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The convergence theorems of Ishikawa iterative process with errors for Φ-hemi-contractive mappings in uniformly smooth Banach spaces

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Abstract
Let D be a nonempty closed convex subset of an arbitrary uniformly smooth real Banach space E, and $T: D \to D$ be a generalized Lipschitz Φ-hemi-contractive mapping with $q \in F(T) \neq \emptyset$. Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{d_n\}$ be four real sequences in $[0,1]$ and satisfy the conditions (i) $a_n, b_n, d_n \to 0$ as $n \to \infty$ and $c_n = o(a_n)$; (ii) $\sum_{n=0}^{\infty} a_n = \infty$. For some $x_0 \in D$, let $\{u_n\}$, $\{v_n\}$ be any bounded sequences in D, and $\{x_n\}$ be an Ishikawa iterative sequence with errors defined by (1.1). Then (1.1) converges strongly to the fixed point q of T. A related result deals with the operator equations for a generalized Lipschitz and Φ-quasi-accretive mapping.
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1 Introduction and preliminary

Let E be a real Banach space and E^* be its dual space. The normalized duality mapping $J: E \to 2^{E^*}$ is defined by

$$J(x) = \{f \in E^* : \langle x, f \rangle = \|x\|^2 = \|f\|^2\}, \quad \forall x \in E,$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. It is well known that

(i) If E is a smooth Banach space, then the mapping J is single-valued;
(ii) $J(\alpha x) = \alpha J(x)$ for all $x \in E$ and $\alpha \in \Re$;
(iii) If E is a uniformly smooth Banach space, then the mapping J is uniformly continuous on any bounded subset of E. We denote the single-valued normalized duality mapping by j.

Definition 1.1 ([1]) Let D be a nonempty closed convex subset of E, $T: D \to D$ be a mapping.

(1) T is called strongly pseudocontractive if there is a constant $k \in (0,1)$ such that for all $x, y \in D$,

$$\langle Tx - Ty, j(x-y) \rangle \leq k\|x-y\|^2;$$

(2) T is called ϕ-strongly pseudocontractive if for all $x, y \in D$, there exist $j(x-y) \in J(x-y)$ and a strictly increasing continuous function $\phi : [0,+\infty) \to [0,+\infty)$ with $\phi(0) = 0$ such that

$$\langle Tx - Ty, j(x-y)\rangle \leq \|x-y\|^2 - \phi(\|x-y\|)\|x-y\|;$$

(3) T is called Φ-pseudocontractive if for all $x, y \in D$, there exist $j(x-y) \in J(x-y)$ and a strictly increasing continuous function $\Phi : [0,+\infty) \to [0,+\infty)$ with $\Phi(0) = 0$ such that

$$\langle Tx - Ty, j(x-y)\rangle \leq \|x-y\|^2 - \Phi(\|x-y\|).$$

It is obvious that Φ-pseudocontractive mappings not only include ϕ-strongly pseudocontractive mappings, but also strongly pseudocontractive mappings.

Definition 1.2 ([1]) Let $T : D \to D$ be a mapping and $F(T) = \{x \in D : Tx = x\} \neq \emptyset$.

(1) T is called ϕ-strongly-hemi-pseudocontractive if for all $x \in D$, $q \in F(T)$, there exist $j(x-q) \in J(x-q)$ and a strictly increasing continuous function $\phi : [0,+\infty) \to [0,+\infty)$ with $\phi(0) = 0$ such that

$$\langle Tx - Tq, j(x-q)\rangle \leq \|x-q\|^2 - \phi(\|x-q\|)\|x-q\|.$$

(2) T is called Φ-hemi-pseudocontractive if for all $x \in D$, $q \in F(T)$, there exist $j(x-q) \in J(x-q)$ and the strictly increasing continuous function $\Phi : [0,+\infty) \to [0,+\infty)$ with $\Phi(0) = 0$ such that

$$\langle Tx - Tq, j(x-q)\rangle \leq \|x-q\|^2 - \Phi(\|x-q\|).$$

Closely related to the class of pseudocontractive-type mappings are those of accretive type.

Definition 1.3 ([1]) Let $N(T) = \{x \in E : Tx = 0\} \neq \emptyset$. The mapping $T : E \to E$ is called strongly quasi-accretive if for all $x \in E$, $x^* \in N(T)$, there exist $j(x-x^*) \in J(x-x^*)$ and a constant $k \in (0,1)$ such that $\langle Tx - Tx^*, j(x-x^*)\rangle \geq k\|x-x^*\|^2$; T is called ϕ-strongly quasi-accretive if for all $x \in E$, $x^* \in N(T)$, there exist $j(x-x^*) \in J(x-x^*)$ and a strictly increasing continuous function $\phi : [0,+\infty) \to [0,+\infty)$ with $\phi(0) = 0$ such that $\langle Tx - Tx^*, j(x-x^*)\rangle \geq \phi(\|x-x^*\|)\|x-x^*\|$; T is called Φ-quasi-accretive if for all $x \in E$, $x^* \in N(T)$, there exist $j(x-x^*) \in J(x-x^*)$ and a strictly increasing continuous function $\Phi : [0,+\infty) \to [0,+\infty)$ with $\Phi(0) = 0$ such that $\langle Tx - Tx^*, j(x-x^*)\rangle \geq \Phi(\|x-x^*\|)$.

