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Common fixed point theorems for expansion mappings in various abstract spaces using the concept of weak reciprocal continuity

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Abstract

In this paper, we prove expansion mapping theorems using the concept of compatible maps, weakly reciprocal continuity, R-weakly commuting mappings, R-weakly commuting of type (A_f), (A_g) and (P) in metric spaces and in G-metric spaces.

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1 Metric spaces

In 1922, Banach proved a common fixed point theorem which ensures, under appropriate conditions, the existence and uniqueness of a fixed point. This result of Banach is known as Banach's fixed point theorem or Banach contraction principle. Many authors have extended, generalized and improved Banach's fixed point theorem in different ways.

Jungck [1] proved a common fixed point theorem for commuting maps, which generalized the Banach fixed point theorem. This theorem has had many applications but suffers from one drawback that the continuity of a map throughout the space is needed. Jungck [2] defined the concept of compatible mappings.

Definition 1 ([2], see also [3]) A pair of self-mappings (f,g) of a metric space (X,d) is said to be compatible if $\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = z$ for some z in X.

In 1994, Pant [4] introduced the notion of pointwise R-weak commutativity in metric spaces.

Definition 2 ([4], see also [5]) A pair of self-mappings (f,g) of a metric space (X,d) is said to be R-weakly commuting at a point x in X if $d(fgx, gfx) \le Rd(fx, gx)$ for some $R > 0$.

Definition 3 ([4]) Two self-maps f and g of a metric space (X,d) are called pointwise R-weakly commuting on X if, given x in X, there exists $R > 0$ such that $d(fgx, gfx) \le Rd(fx, gx)$.

In 1997, Pathak *et al.* [6] generalized the notion of R-weakly commuting mappings to R-weakly commuting mappings of type (A_f) and of type (A_g).

Definition 4 ([6]) Two self-maps f and g of a metric space (X,d) are called R-weakly commuting of type (A_g) if there exists some $R > 0$ such that $d(ffx,gfx) \leq Rd(fx,gx)$ for all x in X.

Similarly, two self-mappings f and g of a metric space (X,d) are called R-weakly commuting of type (A_f) if there exists some $R > 0$ such that $d(fgx,ggx) \leq Rd(fx,gx)$ for all x in X.

It is obvious that pointwise R-weakly commuting maps commute at their coincidence points and pointwise R-weak commutativity is equivalent to commutativity at coincidence points. It may be noted that both compatible and non-compatible mappings can be R-weakly commuting of type (A_g) or (A_f) but converse may not be true.

Definition 5 ([6]) Two self-maps f and g of a metric space (X,d) are called R-weakly commuting of type (P) if there exists some $R > 0$ such that $d(ffx,ggx) \leq Rd(fx,gx)$ for all x in X.

In 1999, Pant [7] introduced a new continuity condition, known as reciprocal continuity, and obtained a common fixed point theorem by using the compatibility in metric spaces. He also showed that in the setting of common fixed point theorems for compatible mappings satisfying contraction conditions, the notion of reciprocal continuity is weaker than the continuity of one of the mappings. The notion of pointwise R-weakly commuting mappings increased the scope of the study of common fixed point theorems from the class of compatible to the wider class of pointwise R-weakly commuting mappings. Subsequently, several common fixed point theorems have been proved by combining the ideas of R-weakly commuting mappings and reciprocal continuity of mappings in different settings.

Definition 6 ([7]) Two self-mappings f and g are called reciprocally continuous if $\lim_{n\to\infty} fgx_n = fz$ and $\lim_{n\to\infty} gfx_n = gz$, whenever $\{x_n\}$ is a sequence such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$ for some z in X.

If f and g are both continuous, then they are obviously reciprocally continuous, but the converse is not true.

Recently, Pant *et al.* [8] generalized the notion of reciprocal continuity to weak reciprocal continuity as follows.

