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Fixed point theorems for a semigroup of generalized asymptotically nonexpansive mappings in CAT(0) spaces

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Abstract
In this paper, we prove the existence of common fixed points for a generalized asymptotically nonexpansive semigroup $\{T_s : s \in S\}$ in CAT(0) spaces, when S is a left reversible semitopological semigroup. We also prove Δ- and strong convergence of such a semigroup when S is a right reversible semitopological semigroup. Our results improve and extend the corresponding results existing in the literature.

MSC: 47H09; 47H10

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1 Introduction

Let S be a semitopological semigroup, *i.e.*, S is a semigroup with a Hausdorff topology such that for each $s \in S$, the mappings $s \mapsto ts$ and $s \mapsto st$ from S to S are continuous, and let $BC(S)$ be the Banach space of all bounded continuous real-valued functions with supremum norm. For $f \in BC(S)$ and $c \in \mathbb{R}$, we write $f(s) \to c$ as $s \to \infty_{\mathbb{R}}$ if for each $\varepsilon > 0$, there exists $w \in S$ such that $|f(tw) - c| < \varepsilon$ for all $t \in S$; see [1].

A semitopological semigroup S is said to be *left* (*resp. right*) *reversible* if any two closed right (resp. left) ideals of S have nonvoid intersection. If S is left reversible, $(S, \succeq)$ is a directed system when the binary relation '$\succeq$' on S is defined by $t \succeq s$ if and only if $\{t\} \cup \overline{tS} \subseteq \{s\} \cup \overline{sS}$, for $t, s \in S$. Similarly, we can define the binary relation '$\succeq$' on a right reversible semitopological semigroup S. Left reversible semitopological semigroups include all commutative semigroups and all semitopological semigroups which are left amenable as discrete semigroups; see [2]. S is called *reversible* if it is both left and right reversible.

In 1969, Takahashi [3] proved the first fixed point theorem for a noncommutative semigroup of nonexpansive mappings which generalizes De Marr's fixed point theorem [4]. He proved that any discrete left amenable semigroup has a common fixed point. In 1970, Mitchell [5] generalized Takahashi's result by showing that any discrete left reversible semigroup has a common fixed point. In 1981, Takahashi [6] proved a nonlinear ergodic theorem for an amenable semigroup of nonexpansive mappings in a Hilbert space. In 1987, Lau and Takahashi [7] considered the problem of weak convergence of a nonexpansive semigroup of a right reversible semitopological semigroup in a uniformly convex Banach space with Fréchet differentiable norm. After that Lau [8–12] proved the existence of common fixed points for nonexpansive maps related to reversibility or amenability of a semigroup. Takahashi and Zhang [13, 14] established the weak convergence of an almost-orbit of Lipschitzian semigroups of a noncommutative semitopological semigroup. Kim and Kim [15] proved weak convergence for semigroups of asymptotically nonexpansive type of a right reversible semitopological semigroup and strong convergence for a commutative case. In [16], Kakavandi and Amini proved a nonlinear ergodic theorem for a nonexpansive semigroup in CAT(0) spaces as well as a strong convergence theorem for a commutative semitopological semigroup. In 2011, Anakkanmatee and Dhompongsa [17] extended Rodé's theorem [18] on common fixed points of semigroups of nonexpansive mappings in Hilbert spaces to the CAT(0) space setting. For works related to semigroups of nonexpansive, asymptotically nonexpansive, and asymptotically nonexpansive type related to reversibility of a semigroup, we refer the reader to [19–26].

In this paper, we introduce a new semigroup for a left (or right) reversible semitopological semigroup on metric spaces, *called a generalized asymptotically nonexpansive semigroup*, and prove the existence and convergence theorems for this semigroup in CAT(0) spaces.

2 Preliminaries

Let S be a semitopological semigroup and C be a nonempty closed subset of a metric space (X,d). A family $\mathfrak{T} = \{T_s : s \in S\}$ of mappings of C into itself is said to be a *semigroup* if it satisfies the following:

(S1) $T_{st}x = T_sT_tx$ for all $s,t \in S$ and $x \in C$;

(S2) for every $x \in C$, the mapping $s \mapsto T_sx$ from S into C is continuous.