Definition 1.4 ([2]) For arbitrary given $x_0 \in D$, the Ishikawa iterative process with errors $\{x_n\}_{n=0}^{\infty}$ is defined by

$$\begin{cases} y_n = (1-b_n-d_n)x_n + b_nTx_n + d_nv_n, & n \geq 0, \\ x_{n+1} = (1-a_n-c_n)x_n + a_nTy_n + c_nu_n, & n \geq 0, \end{cases} \tag{1.1}$$

where $\{u_n\}$, $\{v_n\}$ are any bounded sequences in D; $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{d_n\}$ are four real sequences in $[0,1]$ and satisfy $a_n + c_n \leq 1$, $b_n + d_n \leq 1$, for all $n \geq 0$. If $b_n = d_n = 0$, then the sequence $\{x_n\}$ defined by

$$x_{n+1} = (1 - a_n - c_n)x_n + a_n Tx_n + c_n u_n, \quad n \geq 0 \tag{1.2}$$

is called the Mann iterative process with errors.

Definition 1.5 ([3, 4]) A mapping $T : D \to D$ is called generalized Lipschitz if there exists a constant $L > 0$ such that $\|Tx - Ty\| \leq L(1 + \|x - y\|)$, $\forall x, y \in D$.

The aim of this paper is to prove the convergent results of the above Ishikawa and Mann iterations with errors for generalized Lipschitz Φ-hemi-contractive mappings in uniformly smooth real Banach spaces. For this, we need the following lemmas.

Lemma 1.6 ([5]) *Let E be a uniformly smooth real Banach space, and let $J : E \to 2^{E^*}$ be a normalized duality mapping. Then*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, J(x + y)\rangle \tag{1.3}$$

for all $x, y \in E$.

Lemma 1.7 ([6]) *Let $\{\rho_n\}_{n=0}^{\infty}$ be a nonnegative sequence which satisfies the following inequality:*

$$\rho_{n+1} \leq (1 - \lambda_n)\rho_n + \sigma_n, \quad n \geq 0, \tag{1.4}$$

where $\lambda_n \in [0, 1]$ with $\sum_{n=0}^{\infty} \lambda_n = \infty$, $\sigma_n = o(\lambda_n)$. Then $\rho_n \to 0$ as $n \to \infty$.

2 Main results

Theorem 2.1 *Let E be an arbitrary uniformly smooth real Banach space, D be a nonempty closed convex subset of E, and $T : D \to D$ be a generalized Lipschitz Φ-hemi-contractive mapping with $q \in F(T) \neq \emptyset$. Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{d_n\}$ be four real sequences in* $[0, 1]$ *and satisfy the conditions* (i) *$a_n, b_n, d_n \to 0$ as $n \to \infty$ and $c_n = o(a_n)$;* (ii) *$\sum_{n=0}^{\infty} a_n = \infty$. For some $x_0 \in D$, let $\{u_n\}$, $\{v_n\}$ be any bounded sequences in D, and $\{x_n\}$ be an Ishikawa iterative sequence with errors defined by* (1.1). *Then* (1.1) *converges strongly to the unique fixed point q of T.*

Proof Since $T : D \to D$ is a generalized Lipschitz Φ-hemi-contractive mapping, there exists a strictly increasing continuous function $\Phi : [0, +\infty) \to [0, +\infty)$ with $\Phi(0) = 0$ such that

$$\langle Tx - Tq, J(x - q)\rangle \leq \|x - q\|^2 - \Phi(\|x - q\|),$$

i.e.,

$$-\langle x - Tx, J(x - q)\rangle \leq -\Phi(\|x - q\|),$$

and

$$\|Tx - Ty\| \leq L(1 + \|x - y\|),$$

for any $x, y \in D$ and $q \in F(T)$.