Definition 7 ([8]) Two self-mappings f and g are called weakly reciprocally continuous if $\lim_{n\to\infty} fgx_n = fz$ or $\lim_{n\to\infty} gfx_n = gz$ whenever $\{x_n\}$ is a sequence such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$ for some z in X.

If f and g are reciprocally continuous, then they are obviously weak reciprocally continuous, but the converse is not true. Now, as an application of weak reciprocal continuity, we prove common fixed point theorems under contractive conditions that extend the scope of the study of common fixed point theorems from the class of compatible continuous mappings to a wider class of mappings which also includes non-compatible mappings.

Theorem 1 *Let f and g be two weakly reciprocally continuous self-mappings of a complete metric space* (X,d) *satisfying the following conditions*:

$$g(X) \subseteq f(X) \tag{1.1}$$

for any $x, y \in X$ *and* $q > 1$, *we have that*

$$d(fx, fy) \geq q d(gx, gy). \tag{1.2}$$

If f and g are either compatible or R-weakly commuting of type (A_g) *or R-weakly commuting of type* (A_f) *or R-weakly commuting of type* (P), *then f and g have a unique common fixed point.*

Proof Let x_0 be any point in X. Since $g(X) \subseteq f(X)$, there exists a sequence of points $\{x_n\}$ such that $g(x_n) = f(x_{n+1})$.

Define a sequence $\{y_n\}$ in X by

$$y_n = g(x_n) = f(x_{n+1}). \tag{1.3}$$

Now, we will show that $\{y_n\}$ is a Cauchy sequence in X. For proving this, from (1.2), we have

$$d(y_n, y_{n+1}) = d(gx_n, gx_{n+1}) \leq \frac{1}{q} d(fx_n, fx_{n+1}) = \frac{1}{q} d(y_{n-1}, y_n).$$

Hence,

$$d(y_n, y_{n+1}) \leq \frac{1}{q} d(y_{n-1}, y_n) \leq \frac{1}{q^2} d(y_{n-2}, y_{n-1}) \leq \cdots \leq \frac{1}{q^n} d(y_0, y_1).$$

Therefore, for all $n, m \in \mathbb{N}$ (a set of natural numbers), $n < m$, we have

$$\begin{aligned} d(y_n, y_m) &\leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + d(y_{n+2}, y_{n+3}) + \cdots + d(y_{m-1}, y_m) \\ &\leq \big(1/q^n + 1/q^{n+1} + 1/q^{n+2} + \cdots + 1/q^{m-1}\big) d(y_0, y_1) \\ &\leq \big(1/q^n + 1/q^{n+1} + 1/q^{n+2} + \cdots\big) d(y_0, y_1) \\ &= \frac{1}{q^{n-1}(q-1)} d(y_0, y_1) \to 0 \quad \text{as } n \to \infty. \end{aligned}$$

Thus, $\{y_n\}$ is a Cauchy sequence in X. Since X is complete, there exists a point z in X such that $\lim_{n\to\infty} y_n = z$. Therefore, by (1.3), we have $\lim_{n\to\infty} y_n = \lim_{n\to\infty} g(x_n) = \lim_{n\to\infty} f(x_{n+1}) = z$.

Suppose that f and g are compatible mappings. Now, by the weak reciprocal continuity of f and g, we obtain $\lim_{n\to\infty} fg(x_n) = fz$ or $\lim_{n\to\infty} gf(x_n) = gz$. Let $\lim_{n\to\infty} fg(x_n) = fz$, then the compatibility of f and g gives $\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$, that is, $\lim_{n\to\infty} d(gfx_n, fz) = 0$.

Hence, $\lim_{n\to\infty} gf(x_n) = fz$. From (1.3), we get $\lim_{n\to\infty} gf(x_{n+1}) = \lim_{n\to\infty} gg(x_n) = fz$.