We denote by $F(\mathfrak{T})$ the set of common fixed points of $\mathfrak{T}$, *i.e.*,

$$F(\mathfrak{T}) = \bigcap_{s\in S} F(T_s) = \bigcap_{s\in S} \{x \in C : T_sx = x\}.$$

Remark 2.1 If $\mathfrak{T} = \{T_s : s \in S\}$ is a semigroup of continuous mappings of C into itself and $d(T_sx, y) \to 0$ as $s \to \infty_{\mathbb{R}}$ for $x, y \in C$, then $y \in F(\mathfrak{T})$.

Proof Let $\varepsilon > 0$ be given. Fix $t \in S$. By the continuity of T_t at y, there exists $\delta > 0$ such that $d(x,y) < \delta$ implies $d(T_tx, T_ty) < \frac{\varepsilon}{2}$ for $x \in C$. Since $d(T_sx, y) \to 0$ as $s \to \infty_{\mathbb{R}}$, there exists $w \in S$ such that $d(T_{aw}x, y) < \min\{\frac{\varepsilon}{2}, \delta\}$ for each $a \in S$. Then $d(T_tT_{aw}x, T_ty) < \frac{\varepsilon}{2}$. Therefore, we have

$$\begin{aligned} d(T_ty, y) &\le d(T_ty, T_{taw}x) + d(T_{taw}x, y) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Since ε is arbitrary, we get $T_ty = y$ for each $t \in S$, so $y \in F(\mathfrak{T})$. □

Let S be a left (or right) reversible semitopological semigroup. A semigroup $\mathfrak{T} = \{T_s : s \in S\}$ of mappings of C into itself is said to be

(i) *nonexpansive* if $d(T_sx, T_sy) \le d(x,y)$ for all $x, y \in C$ and $s \in S$;

(ii) *asymptotically nonexpansive* if there exists a nonnegative real number $k_s \ge 0$ with $\lim_s k_s = 0$ such that $d(T_sx, T_sy) \le (1 + k_s)d(x,y)$ for each $x, y \in C$ and $s \in S$.

(iii) *generalized asymptotically nonexpansive* if each T_s is continuous and there exist nonnegative real numbers $k_s, \mu_s \geq 0$ with $\lim_s k_s = 0$ and $\lim_s \mu_s = 0$ such that

$$d(T_s x, T_s y) \leq (1 + k_s) d(x, y) + \mu_s \quad \text{for each } x, y \in C \text{ and } s \in S.$$

Remark 2.2 If $\mu_s = 0$ for all $s \in S$, a generalized asymptotically nonexpansive semigroup reduces to an asymptotically nonexpansive semigroup. If $k_s = 0$ and $\mu_s = 0$ for all $s \in S$, a generalized asymptotically nonexpansive semigroup reduces to a nonexpansive semigroup.

We recall a CAT(0) space; see more details in [27]. Let (X, d) be a metric space. A *geodesic path* joining $x \in X$ to $y \in X$ (or, more briefly, a *geodesic* from x to y) is a map c from a closed interval $[0, l] \subset \mathbb{R}$ to X such that $c(0) = x$, $c(l) = y$ and $d(c(t_1), c(t_2)) = |t_1 - t_2|$ for all $t_1, t_2 \in [0, l]$. In particular, c is an isometry and $d(x, y) = l$. The image α of c is called a *geodesic* (or *metric*) *segment* joining x and y. When unique, this geodesic is denoted $[x, y]$. The space (X, d) is said to be a *geodesic metric space* if every two points of X are joined by a geodesic, and X is said to be *uniquely geodesic* if there is exactly one geodesic joining x and y for each $x, y \in X$. A subset C of X is said to be *convex* if C includes every geodesic segment joining any two of its points.