Step 1. There exists $x_0 \in D$ and $x_0 \neq Tx_0$ such that $r_0 = \|x_0 - Tx_0\| \cdot \|x_0 - q\| \in R(\Phi)$ (range of Φ). Indeed, if $\Phi(r) \to +\infty$ as $r \to +\infty$, then $r_0 \in R(\Phi)$; if $\sup\{\Phi(r) : r \in [0, +\infty)\} = r_1 < +\infty$ with $r_0 < r_1$, then for $q \in D$, there exists a sequence $\{w_n\}$ in D such that $w_n \to q$ as $n \to \infty$ with $w_n \neq q$. Furthermore, we obtain that $\{w_n - Tw_n\}$ is bounded. Hence, there exists a natural number n_0 such that $\|w_n - Tw_n\| \cdot \|w_n - q\| < \frac{r_1}{2}$ for $n \geq n_0$, then we redefine $x_0 = w_{n_0}$ and $\|x_0 - Tx_0\| \cdot \|x_0 - q\| \in R(\Phi)$.

Step 2. For any $n \geq 0$, $\{x_n\}$ is bounded. Set $R = \Phi^{-1}(r_0)$, then from Definition 1.2(2), we obtain that $\|x_0 - q\| \leq R$. Denote $B_1 = \{x \in D : \|x - q\| \leq R\}$, $B_2 = \{x \in D : \|x - q\| \leq 2R\}$. Since T is generalized Lipschitz, so T is bounded. We may define $M = \sup_{x \in B_2}\{\|Tx - q\| + 1\} + \sup_n\{\|u_n - q\|\} + \sup_n\{\|v_n - q\|\}$. Next, we want to prove that $x_n \in B_1$. If $n = 0$, then $x_0 \in B_1$. Now, assume that it holds for some n, *i.e.*, $x_n \in B_1$. We prove that $x_{n+1} \in B_1$. Suppose it is not the case, then $\|x_{n+1} - q\| > R$. Since J is uniformly continuous on a bounded subset of E, then for $\epsilon_0 = \frac{\Phi(\frac{R}{4})}{24L(1+2R)}$, there exists $\delta > 0$ such that $\|Jx - Jy\| < \epsilon$ when $\|x - y\| < \delta$, $\forall x, y \in B_2$. Now, denote

$$\begin{aligned} \tau_0 = \min\biggl\{&1, \frac{R}{2[L(1+2R)+2R+M]}, \frac{R}{4[L(1+R)+2R+M]}, \\ &\frac{\delta}{2[L(1+2R)+2R+M]}, \frac{\Phi(\frac{R}{4})}{24R^2}, \frac{\Phi(\frac{R}{4})}{24L(1+2R)}, \frac{\Phi(\frac{R}{4})}{48MR}\biggr\}. \end{aligned}$$

Owing to $a_n, b_n, c_n, d_n \to 0$ as $n \to \infty$, without loss of generality, assume that $0 \leq a_n, b_n, c_n, d_n \leq \tau_0$ for any $n \geq 0$. Since $c_n = o(a_n)$, denote $c_n < a_n \tau_0$. So, we have

$$\begin{aligned} &\|Tx_n - q\| \\ &\quad\leq L\bigl(1 + \|x_n - q\|\bigr) \\ &\quad\leq L(1+R), \end{aligned} \tag{2.1}$$

$$\begin{aligned} &\|y_n - q\| \\ &\quad\leq (1 - b_n - d_n)\|x_n - q\| + b_n\|Tx_n - q\| + d_n\|v_n - q\| \\ &\quad\leq R + b_n L\bigl(1 + \|x_n - q\|\bigr) + d_n M \\ &\quad\leq R + b_n L(1+R) + d_n M \\ &\quad\leq R + \tau_0\bigl[L(1+R) + M\bigr] \\ &\quad\leq 2R, \end{aligned} \tag{2.2}$$

$$\begin{aligned} &\|Ty_n - q\| \\ &\quad\leq L\bigl(1 + \|y_n - q\|\bigr) \\ &\quad\leq L(1+2R), \end{aligned} \tag{2.3}$$

$$\begin{aligned} &\|x_n - Tx_n\| \\ &\quad\leq L + (1+L)\|x_n - q\| \\ &\quad\leq L + (1+L)R, \end{aligned} \tag{2.4}$$

and

$$\begin{aligned}
&\big\|(x_n-q)-(y_n-q)\big\| \\
&\quad \le b_n\|x_n-Tx_n\|+d_n\big[\|v_n-q\|+\|x_n-q\|\big] \\
&\quad \le b_n\big[L+(1+L)R\big]+d_n(M+R) \\
&\quad \le \tau_0\big[L(1+R)+2R+M\big] \\
&\quad \le \tau_0\big[L(1+2R)+2R+M\big] \\
&\quad \le \frac{\delta}{2}<\delta; \qquad (2.5)
\end{aligned}$$