Therefore, from (1.2), we get

$$d(gz, ggx_n) \leq \frac{1}{q} d(fz, fgx_n).$$

Taking the limit as $n \to \infty$, we get

$$d(gz, fz) \leq \frac{1}{q} d(fz, fz) = 0.$$

Hence, $fz = gz$. Again, the compatibility of f and g implies the commutativity at a coincidence point. Hence, $gfz = fgz = ffz = ggz$. Using (1.2), we obtain

$$d(gz, ggz) \leq \frac{1}{q} d(fz, fgz) = \frac{1}{q} d(gz, ggz),$$

which proves that $gz = ggz$. We also get $gz = ggz = fgz$ and then gz is a common fixed point of f and g.

Next, suppose that $\lim_{n\to\infty} gf(x_n) = gz$. The assumption $g(X) \subseteq f(X)$ implies that $gz = fu$ for some $u \in X$ and therefore, $\lim_{n\to\infty} gf(x_n) = fu$.

The compatibility of f and g implies that $\lim_{n\to\infty} fg(x_n) = fu$. By virtue of (1.3), we have $\lim_{n\to\infty} gf(x_{n+1}) = \lim_{n\to\infty} gg(x_n) = fu$. Using (1.2), we get

$$d(gu, ggx_n) \leq \frac{1}{q} d(fu, fgx_n).$$

Taking the limit as $n \to \infty$, we get

$$d(gu, fu) \leq \frac{1}{q} d(fu, fu) = 0.$$

Then we get $fu = gu$. The compatibility of f and g yields $fgu = ggu = ffu = gfu$. Finally, using (1.2), we obtain

$$d(gu, ggu) \leq \frac{1}{q} d(fu, fgu) = \frac{1}{q} d(gu, ggu),$$

that is, $gu = ggu$. We also have $gu = ggu = fgu$ and gu is a common fixed point of f and g.

Now, suppose that f and g are R-weakly commuting of type (A_f). Now, the weak reciprocal continuity of f and g implies that $\lim_{n\to\infty} fg(x_n) = fz$ or $\lim_{n\to\infty} gf(x_n) = gz$. Let us first assume that $\lim_{n\to\infty} fg(x_n) = fz$. Then the R-weak commutativity of type (A_f) of f and g yields

$$d(ggx_n, fgx_n) \leq Rd(fx_n, gx_n)$$

and therefore

$$\lim_{n\to\infty} d(ggx_n, fz) \leq Rd(z, z) = 0.$$

This proves that $\lim_{n\to\infty} ggx_n = fz$. Again, using (1.2), we get

$$d(gz, ggx_n) \leq \frac{1}{q} d(fz, fgx_n).$$

Taking the limit as $n \to \infty$, we get

$$d(gz,fz) \leq \frac{1}{q} d(fz,fz) = 0.$$

Hence, we get $fz = gz$. Again, by using the R-weak commutativity of type (A_f), we have

$$d(ggz,fgz) \leq \frac{1}{q} d(gz,fz) = \frac{1}{q} d(fz,fz) = 0.$$

This yields $ggz = fgz$. Therefore, $ffz = fgz = gfz = ggz$. Using (1.2), we get

$$d(gz,ggz) \leq \frac{1}{q} d(fz,fgz) = \frac{1}{q} d(gz,ggz),$$

that is, $gz = ggz$. Then we also get $gz = ggz = fgz$ and gz is a common fixed point of f and g.

Similar proof works in the case where $\lim_{n\to\infty} gf(x_n) = gz$.

Suppose that f and g are R-weakly commuting of type (A_g). Again, as done above, we can easily prove that fz is a common fixed point of f and g.

Finally, suppose that f and g are R-weakly commuting of type (P). The weak reciprocal continuity of f and g implies that $\lim_{n\to\infty} fg(x_n) = fz$ or $\lim_{n\to\infty} gf(x_n) = gz$. Let us assume that $\lim_{n\to\infty} fg(x_n) = fz$. Then the R-weak commutativity of type (P) of f and g yields

$$d(ffx_n, ggx_n) \leq Rd(fx_n, gx_n).$$

Taking the limit as $n \to \infty$, we get

$$\lim_{n\to\infty} d(ffx_n, ggx_n) \leq Rd(z,z) = 0,$$

that is, $\lim_{n\to\infty} d(ffx_n, ggx_n) = 0$.

