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Best proximity point theorems for reckoning optimal approximate solutions

S Sadiq Basha[1], N Shahzad[2*] and R Jeyaraj[3]

*Correspondence: naseer_shahzad@hotmail.com
[2]Department of Mathematics, King Abdulaziz University, P.O. Box 80203, Jeddah, 21589, Saudi Arabia
Full list of author information is available at the end of the article

Abstract

Given a non-self mapping from A to B, where A and B are subsets of a metric space, in order to compute an optimal approximate solution of the equation $Sx = x$, a best proximity point theorem probes into the global minimization of the error function $x \longrightarrow d(x, Sx)$ corresponding to approximate solutions of the equation $Sx = x$. This paper presents a best proximity point theorem for generalized contractions, thereby furnishing optimal approximate solutions, called best proximity points, to some non-linear equations. Also, an iterative algorithm is presented to compute such optimal approximate solutions.

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1 Introduction

Best proximity point theory involves an intertwining of approximation and global optimization. Indeed, it explores the existence and computation of an optimal approximate solution of non-linear equations of the form $Sx = x$, where S is a non-self mapping in some framework. Such equations are confronted when we attempt the mathematical formulation of several problems. Given a non-self mapping $S : A \longrightarrow B$, where A and B are non-empty subsets of a metric space, the equation $Sx = x$ does not necessarily have a solution because of the fact that a solution of the preceding equation constrains the equality between an element in the domain and an element in the range of the mapping. In such circumstances, one raises the following questions:

- Is it possible to find an optimal approximate solution with the least possible error?
- If an approximate solution exists, is there any iterative algorithm to compute such a solution?
- Can one have more than one approximate solution with the least possible error?

Best proximity point theory is an outgrowth of attempts in many directions to answer previously posed questions for various families of non-self mappings. In fact, a best proximity point theorem furnishes sufficient conditions for the existence and computation of an approximate solution x^* that is optimal in the sense that the error $d(x^*, Sx^*)$ assumes the global minimum value $d(A, B)$. Such an optimal approximate solution is known as a best proximity point of the mapping S. It is straightforward to observe that a best proximity point becomes a solution of the equation in the special case that the domain of the mapping intersects the co-domain of the mapping. In essence, a best proximity point theorem delves into the global minimization of the error function $x \longrightarrow d(x, Sx)$ corresponding to the approximate solutions of the equation $Sx = x$. Many interesting best proximity point theorems for various classes of non-self mappings in different frameworks and best approximation theorems have been elicited in [1–17] and [18–40]. The main objective of this article is to present, in the framework of complete metric spaces, a best proximity point theorem for a new family of non-self mappings, known as generalized contractions, thereby computing an optimal approximate solution to the equation $Sx = x$, where S is a generalized contraction. Further, some results in the literature are realizable as special cases from the preceding result.

2 Preliminaries

Throughout this section, we assume that A and B are non-empty subsets of a metric space. We recall the following notions that will be used in the sequel.

Definition 2.1 ([41]) A mapping $S : A \longrightarrow B$ is called a *generalized contraction* if, given real numbers a and b with $0 < a \leq b$, there exists a real number $\alpha(a, b) \in [0, 1)$ such that

$$a \leq d(x_1, x_2) \leq b \quad \Longrightarrow \quad d(Sx_1, Sx_2) \leq \alpha(a, b)d(x_1, x_2)$$

for all x_1, x_2 in A.

It is apparent that every generalized contraction is a contractive mapping and hence it must be continuous.

Definition 2.2 An element x^* in A is called a best proximity point of a mapping $S : A \longrightarrow B$ if it satisfies the condition that

$$d\left(x^*, Sx^*\right) = d(A, B).$$

Due to the fact that $d(x, Sx) \geq d(A, B)$ for all x in A, the global minimum of the error function $x \longrightarrow d(x, Sx)$ corresponding to approximate solutions of the equation $Sx = x$ is attained at any best proximity point. Moreover, if the mapping under consideration is a self-mapping, a best proximity point reduces to a fixed point.