$$\begin{aligned}
&\|x_n-q\| \\
&\quad \ge \|x_{n+1}-q\|-a_n\|Ty_n-x_n\|-c_n\|u_n-x_n\| \\
&\quad \ge \|x_{n+1}-q\|-a_n\big[\|Ty_n-q\|+\|x_n-q\|\big]-c_n\big[\|x_n-q\|+\|u_n-q\|\big] \\
&\quad \ge R-a_n\big[L(1+2R)+R\big]-c_n(R+M) \\
&\quad \ge R-\tau_0\big[L(1+2R)+M+2R\big] \\
&\quad \ge R-\frac{R}{2} \\
&\quad =\frac{R}{2}, \qquad (2.6)
\end{aligned}$$

$$\begin{aligned}
&\|y_n-q\| \\
&\quad \ge \|x_n-q\|-b_n\|Tx_n-x_n\|-d_n\|x_n-v_n\| \\
&\quad \ge \|x_n-q\|-b_n\big[L+(1+L)R\big]-d_n\big[\|x_n-q\|+\|v_n-q\|\big] \\
&\quad \ge \|x_n-q\|-b_n\big[L+(1+L)R\big]-d_n(R+M) \\
&\quad \ge \|x_n-q\|-\tau_0\big[L+(2+L)R+M\big] \\
&\quad >\frac{R}{2}-\frac{R}{4}=\frac{R}{4}, \qquad (2.7)
\end{aligned}$$

$$\begin{aligned}
&\|x_{n+1}-q\| \\
&\quad \le (1-a_n-c_n)\|x_n-q\|+a_n\|Ty_n-q\|+c_n\|u_n-q\| \\
&\quad \le R+\tau_0\big[L(1+2R)+M\big] \\
&\quad \le 2R, \qquad (2.8)
\end{aligned}$$

$$\begin{aligned}
&\big\|(x_{n+1}-q)-(x_n-q)\big\| \\
&\quad \le a_n\|Ty_n-x_n\|+c_n\|u_n-x_n\| \\
&\quad \le a_n\big[\|Ty_n-q\|+\|x_n-q\|\big]+c_n\big[\|u_n-q\|+\|x_n-q\|\big] \\
&\quad \le a_n\big[L(1+2R)+R\big]+c_n(M+R) \\
&\quad \le \tau_0\big[L(1+2R)+2R+M\big] \\
&\quad \le \frac{\delta}{2}<\delta. \qquad (2.9)
\end{aligned}$$

Therefore,

$$\begin{aligned}&\big\|J(x_n-q)-J(y_n-q)\big\|<\epsilon_0;\\&\big\|J(x_{n+1}-q)-J(x_n-q)\big\|<\epsilon_0.\end{aligned}$$

Using Lemma 1.6 and the above formulas, we obtain

$$\begin{aligned}&\|x_{n+1}-q\|^2\\&\quad\le(1-a_n-c_n)^2\|x_n-q\|^2+2a_n\big\langle Ty_n-q,J(x_{n+1}-q)\big\rangle\\&\qquad+2c_n\big\langle u_n-q,J(x_{n+1}-q)\big\rangle\\&\quad\le(1-a_n)^2\|x_n-q\|^2+2a_n\big\langle Ty_n-q,J(x_{n+1}-q)-J(x_n-q)\big\rangle\\&\qquad+2a_n\big\langle Ty_n-q,J(x_n-q)-J(y_n-q)\big\rangle+2a_n\big\langle Ty_n-q,J(y_n-q)\big\rangle\\&\qquad+2c_n\big\langle u_n-q,J(x_{n+1}-q)\big\rangle\\&\quad\le(1-a_n)^2\|x_n-q\|^2+2a_n\|Ty_n-q\|\cdot\big\|J(x_{n+1}-q)-J(x_n-q)\big\|\\&\qquad+2a_n\|Ty_n-q\|\cdot\big\|J(x_n-q)-J(y_n-q)\big\|\\&\qquad+2a_n\big[\|y_n-q\|^2-\Phi\big(\|y_n-q\|\big)\big]+2c_n\|u_n-q\|\cdot\|x_{n+1}-q\|\\&\quad\le(1-a_n)^2R^2+4a_nL(1+2R)\epsilon_0+2a_n\big[\|y_n-q\|^2-\Phi\big(\|y_n-q\|\big)\big]+4c_nMR,\end{aligned}\tag{2.10}$$

and

$$\begin{aligned}&\|y_n-q\|^2\\&\quad\le(1-b_n-d_n)^2\|x_n-q\|^2+2b_n\big\langle Tx_n-q,J(y_n-q)\big\rangle\\&\qquad+2d_n\big\langle v_n-q,J(y_n-q)\big\rangle\\&\quad\le\|x_n-q\|^2+2b_n\big\langle Tx_n-q,J(y_n-q)-J(x_n-q)\big\rangle\\&\qquad+2b_n\big\langle Tx_n-q,J(x_n-q)\big\rangle+2d_n\|v_n-q\|\cdot\|y_n-q\|\\&\quad\le\|x_n-q\|^2+2b_n\|Tx_n-q\|\cdot\big\|J(y_n-q)-J(x_n-q)\big\|\\&\qquad+2b_n\big[\|x_n-q\|^2-\Phi\big(\|x_n-q\|\big)\big]+2d_n\|v_n-q\|\cdot\|y_n-q\|\\&\quad\le R^2+2b_nL(1+R)\epsilon_0+2b_nR^2+4d_nMR.\end{aligned}\tag{2.11}$$