Using (1.1) and (1.3), we have, $fgx_{n-1} = ffx_n \to fz$ as $n \to \infty$, which gives $ggx_n \to fz$ as $n \to \infty$. Also, using (1.2), we get

$$d(gz, ggx_n) \leq \frac{1}{q} d(fz, fgx_n).$$

Taking the limit as $n \to \infty$, we get

$$d(gz,fz) \leq \frac{1}{q} d(fz,fz) = 0.$$

Hence, $fz = gz$. Again, by using the R-weak commutativity of type (P),

$$d(ffz, ggz) \leq Rd(fz, gz) = 0.$$

This yields $ffz = ggz$.

Therefore, $ffz = fgz = gfz = ggz$. Using (1.2), we get

$$d(gz, ggz) \leq \frac{1}{q} d(fz, fgz) = \frac{1}{q} d(gz, ggz).$$

This proves that $gz = ggz$. Hence, $gz = ggz = fgz$ and gz is a common fixed point of f and g.

Similar proof works in the case where $\lim_{n\to\infty} gf(x_n) = gz$.

Uniqueness of the common fixed point theorem follows easily in each of the four cases by using (1.2). □

2 *G*-metric spaces

In 1963, Gahler [9] introduced the concept of 2-metric spaces and claimed that a 2-metric is a generalization of the usual notion of a metric, but some authors proved that there is no relation between these two functions. It is clear that in 2-metric, $d(x, y, z)$ is to be taken as the area of the triangle with vertices x, y and z in R^2. However, Hsiao [10] showed that for every contractive definition, with $x_n = T^n x_0$, every orbit is linearly dependent, thus rendering fixed point theorems in such spaces trivial.

In 1992, Dhage [11] introduced the concept of a D-metric space. The situation for a D-metric space is quite different from that for 2-metric spaces. Geometrically, a D-metric $D(x, y, z)$ represents the perimeter of the triangle with vertices x, y and z in R^2. Recently, Mustafa and Sims [10] have shown that most of the results concerning Dhage's D-metric spaces are invalid. Therefore, they introduced an improved version of the generalized metric space structure, which they called G-metric spaces.

In 2006, Mustafa and Sims [12] introduced the concept of G-metric spaces as follows.

Definition 8 ([12]) Let X be a nonempty set, and let $G : X \times X \times X \to R^+$ be a function satisfying the following axioms:

(G1) $G(x, y, z) = 0$ if $x = y = z$,
(G2) $0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$,
(G3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $z \neq y$,
(G4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$ (symmetry in all three variables),
(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$ (rectangle inequality).

Then the function G is called a generalized metric or, more specifically, a G-metric on X and the pair (X, G) is called a G-metric space.

Definition 9 ([12]) Let (X, G) be a G-metric space and let $\{x_n\}$ be a sequence of points in X. A point x in X is said to be the limit of the sequence $\{x_n\}$, $\lim_{m,n\to\infty} G(x, x_n, x_m) = 0$, and one says that the sequence $\{x_n\}$ is G-convergent to x.

Thus, $x_n \to x$, $n \to \infty$ or $\lim_{n\to\infty} x_n = x$ in a G-metric space (X, G) if for each $\varepsilon > 0$, there exists a positive integer N such that $G(x, x_n, x_m) < \varepsilon$ for all $m, n \geq N$.

Now, we state some results from the papers [10, 12–15] which are helpful for proving our main results.