A *geodesic triangle* $\triangle(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three points x_1, x_2, x_3 in X (the vertices of $\triangle$) and a geodesic segment between each pair of vertices (the edges of $\triangle$). A *comparison triangle* for the geodesic triangle $\triangle(x_1, x_2, x_3)$ in (X, d) is a triangle $\overline{\triangle}(x_1, x_2, x_3) := \triangle(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ in the Euclidean plane $\mathbb{E}^2$ such that $d_{\mathbb{E}^2}(\bar{x}_i, \bar{x}_j) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$.

A geodesic metric space is said to be a CAT(0) space if all geodesic triangles satisfy the following comparison axiom: Let $\triangle$ be a geodesic triangle in X and let $\overline{\triangle}$ be a comparison triangle for $\triangle$. Then $\triangle$ is said to satisfy the CAT(0) inequality if for all $x, y \in \triangle$ and all comparison points $\bar{x}, \bar{y} \in \overline{\triangle}$, $d(x, y) \leq d_{\mathbb{E}^2}(\bar{x}, \bar{y})$.

If z, x, y are points in a CAT(0) space and if m is the midpoint of the segment $[x, y]$, then the CAT(0) inequality implies

$$d(z, m)^2 \leq \frac{1}{2} d(z, x)^2 + \frac{1}{2} d(z, y)^2 - \frac{1}{4} d(x, y)^2. \tag{CN}$$

This is the (CN) inequality of Bruhat and Tits [28]. By using the (CN) inequality, it is easy to see the CAT(0) spaces are uniformly convex. In fact [27], a geodesic metric space is a CAT(0) space if and only if it satisfies the (CN) inequality. Moreover, for each $x, y \in X$ and $\lambda \in [0, 1]$, there exists a unique point $\lambda x \oplus (1 - \lambda) y \in [x, y]$ such that $d(x, \lambda x \oplus (1 - \lambda) y) = (1 - \lambda) d(x, y)$, $d(y, \lambda x \oplus (1 - \lambda) y) = \lambda d(x, y)$ and the following inequality holds:

$$d\bigl(z, \lambda x \oplus (1 - \lambda) y\bigr) \leq \lambda d(z, x) + (1 - \lambda) d(z, y) \quad \text{for each } z \in X.$$

For any nonempty subset C of a CAT(0) space X, let $\pi := \pi_D$ be the nearest point projection mapping from C to a subset D of C. In [27], it is known that if D is closed and convex, the mapping π is well defined, nonexpansive, and the following inequality holds:

$$d(x, y)^2 \geq d(x, \pi x)^2 + d(\pi x, y)^2 \quad \text{for all } x \in C \text{ and } y \in D.$$

Let $\{x_\alpha\}$ be a bounded net in a nonempty closed convex subset C of a CAT(0) space X. For $x \in X$, we set

$$r(x, \{x_\alpha\}) = \limsup_\alpha d(x, x_\alpha).$$

The *asymptotic radius* of $\{x_\alpha\}$ on C is given by

$$r(C, \{x_\alpha\}) = \inf_{x\in C} r(x, \{x_\alpha\}),$$

and the *asymptotic center* of $\{x_\alpha\}$ on C is given by

$$A(C, \{x_\alpha\}) = \{x \in C : r(x, \{x_\alpha\}) = r(C, \{x_\alpha\})\}.$$

It is known that a CAT(0) space X, $A(C, \{x_\alpha\})$ consists of exactly one point; see [29].

In 1976, Lim [30] introduced the concept of Δ-convergence in a general metric space. Later, Kirk and Panyanak [31] extended the concept of Lim to a CAT(0) space.

Definition 2.3 ([31]) A net $\{x_\alpha\}$ in a CAT(0) space X is said to Δ-*converge* to $x \in X$ if x is the unique asymptotic center of $\{u_\alpha\}$ for every subnet $\{u_\alpha\}$ of $\{x_\alpha\}$. In this case, we write $\Delta\text{-}\lim_\alpha x_\alpha = x$ and call x the Δ-*limit* of $\{x_\alpha\}$.