Substitute (2.11) into (2.10)

$$\begin{aligned}&\|x_{n+1}-q\|^2\\&\quad\le(1-a_n)^2R^2+4a_nL(1+2R)\epsilon_0+2a_n\big[R^2+2b_nL(1+R)\epsilon_0\\&\qquad+2b_nR^2+4d_nMR\big]-2a_n\Phi\big(\|y_n-q\|\big)+4c_nMR\\&\quad\le R^2+a_n^2R^2+4a_nL(1+2R)\epsilon_0+2a_n\big[2b_nL(1+R)\epsilon_0\\&\qquad+2b_nR^2+4d_nMR\big]-2a_n\Phi\left(\frac{R}{4}\right)+4c_nMR\end{aligned}$$

$$\begin{aligned}
&= R^2 + 2a_n\left[\frac{a_n}{2}R^2 + 2L(1+2R)\epsilon_0 + 2b_nL(1+R)\epsilon_0\right.\\
&\quad \left.+ 2b_nR^2 + 4d_nMR + \frac{2c_nMR}{a_n}\right] - 2a_n\Phi\left(\frac{R}{4}\right)\\
&\le R^2 + 2a_n\left[\frac{\Phi(\frac{R}{4})}{2} - \Phi\left(\frac{R}{4}\right)\right]\\
&\le R^2 - \Phi\left(\frac{R}{4}\right)a_n\\
&\le R^2, \qquad (2.12)
\end{aligned}$$

this is a contradiction. Thus, $x_{n+1} \in B_1$, *i.e.*, $\{x_n\}$ is a bounded sequence. So, $\{y_n\}$, $\{Ty_n\}$, $\{Tx_n\}$ are all bounded sequences.

Step 3. We want to prove $\|x_n - q\| \to 0$ as $n \to \infty$. Set $M_1 = \max\{\sup_n \|x_n - q\|, \sup_n \|y_n - q\|, \sup_n \|Tx_n - q\|, \sup_n \|Ty_n - q\|, \sup_n \|u_n - q\|, \sup_n \|v_n - q\|\}$.

By (2.10), (2.11), we have

$$\begin{aligned}
&\|x_{n+1} - q\|^2\\
&\quad\le (1-a_n)^2\|x_n - q\|^2 + 2a_n\|Ty_n - q\| \cdot \big\|J(x_{n+1} - q) - J(x_n - q)\big\|\\
&\qquad + 2a_n\|Ty_n - q\| \cdot \big\|J(x_n - q) - J(y_n - q)\big\|\\
&\qquad + 2a_n\big[\|y_n - q\|^2 - \Phi\big(\|y_n - q\|\big)\big] + 2c_n\|u_n - q\| \cdot \|x_{n+1} - q\|\\
&\quad\le (1-a_n)^2\|x_n - q\|^2 + 2a_nM_1A_n + 2a_nM_1B_n\\
&\qquad + 2a_n\big[\|y_n - q\|^2 - \Phi\big(\|y_n - q\|\big)\big] + 2c_nM_1^2, \qquad (2.13)
\end{aligned}$$

and

$$\begin{aligned}
&\|y_n - q\|^2\\
&\quad\le \|x_n - q\|^2 + 2b_n\|Tx_n - q\| \cdot \big\|J(y_n - q) - J(x_n - q)\big\|\\
&\qquad + 2b_n\big[\|x_n - q\|^2 - \Phi\big(\|x_n - q\|\big)\big] + 2d_n\|v_n - q\| \cdot \|y_n - q\|\\
&\quad\le \|x_n - q\|^2 + 2M_1B_n + 2b_nM_1^2 + 2d_nM_1^2, \qquad (2.14)
\end{aligned}$$

where $A_n = \|J(x_{n+1} - q) - J(x_n - q)\|$, $B_n = \|J(x_n - q) - J(y_n - q)\|$ and $A_n, B_n \to 0$ as $n \to \infty$.