Proposition 1 ([12]) *Let* (X, G) *be a G-metric space. Then the following are equivalent*:

(1) $\{x_n\}$ *is G convergent to* x,
(2) $G(x_n, x_n, x) \to 0$ *as* $n \to \infty$,
(3) $G(x_n, x, x) \to 0$ *as* $n \to \infty$,
(4) $G(x_m, x_n, x) \to 0$ *as* $m, n \to \infty$.

Definition 10 ([12]) Let (X, G) be a G-metric space. A sequence $\{x_n\}$ is called G-Cauchy if, for each $\varepsilon > 0$, there exists a positive integer N such that $G(x_n, x_m, x_l) < \varepsilon$ for all $n, m, l \geq N$; *i.e.*, if $G(x_n, x_m, x_l) \to 0$ as $n, m, l \to \infty$.

Proposition 2 ([15]) *If (X, G) is a G-metric space, then the following are equivalent*:

(1) *the sequence $\{x_n\}$ is G-Cauchy,*

(2) *for each $\varepsilon > 0$, there exists a positive integer N such that $G(x_n, x_m, x_m) < \varepsilon$ for all $n, m \geq N$.*

Proposition 3 ([12]) *Let (X, G) be a G-metric space. Then the function $G(x, y, z)$ is jointly continuous in all three of its variables.*

Definition 11 ([12]) A G-metric space (X, G) is called a symmetric G-metric space if

$$G(x, y, y) = G(y, x, x) \quad \text{for all } x, y \text{ in } X.$$

Proposition 4 ([14]) *Every G-metric space (X, G) will define a metric space (X, d_G) by*

(i) $d_G(x, y) = G(x, y, y) + G(y, x, x)$ *for all x, y in X.*

If (X, G) is a symmetric G-metric space, then

(ii) $d_G(x, y) = 2G(x, y, y)$ *for all x, y in X.*

However, if (X, G) is not symmetric, then it follows from the G-metric properties that

(iii) $\frac{3}{2}G(x, y, y) \leq d_G(x, y) \leq 3G(x, y, y)$ *for all x, y in X.*

Definition 12 ([14]) A G-metric space (X, G) is said to be G-complete if every G-Cauchy sequence in (X, G) is G-convergent in X.

Proposition 5 ([14]) *A G-metric space (X, G) is G-complete if and only if (X, d_G) is a complete metric space.*

Proposition 6 ([15]) *Let (X, G) be a G-metric space. Then, for any $x, y, z, a \in X$, it follows that*

(i) *if $G(x, y, z) = 0$, then $x = y = z$,*

(ii) $G(x, y, z) \leq G(x, x, y) + G(x, x, z)$,

(iii) $G(x, y, y) \leq 2G(y, x, x)$,

(iv) $G(x, y, z) \leq G(x, a, z) + G(a, y, z)$,

(v) $G(x, y, z) \leq \frac{2}{3}(G(x, y, a) + G(x, a, z) + G(a, y, z))$,

(vi) $G(x, y, z) \leq (G(x, a, a) + G(y, a, a) + G(z, a, a))$.

Definition 13 ([16]) A pair of self-mappings (f, g) of a G-metric space (X, G) is said to be compatible if $\lim_{n\to\infty} G(fgx_n, gfx_n, gfx_n) = 0$ or $\lim_{n\to\infty} G(gfx_n, fgx_n, fgx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = z$ for some z in X.

Definition 14 ([17]) A pair of self-mappings (f, g) of a G-metric space (X, G) is said to be R-weakly commuting at a point x in X if $G(fgx, gfx, gfx) \leq RG(fx, gx, gx)$ for some $R > 0$.

Definition 15 ([17]) Two self-maps f and g of a G-metric space (X, G) are called pointwise R-weakly commuting on X if, given x in X, there exists $R > 0$ such that $G(fgx, gfx, gfx) \leq RG(fx, gx, gx)$.

