minimization problem have extensively been studied respectively in a Hilbert space. But to the best of our knowledge, it would probably be the first time in the literature that we introduce implicit and explicit algorithms for finding the common element of the set of solutions of an equilibrium problem and the set of solutions of a constrained convex minimization problem, which also solves a certain variational inequality.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All the authors read and approved the final manuscript.

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