Taking (2.14) into (2.13),

$$\begin{aligned}
&\|x_{n+1} - q\|^2\\
&\quad\le \|x_n - q\|^2 + a_n^2M_1^2 + 2a_nM_1A_n + 2a_nM_1B_n\\
&\qquad + 2a_n\big[2M_1B_n + 2b_nM_1^2 + 2d_nM_1^2 - \Phi\big(\|y_n - q\|\big)\big] + 2c_nM_1^2\\
&\quad\le \|x_n - q\|^2 + 2a_n\left[\frac{a_n}{2}M_1^2 + M_1A_n + 3M_1B_n + 2b_nM_1^2\right.\\
&\qquad \left.+ 2d_nM_1^2 + \frac{c_nM_1^2}{a_n} - \Phi\big(\|y_n - q\|\big)\right]\\
&\quad\le \|x_n - q\|^2 + 2a_n\big[C_n - \Phi\big(\|y_n - q\|\big)\big], \qquad (2.15)
\end{aligned}$$

where $C_n = \frac{a_n}{2}M_1^2 + M_1A_n + 3M_1B_n + 2b_nM_1^2 + 2d_nM_1^2 + \frac{c_nM_1^2}{a_n} \to 0$ as $n \to \infty$.

Set $\inf_{n\geq 0}\frac{\Phi(\|y_n-q\|)}{1+\|x_{n+1}-q\|^2}=\lambda$, then $\lambda=0$. If it is not the case, we assume that $\lambda>0$. Let $0<\gamma<\min\{1,\lambda\}$, then $\frac{\Phi(\|y_n-q\|)}{1+\|x_{n+1}-q\|^2}\geq\gamma$, *i.e.*, $\Phi(\|y_n-q\|)\geq\gamma+\gamma\|x_{n+1}-q\|^2\geq\gamma\|x_{n+1}-q\|^2$. Thus, from (2.15) it follows that

$$\begin{aligned}&\|x_{n+1}-q\|^2\\&\quad\leq\|x_n-q\|^2+2a_n\bigl(C_n-\gamma\|x_{n+1}-q\|^2\bigr).\end{aligned}\tag{2.16}$$

This implies that

$$\begin{aligned}&\|x_{n+1}-q\|^2\\&\quad\leq\frac{1}{1+2a_n\gamma}\|x_n-q\|^2+\frac{2a_nC_n}{1+2a_n\gamma}\\&\quad=\left(1-\frac{2a_n\gamma}{1+2a_n\gamma}\right)\|x_n-q\|^2+\frac{2a_nC_n}{1+2a_n\gamma}.\end{aligned}\tag{2.17}$$

Let $\rho_n=\|x_n-q\|^2$, $\lambda_n=\frac{2a_n\gamma}{1+2a_n\gamma}$, $\sigma_n=\frac{2a_nC_n}{1+2a_n\gamma}$. Then we get that

$$\rho_{n+1}\leq(1-\lambda_n)\rho_n+\sigma_n.$$

Applying Lemma 1.7, we get that $\rho_n\to 0$ as $n\to\infty$. This is a contradiction and so $\lambda=0$. Therefore, there exists an infinite subsequence such that $\frac{\Phi(\|y_{n_i}-q\|)}{1+\|x_{n_i+1}-q\|^2}\to 0$ as $i\to\infty$. Since $0\leq\frac{\Phi(\|y_{n_i}-q\|)}{1+M_1^2}\leq\frac{\Phi(\|y_{n_i}-q\|)}{1+\|x_{n_i+1}-q\|^2}$, then $\Phi(\|y_{n_i}-q\|)\to 0$ as $i\to\infty$. In view of the strictly increasing continuity of Φ, we have $\|y_{n_i}-q\|\to 0$ as $i\to\infty$. Hence, $\|x_{n_i}-q\|\to 0$ as $i\to\infty$. Next, we want to prove $\|x_n-q\|\to 0$ as $n\to\infty$. Let $\forall\varepsilon\in(0,1)$, there exists n_{i_0} such that $\|x_{n_i}-q\|<\epsilon$, $a_n,c_n<\frac{\epsilon}{8M_1}$, $b_n,d_n<\frac{\epsilon}{16M_1}$, $C_n<\frac{\Phi(\epsilon)}{2}$, for any $n_i,n\geq n_{i_0}$. First, we want to prove $\|x_{n_i+1}-q\|<\epsilon$. Suppose it is not the case, then $\|x_{n_i+1}-q\|\geq\epsilon$. Using (1.1), we may get the following estimates:

$$\begin{aligned}&\|x_{n_i}-q\|\\&\quad\geq\|x_{n_i+1}-q\|-a_{n_i}\|Ty_{n_i}-x_{n_i}\|-c_{n_i}\|u_{n_i}-x_{n_i}\|\\&\quad>\epsilon-a_{n_i}\bigl[\|Ty_{n_i}-q\|+\|x_{n_i}-q\|\bigr]-c_{n_i}\bigl[\|u_{n_i}-q\|+\|x_{n_i}-q\|\bigr]\\&\quad\geq\epsilon-a_{n_i}2M_1-c_{n_i}2M_1\\&\quad\geq\epsilon-a_{n_i}2M_1-c_{n_i}2M_1\\&\quad>\frac{\epsilon}{2},\end{aligned}\tag{2.18}$$

$$\begin{aligned}&\|y_{n_i}-q\|\\&\quad\geq\|x_{n_i}-q\|-b_{n_i}\|Tx_{n_i}-x_{n_i}\|-d_{n_i}\|v_{n_i}-x_{n_i}\|\\&\quad>\frac{\epsilon}{2}-b_{n_i}\bigl[\|Tx_{n_i}-q\|+\|x_{n_i}-q\|\bigr]-d_{n_i}\bigl[\|v_{n_i}-q\|+\|x_{n_i}-q\|\bigr]\\&\quad\geq\frac{\epsilon}{2}-b_{n_i}2M_1-d_{n_i}2M_1\\&\quad\geq\frac{\epsilon}{4}.\end{aligned}\tag{2.19}$$

Since Φ is strictly increasing, then (2.19) leads to $\Phi(\|y_{n_i} - q\|) \geq \Phi(\frac{\epsilon}{4})$. From (2.15), we have

$$\begin{aligned}
&\|x_{n_i+1} - q\|^2 \\
&\quad \leq \|x_{n_i} - q\|^2 + 2a_{n_i}\bigl[C_{n_i} - \Phi\bigl(\|y_{n_i} - q\|\bigr)\bigr] \\
&\quad < \epsilon^2 + 2a_{n_i}\left[\frac{1}{2}\Phi\left(\frac{\epsilon}{4}\right) - \Phi\left(\frac{\epsilon}{4}\right)\right] \\
&\quad \leq \epsilon^2 - \Phi\left(\frac{\epsilon}{4}\right)a_{n_i} \\
&\quad \leq \epsilon^2,
\end{aligned} \tag{2.20}$$

which is a contradiction. Hence, $\|x_{n_i+1} - q\| < \epsilon$. Suppose that $\|x_{n_i+m} - q\| < \epsilon$ holds. Repeating the above course, we can easily prove that $\|x_{n_i+m+1} - q\| < \epsilon$ holds. Therefore, for any m, we obtain that $\|x_{n_i+m} - q\| < \epsilon$, which means $\|x_n - q\| \to 0$ as $n \to \infty$. This completes the proof. □

Theorem 2.2 *Let E be an arbitrary uniformly smooth real Banach space, and $T : E \to E$ be a generalized Lipschitz Φ-quasi-accretive mapping with $N(T) \neq \emptyset$. Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{d_n\}$ be four real sequences in $[0,1]$ and satisfy the conditions* (i) $a_n, b_n, d_n \to 0$ *as* $n \to \infty$ *and* $c_n = o(a_n)$; (ii) $\sum_{n=0}^{\infty} a_n = \infty$. *For some $x_0 \in D$, let $\{u_n\}$, $\{v_n\}$ be any bounded sequences in E, and $\{x_n\}$ be an Ishikawa iterative sequence with errors defined by*

$$\begin{cases} y_n = (1 - b_n - d_n)x_n + b_nSx_n + d_nv_n, & n \geq 0, \\ x_{n+1} = (1 - a_n - c_n)x_n + a_nSy_n + c_nu_n, & n \geq 0, \end{cases} \tag{2.21}$$

where $S : E \to E$ is defined by $Sx = x - Tx$ for any $x \in E$. Then $\{x_n\}$ converges strongly to the unique solution of the equation $Tx = 0$ (or the unique fixed point of S).

Proof Since T is a generalized Lipschitz and Φ-quasi-accretive mapping, it follows that

$$\|Tx - Ty\| \leq L\bigl(1 + \|x - y\|\bigr),$$

i.e.,

$$\begin{aligned}
&\|Sx - Sy\| \leq L_1\bigl(1 + \|x - y\|\bigr), \quad L_1 = 1 + L; \\
&\bigl\langle Tx - Tq, J(x - q)\bigr\rangle \geq \Phi\bigl(\|x - q\|\bigr),
\end{aligned}$$

i.e.,

$$\bigl\langle Sx - Sq, J(x - q)\bigr\rangle \leq \|x - q\|^2 - \Phi\bigl(\|x - q\|\bigr),$$

for all $x, y \in E$, $q \in N(T)$. The rest of the proof is the same as that of Theorem 2.1. □