Definition 16 ([6]) Two self-maps f and g of a G-metric space (X, G) are called R-weakly commuting of type (A_g) if there exists some $R > 0$ such that $G(ffx, gfx, gfx) \leq RG(fx, gx, gx)$

for all x in X. Similarly, two self-mappings f and g of a G-metric space (X, G) are called R-weakly commuting of type (A_f) if there exists some $R > 0$ such that $G(fgx, ggx, ggx) \leq RG(fx, gx, gx)$ for all x in X.

Definition 17 ([6]) Two self-mappings f and g of a G-metric space (X, G) are called R-weakly commuting of type (P) if there exists some $R > 0$ such that $G(ffx, ggx, ggx) \leq RG(fx, gx, gx)$ for all x in X.

Theorem 2 *Let f and g be two weakly reciprocally continuous self-mappings of a complete G-metric space (X, G) satisfying the following conditions*:

$$g(X) \subseteq f(X) \tag{2.1}$$

for any $x, y, z \in X$ and $q > 1$, we have that

$$qG(gx, gy, gz) \leq G(fx, fy, fz). \tag{2.2}$$

If f and g are either compatible or R-weakly commuting of type (A_g) or R-weakly commuting of type (A_f) or R-weakly commuting of type (P), then f and g have a unique common fixed point.

Proof Let x_0 be any point in X. Since $g(X) \subseteq f(X)$, there exists a sequence of points $\{x_n\}$ such that $g(x_n) = f(x_{n+1})$.

Define a sequence $\{y_n\}$ in X by

$$y_n = g(x_n) = f(x_{n+1}). \tag{2.3}$$

Now, we will show that $\{y_n\}$ is a G-Cauchy sequence in X. For proving this, by (2.2) take $x = x_n$, $y = x_{n+1}$, $z = x_{n+1}$, we have

$$G(gx_n, gx_{n+1}, gx_{n+1}) \leq 1/qG(fx_n, fx_{n+1}, fx_{n+1}) = 1/qG(gx_{n-1}, gx_n, gx_n).$$

Continuing in the same way, we have

$$G(gx_n, gx_{n+1}, gx_{n+1}) \leq 1/q^n G(gx_0, gx_1, gx_1) \quad \Rightarrow \quad G(y_n, y_{n+1}, y_{n+1}) \leq 1/q^n G(y_0, y_1, y_1).$$

Therefore, for all $n, m \in \mathbb{N}$ (a set of natural numbers), $n < m$, we have by using (G5)

$$\begin{aligned} G(y_n, y_m, y_m) &\leq G(y_n, y_{n+1}, y_{n+1}) + G(y_{n+1}, y_{n+2}, y_{n+2}) + G(y_{n+2}, y_{n+3}, y_{n+3}) + \cdots \\ &\quad + G(y_{m-1}, y_m, y_m) \\ &\leq \left(1/q^n + 1/q^{n+1} + 1/q^{n+2} + \cdots + 1/q^{m-1}\right) G(y_0, y_1, y_1) \\ &\leq \left(1/q^n + 1/q^{n+1} + 1/q^{n+2} + \cdots\right) G(y_0, y_1, y_1) \\ &= \frac{1}{q^{n-1}(q-1)} G(y_0, y_1, y_1) \to 0 \quad \text{as } n \to \infty. \end{aligned}$$

Thus, $\{y_n\}$ is a G-Cauchy sequence in X. Since (X, G) is a complete G-metric space, there exists a point z in X such that $\lim_{n\to\infty} y_n = z$. Therefore, by (2.3), we have $\lim_{n\to\infty} y_n = \lim_{n\to\infty} g(x_n) = \lim_{n\to\infty} f(x_{n+1}) = z$.

Suppose that f and g are compatible mappings. Now, the weak reciprocal continuity of f and g implies that $\lim_{n\to\infty} fg(x_n) = fz$ or $\lim_{n\to\infty} gf(x_n) = gz$. Let $\lim_{n\to\infty} fg(x_n) = fz$, then the compatibility of f and g gives $\lim_{n\to\infty} G(gfx_n, fgx_n, fgx_n) = 0$, that is, $G(\lim_{n\to\infty} gfx_n, fz, fz) = 0$.