Definition 2.3 ([21]) Given mappings $S : A \longrightarrow B$ and $T : B \longrightarrow A$, it is stated that the pair (S, T) satisfies the *min-max* condition if, for all $x \in A$ and $y \in B$, we have

$$d(A, B) < d(x, y) \quad \Longrightarrow \quad \max(Sx, Ty) \neq \min(Sx, Ty),$$

where $\min(Sx, Ty)$ and $\max(Sx, Ty)$ are defined as

$$\begin{aligned}
\min(Sx, Ty) = \min\bigl\{&d(x, y), d(x, Sx), d(y, Ty), d(Sx, Ty), d(x, STy),\\
&d(y, TSx), d(Sx, TSx), d(Ty, STy), d(TSx, STy)\bigr\},\\
\max(Sx, Ty) = \max\bigl\{&d(x, y), d(x, Sx), d(y, Ty), d(x, Ty), d(y, Sx), d(Sx, Ty)\\
&d(x, TSx), d(y, STy), d(x, STy), d(y, TSx), d(Sx, TSx),\\
&d(Ty, STy), d(TSx, STy)\bigr\}.
\end{aligned}$$

It is quite easy to observe that the min-max condition is less restrictive so that several classes of pairs of mappings meet this requirement.

Definition 2.4 ([21]) Given non-self mappings $S : A \longrightarrow B$ and $T : B \longrightarrow A$, the pair (S, T) is said to be

(a) a cyclic inequality pair if $d(A,B) < d(x,y) \Longrightarrow d(Sx,Ty) \neq d(x,y)$
(b) a cyclic contractive pair if $d(A,B) < d(x,y) \Longrightarrow d(Sx,Ty) < d(x,y)$
(c) a cyclic expansive pair if $d(A,B) < d(x,y) \Longrightarrow d(Sx,Ty) > d(x,y)$

for all $x \in A$ and $y \in B$.

It is remarked that cyclic inequality pairs, cyclic contractive pairs, and cyclic expansive pairs satisfy the min-max condition.

Definition 2.5 Given non-self mappings $S : A \longrightarrow B$ and $T : B \longrightarrow A$, the pair (S, T) is said to form a generalized cyclic contraction if, given real numbers a and b with $0 < a \leq b$, there exists a real number $\alpha(a,b) \in [0,1)$ such that

$$\begin{aligned} &a \leq d(x,y) \leq b \\ &\quad \Longrightarrow \quad d(Sx,Ty) \leq \alpha(a,b)d(x,y) + \bigl[1 - \alpha(a,b)\bigr]d(A,B) \end{aligned}$$

for all x in A and y in B.

It is straightforward to see that every generalized cyclic contraction forms a cyclic contractive pair and hence satisfies the min-max condition.

3 Generalized contractions

We are now ready to establish the following interesting best proximity point theorem for non-self generalized contractions.

Theorem 3.1 *Let A and B be non-empty, closed subsets of a complete metric space. Let $S : A \longrightarrow B$ and $T : B \longrightarrow A$ satisfy the following conditions.*

(a) *S is a generalized contraction.*
(b) *T is a non-expansive mapping.*
(c) *The pair (S, T) satisfies the min-max condition.*

Further, for a fixed element x_0 in A, let

$$x_{2n+1} = Sx_{2n} \quad \textit{and} \quad x_{2n} = Tx_{2n-1}.$$

Then the sequence $\{x_{2n}\}$ must converge to a best proximity point x^* of S and the sequence $\{x_{2n+1}\}$ must converge to a best proximity point y^* of T such that

$$d\bigl(x^*, y^*\bigr) = d(A,B).$$

Further, if S has two distinct best proximity points, then $d(A,B)$ does not vanish and hence the sets A and B should be disjoint.

Proof We define a sequence $\{b_n\}$ of real numbers as follows:

$$b_n := d(x_{2n}, x_{2n+2}) \quad \text{for } n \geq 0.$$

In light of the fact that S is a generalized contraction mapping and T is a non-expansive mapping, it follows that $\{b_n\}$ is a bounded below, decreasing sequence of non-negative real numbers and hence converges to some non-negative real number, say b. Next, we shall prove that b vanishes. If b is positive, then choose a positive integer N such that

$$b \leq b_n \leq (b+1) \quad \text{for all } n \geq N.$$

Because S is a generalized contraction mapping and T is a non-expansive mapping, we have

$$\begin{aligned} b_{N+1} &= d(x_{2N+2}, x_{2N+4}) \\ &= d(Tx_{2N+1}, Tx_{2N+3}) \\ &\leq d(x_{2N+1}, x_{2N+3}) \\ &= d(Sx_{2N}, Sx_{2N+2}) \\ &\leq \alpha(b, b+1) d(x_{2N}, x_{2N+2}) \\ &= \alpha(b, b+1) b_N. \end{aligned}$$

Similarly, we can prove that

$$b_{N+2} \leq \alpha(b, b+1) b_{N+1} \leq [\alpha(b, b+1)]^2 b_N.$$

In general, we shall obtain that

$$b_{N+k} \leq [\alpha(b, b+1)]^k b_N.$$

Letting $k \longrightarrow \infty$, we deduce that b vanishes, which is incompatible with our assumption. Therefore, it can be concluded that $b_n \longrightarrow 0$ as $n \longrightarrow \infty$.