Corollary 2.3 *Let E be an arbitrary uniformly smooth real Banach space, D be a nonempty closed convex subset of E, and $T : D \to D$ be a generalized Lipschitz Φ-hemi-contractive* *mapping with $q \in F(T) \neq \emptyset$. Let $\{a_n\}$, $\{c_n\}$ be two real sequences in $[0,1]$ and satisfy the conditions* (i) *$a_n \to 0$ as $n \to \infty$ and $c_n = o(a_n)$*; (ii) *$\sum_{n=0}^{\infty} a_n = \infty$. For some $x_0 \in D$, let $\{u_n\}$ be any bounded sequence in D, and $\{x_n\}$ be the Mann iterative sequence with errors defined by* (1.2). *Then* (1.2) *converges strongly to the unique fixed point q of T.*

Corollary 2.4 *Let E be an arbitrary uniformly smooth real Banach space, and $T : E \to E$ be a generalized Lipschitz Φ-quasi-accretive mapping with $N(T) \neq \emptyset$. Let $\{a_n\}$, $\{d_n\}$ be two real sequences in $[0,1]$ and satisfy the conditions* (i) *$a_n \to 0$ as $n \to \infty$ and $c_n = o(a_n)$*; (ii) *$\sum_{n=0}^{\infty} a_n = \infty$. For some $x_0 \in D$, let $\{u_n\}$ be any bounded sequence in E, and $\{x_n\}$ be the Mann iterative sequence with errors defined by*

$$x_{n+1} = (1 - a_n - c_n)x_n + a_n S x_n + c_n u_n, \quad n \geq 0, \tag{2.22}$$

where $S : E \to E$ is defined by $Sx = x - Tx$ for any $x \in E$. Then $\{x_n\}$ converges strongly to the unique solution of the equation $Tx = 0$ (*or the unique fixed point of S*).

Remark 2.5 It is mentioned that in 2006, Chidume and Chidume [1] proved the approximative theorem for zeros of generalized Lipschitz generalized Φ-quasi-accretive operators. This result provided significant improvements of some recent important results. Their result is as follows.

Theorem CC ([1, Theorem 3.1]) *Let E be a uniformly smooth real Banach space and $A : E \to E$ be a mapping with $N(A) \neq \emptyset$. Suppose A is a generalized Lipschitz Φ-quasi-accretive mapping. Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be real sequences in $[0,1]$ satisfying the following conditions*: (i) $a_n + b_n + c_n = 1$; (ii) $\sum_{n=0}^{\infty}(b_n + c_n) = \infty$; (iii) $\sum_{n=0}^{\infty} c_n < \infty$; (iv) $\lim_{n\to\infty} b_n = 0$. *Let $\{x_n\}$ be generated iteratively from arbitrary $x_0 \in E$ by*

$$x_{n+1} = a_n x_n + b_n S x_n + c_n u_n, \quad n \geq 0, \tag{2.23}$$

where $S : E \to E$ is defined by $Sx := f + x - Ax$, $\forall x \in E$ and $\{u_n\}$ is an arbitrary bounded sequence in E. Then, there exists $\gamma_0 \in \Re$ such that if $b_n + c_n \leq \gamma_0$, $\forall n \geq 0$, the sequence $\{x_n\}$ converges strongly to the unique solution of the equation $Au = 0$.

However, there exists a gap in the proof process of above Theorem CC. Here, $c_n = \min\{\frac{\epsilon}{4\beta}, \frac{1}{2\sigma}\Phi(\frac{\epsilon}{2})\alpha_n\}$ ($\alpha_n = b_n + c_n$) does not hold in line 14 of Claim 2 on page 248, *i.e.*, $c_n \leq \frac{1}{2\sigma}\Phi(\frac{\epsilon}{2})\alpha_n$ is a wrong case. For instance, set the iteration parameters: $a_n = 1 - b_n - c_n$, where $\{b_n\} : b_1 = \frac{1}{4}$, $b_n = \frac{1}{n}$, $n \geq 2$; $\{c_n\} : \frac{1}{4}, \frac{1}{2^2}, \frac{1}{3^2}, \frac{1}{4}, \frac{1}{5^2}, \ldots, \frac{1}{8^2}, \frac{1}{9}, \frac{1}{10^2}, \ldots, \frac{1}{15^2}, \frac{1}{16}, \frac{1}{17^2}, \ldots, \frac{1}{24^2}, \frac{1}{25}, \frac{1}{26^2}, \ldots, \frac{1}{35^2}, \frac{1}{36}, \frac{1}{37^2}, \ldots$. Then $\sum_{n=0}^{\infty} c_n < +\infty$, but $c_n \neq o(b_n + c_n)$. Therefore, the proof of above Theorem CC is not reasonable. Up to now, we do not know the validity of Theorem CC. This will be an open question left for the readers!

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally in writing this paper. All authors read and approved the final manuscript.

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