Hence, $\lim_{n\to\infty} gf(x_n) = fz$. From (2.3), we get $\lim_{n\to\infty} gf(x_{n+1}) = \lim_{n\to\infty} gg(x_n) = fz$.

Therefore, by (2.2), we get

$$G(gz, ggx_n, ggx_n) \leq \frac{1}{q} G(fz, fgx_n, fgx_n).$$

Taking the limit as $n \to \infty$, we get

$$G(gz, fz, fz) \leq \frac{1}{q} G(fz, fz, fz) = 0.$$

Hence, $fz = gz$. Again, the compatibility of f and g implies the commutativity at a coincidence point. Hence, $gfz = fgz = ffz = ggz$. Now, we claim that $gz = ggz$. Suppose not, then by using (2.2), we obtain

$$G(gz, ggz, ggz) \leq \frac{1}{q} G(fz, fgz, fgz)$$

and

$$G(gz, ggz, ggz) \leq \frac{1}{q} G(gz, ggz, ggz),$$

which gives contradiction because $q > 1$. Hence, $gz = ggz$. Hence, $gz = ggz = fgz$ and gz is a common fixed point of f and g.

Next suppose that $\lim_{n\to\infty} gf(x_n) = gz$. The assumption $g(X) \subseteq f(X)$ implies that $gz = fu$ for some $u \in X$ and therefore, $\lim_{n\to\infty} gf(x_n) = fu$.

The compatibility of f and g implies that $\lim_{n\to\infty} fg(x_n) = fu$. By virtue of (2.3), this gives $\lim_{n\to\infty} gf(x_{n+1}) = \lim_{n\to\infty} gg(x_n) = fu$. Using (2.2), we get

$$G(gu, ggx_n, ggx_n) \leq \frac{1}{q} G(fu, fgx_n, fgx_n).$$

Taking the limit as $n \to \infty$, we get

$$G(gu, fu, fu) \leq \frac{1}{q} G(fu, fu, fu) = 0,$$

which gives $fu = gu$. The compatibility of f and g yields $fgu = ggu = ffu = gfu$. Finally, we claim that $gu = ggu$. Suppose not, then by using (2.2), we obtain

$$G(gu, ggu, ggu) \leq \frac{1}{q} G(fu, fgu, fgu)$$

and

$$G(gu, ggu, ggu) \leq \frac{1}{q} G(gu, ggu, ggu),$$

that is, $gu = ggu$. Hence, $gu = ggu = fgu$ and gu is a common fixed point of f and g.

Now, suppose that f and g are R-weakly commuting of type (A_f). Now, the weak reciprocal continuity of f and g implies that $\lim_{n\to\infty} fg(x_n) = fz$ or $\lim_{n\to\infty} gf(x_n) = gz$. Let us first assume that $\lim_{n\to\infty} fg(x_n) = fz$. Then the R-weak commutativity of type (A_f) of f and g yields

$$G(ggx_n, fgx_n, fgx_n) \leq RG(gx_n, fx_n, fx_n).$$

Taking the limit as $n \to \infty$, we get

$$G\Big(\lim_{n\to\infty} ggx_n, gz, gz\Big) \leq RG(z, z, z) = 0.$$

This proves that $\lim_{n\to\infty} ggx_n = fz$. Again, using (2.2), we get

$$G(gz, ggx_n, ggx_n) \leq \frac{1}{q} G(fz, fgx_n, fgx_n).$$

Taking the limit as $n \to \infty$, we get

$$G(gz, fz, fz) \leq \frac{1}{q} G(fz, fz, fz) = 0.$$

Hence, we get $fz = gz$. Again, by using the R-weak commutativity of type (A_f),

$$G(ggz, fgz, fgz) \leq G(gz, fz, fz) = 0.$$

This yields $ggz = fgz$. Therefore, $ffz = fgz = gfz = ggz$. We claim that $gz = ggz$. Suppose not, using (2.2), we get

$$G(gz, ggz, ggz) \leq \frac{1}{q} G(fz, fgz, fgz)$$

and

$$G(gz, ggz, ggz) \leq \frac{1}{q} G(gz, ggz, ggz),$$

a contradiction, that is, $gz = ggz$. Hence, $gz = ggz = fgz$ and gz is a common fixed point of f and g.