Lemma 2.4 ([31]) *Every bounded net in a complete CAT(0) space X has a Δ-convergent subnet.*

3 Existence theorems

In this section, we study the existence theorems for a generalized asymptotically nonexpansive semigroup in a complete CAT(0) space.

Theorem 3.1 *Let S be a left reversible semitopological semigroup, C be a nonempty closed convex subset of a complete CAT(0) space X, and $\mathfrak{T} = \{T_s : s \in S\}$ be a generalized asymptotically nonexpansive semigroup of C into itself. If $\{T_s x : s \in S\}$ is bounded for some $x \in C$ and $z \in A(C, \{T_s x\})$, then $z \in F(\mathfrak{T})$.*

Proof Let $\{T_s x : s \in S\}$ be a bounded net and let $z \in A(C, \{T_s x\})$. Then

$$R := r(z, \{T_s x\}) = r(C, \{T_s x\}) = \inf_{y\in C} r(y, \{T_s x\}).$$

If $R = 0$, then $\limsup_s d(z, T_s x) = 0$. This implies $T_s x \to z$. It is obvious by Remark 2.1 that $z \in F(\mathfrak{T})$. Next, we assume $R > 0$. Suppose that $z \notin F(\mathfrak{T})$. By Remark 2.1, $\{T_s z\}$ does not converge to z. Then there exists $\varepsilon > 0$ and a subnet $\{s_\alpha\}$ in S such that

$$s_\alpha \succeq \alpha \quad \text{and} \quad d(z, T_{s_\alpha} z) > \varepsilon \quad \text{for each } \alpha \in S. \tag{3.1}$$

We choose a positive number η such that

$$(R + \eta)^2 - \frac{\varepsilon^2}{4} < (R - \eta)^2.$$

Since $\mathfrak{T}$ is a generalized asymptotically nonexpansive semigroup, there exists $s_0 \in S$ such that

$$\begin{aligned} d(T_s z, T_s y) &\leq \limsup_a d(T_a z, T_a y) + \frac{\eta}{2} \\ &\leq \limsup_a \big((1 + k_a) d(z, y) + \mu_a\big) + \frac{\eta}{2} \\ &= d(z, y) + \frac{\eta}{2} \end{aligned} \tag{3.2}$$

for each $s \in S$ with $s \succeq s_0$, and $y \in C$.

It is known by [1] that $\inf_t \sup_s d(z, T_{ts}x) = \limsup_u d(z, T_u x)$. Then $\inf_t \sup_s d(z, T_{ts}x) = R$. So, there exists $t_0 \in S$ such that for all $t \in S$ with $t \succeq t_0$,

$$d(z, T_{ts}x) < R + \frac{\eta}{2} \quad \text{for each } s \in S. \tag{3.3}$$

Since S is left reversible, there exists $\gamma \in S$ with $\gamma \succeq s_0$ and $\gamma \succeq t_0$. Then, by (3.1), $s_\gamma \succeq \gamma$ and

$$d(z, T_{s_\gamma} z) > \varepsilon. \tag{3.4}$$

Let $t \succeq s_\gamma \gamma$. Since S is left reversible, we have $t \in \{s_\gamma \gamma\} \cup \overline{s_\gamma \gamma S}$. Then we may assume $t \in \overline{s_\gamma \gamma S}$. So, there exists $\{t_\beta\}$ in S such that $s_\gamma \gamma t_\beta \to t$. It follows by (3.2) and (3.3) that

$$d(T_{s_\gamma} z, T_{s_\gamma} T_{\gamma t_\beta} x) \leq d(z, T_{\gamma t_\beta} x) + \frac{\eta}{2} \leq R + \eta \quad \text{for each } \beta.$$

By (3.3) and $s_\gamma \gamma t_\beta \to t$, we have

$$d(T_{s_\gamma} z, T_t x) \leq R + \frac{\eta}{2} \quad \text{for all } t \succeq s_\gamma \gamma. \tag{3.5}$$

It follows by (3.3) that

$$d(z, T_{s_\gamma} T_{\gamma t_\beta} x) < R + \frac{\eta}{2} \quad \text{for each } \beta.$$