Next, we shall prove that $\{x_{2n}\}$ is a Cauchy sequence. Let $\epsilon > 0$ be given. Since $b_n \longrightarrow 0$, it is possible to choose a positive integer N such that

$$b_N = d(x_{2N}, x_{2N+2}) \leq (\epsilon/2)[1 - \alpha(\epsilon/2, \epsilon)].$$

We define

$$B[x_{2N}, \epsilon] := \{x \in A : d(x, x_{2N}) \leq \epsilon\}.$$

It suffices to prove that if x is an element of $B[x_{2N}, \epsilon]$, then TSx must also be an element of $B[x_{2N}, \epsilon]$. We shall consider two different cases to ascertain the preceding assertion. Let x be an element of $B[x_{2N}, \epsilon]$. If x satisfies the condition that

$$d(x, x_{2N}) \leq \epsilon/2,$$

then it follows that

$$\begin{aligned} d(TSx, x_{2N}) &\leq d(TSx, x_{2N+2}) + d(x_{2N+2}, x_{2N}) \\ &\leq d(Sx, Sx_{2N}) + d(x_{2N+2}, x_{2N}) \\ &\leq d(x, x_{2N}) + d(x_{2N+2}, x_{2N}) \\ &= d(x, x_{2N}) + b_N \\ &\leq \epsilon. \end{aligned}$$

On the other hand, if x satisfies the condition that

$$\epsilon/2 < d(x, x_{2N}) \leq \epsilon,$$

then it follows that

$$\begin{aligned} d(TSx, x_{2N}) &\leq d(TSx, x_{2N+2}) + d(x_{2N+2}, x_{2N}) \\ &\leq d(Sx, Sx_{2N}) + d(x_{2N+2}, x_{2N}) \\ &\leq \alpha(\epsilon/2, \epsilon) d(x, x_{2N}) + d(x_{2N+2}, x_{2N}) \\ &= \alpha(\epsilon/2, \epsilon) d(x, x_{2N}) + b_N \\ &\leq \epsilon. \end{aligned}$$

Therefore, TSx should be an element of $B[x_{2N}, \epsilon]$. Consequently, $x_{2n} \in B[x_{2N}, \epsilon]$ for all $n \geq N$. Thus, $\{x_{2n}\}$ should be a Cauchy sequence. In view of the completeness of the space, the sequence $\{x_{2n}\}$ should converge to some element x^* in A. In light of the continuity of S, it results that the sequence $\{x_{2n+1}\}$ should converge to some element y^* in B and $y^* = Sx^*$. Further, because of the continuity of T, $\{x_{2n}\}$ should converge to Ty^*. So, Ty^* and x^* should be identical. Also, we can deduce that $TSx^* = x^*$ and $STy^* = y^*$. As a result, we have

$$\min(Sx^*, Ty^*) = \max(Sx^*, Ty^*) = d(x^*, y^*).$$

On account of the fact that the pair (S, T) satisfies the min-max condition, we obtain that

$$d(x^*, y^*) = d(A, B).$$

Thus, we can deduce that

$$d(x^*, Tx^*) = d(x^*, y^*) = d(A, B), \qquad d(y^*, Sy^*) = d(x^*, y^*) = d(A, B).$$

We shall assume that the non-self mapping S has two distinct best proximity points x' and x''. Then it follows that

$$\begin{aligned} d(x', x'') &\leq d(x', Sx') + d(Sx', Sx'') + d(x'', Sx'') \\ &\leq \alpha(a, a) d(x', x'') + 2d(A, B) \quad \text{where } a = d(x', x'') \\ &< d(x', x'') + 2d(A, B). \end{aligned}$$

Thus, $d(A,B) > 0$ and hence the sets A and B are disjoint. This completes the proof of the theorem. □

We illustrate the preceding best proximity point theorem by means of the following example.