Similar proof works in the case where $\lim_{n\to\infty} gf(x_n) = gz$.

Suppose that f and g are R-weakly commuting of type (A_g). Again, as done above, we can easily prove that gz is a common fixed point of f and g.

Finally, suppose f and g are R-weakly commuting of type (P). The weak reciprocal continuity of f and g implies that $\lim_{n\to\infty} fg(x_n) = fz$ or $\lim_{n\to\infty} gf(x_n) = gz$. Let us assume that

$\lim_{n\to\infty} fg(x_n) = fz$. Then the R-weak commutativity of type (P) of f and g yields

$$G(ffx_n, ggx_n, ggx_n) \le RG(fx_n, gx_n, gx_n).$$

Taking the limit as $n \to \infty$, we get

$$G\Bigl(\lim_{n\to\infty} ffx_n, \lim_{n\to\infty} ggx_n, \lim_{n\to\infty} ggx_n\Bigr) \le RG(z,z,z) = 0.$$

That is,

$$\lim_{n\to\infty} G\Bigl(\lim_{n\to\infty} ffx_n, \lim_{n\to\infty} ggx_n, \lim_{n\to\infty} ggx_n\Bigr) = 0.$$

Using (2.1) and (2.3), we have $fgx_{n-1} = ffx_n \to fz$ as $n \to \infty$, which gives $ggx_n \to fz$ as $n \to \infty$. Also, using (2.2), we get

$$G(gz, ggx_n, ggx_n) \le \frac{1}{q} G(fz, fgx_n, fgx_n).$$

Taking the limit as $n \to \infty$, we get

$$G(gz, fz, fz) \le \frac{1}{q} G(fz, fz, fz) = 0.$$

Hence, $fz = gz$. Again, by using the R-weak commutativity of type (P),

$$G(ffz, ggz, ggz) \le RG(fz, gz, gz) = 0.$$

This yields $ffz = ggz$.

Therefore, $ffz = fgz = gfz = ggz$. Finally, we claim that $gz = ggz$. Suppose not, using (2.2), we get

$$G(gz, ggz, ggz) \le \frac{1}{q} G(fz, fgz, fgz)$$

and

$$G(gz, ggz, ggz) \le \frac{1}{q} G(gz, ggz, ggz),$$

a contradiction, we get $gz = ggz$. Hence, $gz = ggz = fgz$ and gz is a common fixed point of f and g.

Similar proof works in the case where $\lim_{n\to\infty} gf(x_n) = gz$.

Uniqueness of the common fixed point theorem follows easily in each of the four cases by using (2.2). □

We now give an example (see also [18]) to illustrate Theorem 2.

Example 1 Let (X, G) be a G-metric space, where $X = [2, 20]$ and

$$G(x, y, z) = \bigl(|x - y| + |y - z| + |z - x|\bigr),$$

for all $x, y, z \in X$. Define $f, g : X \to X$ by

$$fx = 2 \quad \text{if } x = 2 \text{ or } x > 5,$$
$$fx = 6 \quad \text{if } 2 < x \leq 5,$$
$$g2 = 2, \qquad gx = 11 \quad \text{if } 2 < x \leq 5,$$
$$gx = (x+1)/3 \quad \text{if } x > 5.$$

Let $\{x_n\}$ be a sequence in X such that either $x_n = 2$ or $x_n = 5 + 1/n$ for each n.

Then, clearly, f and g satisfy all the conditions of Theorem 2 and have a unique common fixed point at $x = 2$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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