By $s_\gamma \gamma t_\beta \to t$, we have

$$d(z, T_t x) \leq R + \frac{\eta}{2} < R + \eta \quad \text{for all } t \succeq s_\gamma \gamma. \tag{3.6}$$

So, by the (CN) inequality, (3.4), (3.5), and (3.6), we have

$$\begin{aligned} d^2\left(\frac{z \oplus T_{s_\gamma} z}{2}, T_t x\right) &\leq \frac{1}{2} d^2(z, T_t x) + \frac{1}{2} d^2(T_{s_\gamma} z, T_t x) - \frac{1}{4} d^2(z, T_{s_\gamma} z) \\ &< (R + \eta)^2 - \frac{1}{4}\varepsilon^2 \\ &< (R - \eta)^2. \end{aligned}$$

Thus, $d(\frac{z \oplus T_{s_\gamma} z}{2}, T_t x) < R - \eta$. This implies that

$$r\left(\frac{z \oplus T_{s_\gamma} z}{2}, \{T_t x\}\right) < r\big(C, \{T_t x\}\big),$$

which is a contradiction. Hence, $z \in F(\mathfrak{T})$. □

Theorem 3.2 *Let S be a left reversible semitopological semigroup, C be a nonempty closed convex subset of a complete $CAT(0)$ space X, and $\mathfrak{T} = \{T_s : s \in S\}$ be a generalized asymptotically nonexpansive semigroup of C into itself. Then $F(\mathfrak{T}) \neq \emptyset$ if and only if $\{T_s x : s \in S\}$ is bounded for some $x \in C$.*

Proof Necessity is obvious. Conversely, assume that $x \in C$ such that $\{T_s x : s \in S\}$ is bounded. Then there exists a unique element $z \in C$ such that $z \in A(C, \{T_s x\})$. It follows by Theorem 3.1 that $F(\mathfrak{T}) \neq \emptyset$. □

Theorem 3.3 *Let S be a left or right reversible semitopological semigroup, C be a nonempty closed convex subset of a complete $CAT(0)$ space X, and $\mathfrak{T} = \{T_s : s \in S\}$ be a generalized asymptotically nonexpansive semigroup of C into itself with $F(\mathfrak{T}) \neq \emptyset$. Then $F(\mathfrak{T})$ is a closed convex subset of C.*

Proof First, we show that $F(\mathfrak{T})$ is closed. Let $\{x_t\}$ be a net in $F(\mathfrak{T})$ such that $x_t \to x$. By the definition of T_t, we have

$$\begin{aligned} d(T_t x, x) &\leq d(T_t x, x_t) + d(x, x_t) \\ &\leq (2 + k_t) d(x, x_t) + \mu_t \to 0. \end{aligned}$$

Thus, $T_t x \to x$. This implies $x \in F(\mathfrak{T})$, and so $F(\mathfrak{T})$ is closed.

Next, we show $F(\mathfrak{T})$ is convex. Let $x, y \in F(\mathfrak{T})$ and $z = \frac{x \oplus y}{2}$. For $t \in S$, we have

$$d(T_t z, x) \leq (1 + k_t) d(z, x) + \mu_t = \frac{1 + k_t}{2} d(x, y) + \mu_t$$

and

$$d(T_t z, y) \leq (1 + k_t) d(z, y) + \mu_t = \frac{1 + k_t}{2} d(x, y) + \mu_t.$$

Thus, by the (CN) inequality, we have

$$\begin{aligned} d^2(T_t z, z) &\leq \frac{1}{2} d^2(T_t z, x) + \frac{1}{2} d^2(T_t z, y) - \frac{1}{4} d^2(x, y) \\ &\leq \left(\frac{1 + k_t}{2} d(x, y) + \mu_t \right)^2 - \frac{1}{4} d^2(x, y) \to 0. \end{aligned}$$

Therefore, $T_t z \to z$. This implies $z \in F(\mathfrak{T})$. Hence, $F(\mathfrak{T})$ is convex. □

Taking $S = \mathbb{N}$ in Theorems 3.2 and 3.3, we obtain the following existence theorem of a generalized asymptotically nonexpansive mapping in CAT(0) spaces.