Example 3.2 We shall consider the complete metric space $C[0,2]$ with the supremum metric. For $n = \pm1, \pm2, \pm3, \ldots$, define $f_n : [0,2] \longrightarrow \mathbf{R}$ as

$$f_n(x) = \begin{cases} 1 + (x/n) & \text{if } 0 \le x \le 1, \\ 1 + (2/n) - (x/n) & \text{otherwise.} \end{cases}$$

Let $f_0 : [0,2] \longrightarrow \mathbf{R}$ be defined as

$$f_0(x) = 1 \quad \text{for all } x \in [0,2].$$

Let $A := \{f_n : n = 0, \pm1, \pm2, \pm3, \ldots\}$ and $B := \{-f : f \in A\}$.

Let $S : A \longrightarrow B$ be defined as

$$S(f_n) \begin{cases} -f_{n+1} & \text{if } n > 0, \\ -f_{n-1} & \text{if } n < 0, \\ -f_0 & \text{otherwise.} \end{cases}$$

Let $T : B \longrightarrow A$ be defined as

$$T(-f_n) \begin{cases} f_{n+2} & \text{if } n > 0, \\ f_{n-2} & \text{if } n < 0, \\ f_0 & \text{otherwise.} \end{cases}$$

Then, it is easy to see that S is a generalized contraction and T is a non-expansive mapping. Also, the pair (S,T) satisfies the min-max condition. However, (S,T) is neither a cyclic contractive pair nor a cyclic expansive pair. Finally, we can note that the element $x^* = f_0$ in A is a best proximity point of the mapping S and the element $y^* = -f_0$ in B is a best proximity point of the mapping T such that $d(x^*, y^*) = d(A,B) = 2$.

One can easily see that best proximity point Theorem 3.1 subsumes the following result.

Corollary 3.3 *Let A and B be non-empty, closed subsets of a complete metric space. Let the non-self mappings $S : A \longrightarrow B$ and $T : B \longrightarrow A$ satisfy the following conditions.*

(a) *S is a generalized contraction.*
(b) *T is a non-expansive mapping.*
(c) *(S,T) is a generalized cyclic contraction.*

Further, for a fixed element x_0 in A, let

$$x_{2n+1} = Sx_{2n} \quad \text{and} \quad x_{2n} = Tx_{2n-1}.$$

Then the sequence $\{x_{2n}\}$ converges to a best proximity point x^* of S and the sequence $\{x_{2n+1}\}$ converges to a best proximity point y^* of T such that

$$d(x^*, y^*) = d(A, B).$$

Moreover, if S has two distinct best proximity points, then $d(A, B) > 0$ and hence the sets A and B must be disjoint.

Best proximity point Theorem 3.1 subsumes the following fixed point theorem, due to Krasnoselskii [41], which in turn extends the most interesting and well-known contraction principle.

Corollary 3.4 *Let X be a complete metric space. If the self-mapping $T : X \longrightarrow X$ is a generalized contraction, then it has a unique fixed point x^*, and for every x in X, the sequence $\{T^n(x)\}$ converges to x^*.*

The following best proximity point theorem, due to Basha [19], which extends the contraction principle to the case of non-self mappings, is a special case of Theorem 3.1.

Corollary 3.5 *Let A and B be non-empty, closed subsets of a complete metric space. Let $S : A \longrightarrow B$ and $T : B \longrightarrow A$ be non-self mappings satisfying the following conditions.*

(a) *S is a contraction.*
(b) *T is a non-expansive mapping.*
(c) *(S, T) is a cyclic contractive pair.*

Further, for a fixed element x_0 in A, let

$$x_{2n+1} = Sx_{2n} \quad \text{and} \quad x_{2n} = Tx_{2n-1}.$$

Then the sequence $\{x_{2n}\}$ converges to a best proximity point x^* of S and the sequence $\{x_{2n+1}\}$ converges to a best proximity point y^* of T such that

$$d(x^*, y^*) = d(A, B).$$

Moreover, if S has two distinct best proximity points, then $d(A, B) > 0$ and hence the sets A and B must be disjoint.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

Author details
[1]Department of Mathematics, Anna University, Chennai, 600 025, India. [2]Department of Mathematics, King Abdulaziz University, P.O. Box 80203, Jeddah, 21589, Saudi Arabia. [3]St. Joesph's College Higher Secondary School, Trichy, 620 002, India.

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