Theorem 3.4 *Let C be a nonempty closed convex subset of a complete $CAT(0)$ space X and $T : C \to C$ be a continuous generalized asymptotically nonexpansive mapping. Then $F(T) \neq \emptyset$ if and only if $\{T^n x : n \in \mathbb{N}\}$ is bounded for some $x \in C$. Moreover, $F(T)$ is closed and convex.*

4 Δ- and strong convergence theorems

In this section, we study the Δ-convergence and strong convergence theorems for a generalized asymptotically nonexpansive semigroup in a CAT(0) space.

Lemma 4.1 *Let S be a right reversible semitopological semigroup, C be a nonempty closed convex subset of a complete CAT(0) space X, and $\mathfrak{T} = \{T_s : s \in S\}$ be a generalized asymptotically nonexpansive semigroup of C into itself with $F(\mathfrak{T}) \neq \emptyset$. Then $\lim_s d(T_s x, z)$ exists for each $z \in F(\mathfrak{T})$.*

Proof Let $z \in F(\mathfrak{T})$ and $R = \inf_s d(T_s x, z)$. For $\varepsilon > 0$, there is $s_0 \in S$ such that

$$d(T_{s_0} x, z) < R + \frac{\varepsilon}{2}.$$

Since $\mathfrak{T}$ is a generalized asymptotically nonexpansive semigroup, there exists $t_0 \in S$ such that

$$\begin{aligned} d(T_t T_{s_0} x, z) &\leq \limsup_u d(T_u T_{s_0} x, z) + \frac{\varepsilon}{2} \\ &\leq d(T_{s_0} x, z) + \frac{\varepsilon}{2} \end{aligned}$$

for each $t \succeq t_0$. Let $b \succeq t_0 s_0$. Since S is right reversible, we have $b \in \{t_0 s_0\} \cup \overline{S t_0 s_0}$. Then we may assume $b \in \overline{S t_0 s_0}$. So, there exists $\{s_\alpha\}$ in S such that $s_\alpha t_0 s_0 \to b$. Therefore,

$$d(T_{s_\alpha t_0 s_0} x, z) \leq d(T_{s_0} x, z) + \frac{\varepsilon}{2} \quad \text{for each } \alpha.$$

Hence, $d(T_b x, z) \leq d(T_{s_0} x, z) + \frac{\varepsilon}{2}$. This implies that

$$R \leq \inf_s \sup_{t \succeq s} d(T_t x, z) \leq \sup_{b \succeq t_0 s_0} d(T_b x, z) \leq d(T_{s_0} x, z) + \frac{\varepsilon}{2} < R + \varepsilon.$$

Since ε is arbitrary, we get

$$\inf_s \sup_{t \succeq s} d(T_t x, z) = R = \inf_s d(T_s x, z).$$

Thus, $\lim_s d(T_s x, z)$ exists. □

Theorem 4.2 *Let S be a right reversible semitopological semigroup, C be a nonempty closed convex subset of a complete CAT(0) space X, and $x \in C$. Assume that $\mathfrak{T} = \{T_s : s \in S\}$ is a generalized asymptotically nonexpansive semigroup of C into itself with $F(\mathfrak{T}) \neq \emptyset$. If $\lim_s d(T_s x, T_{ts} x) = 0$ for all $t \in S$, then $\{T_s x : s \in S\}$ Δ-converges to a common fixed point of the semigroup $\mathfrak{T}$.*

Proof By Lemma 4.1, we have $\lim_s d(T_s x, z)$ exists for each $z \in F(\mathfrak{T})$, and so $\{T_s x : s \in S\}$ is bounded. We now let $\omega_\Delta(T_s x) := \bigcup A(C, \{T_{s_\alpha} x\})$, where the union is taken over all subnets $\{T_{s_\alpha} x\}$ of $\{T_s x\}$. We claim that $\omega_\Delta(T_s x) \subset F(\mathfrak{T})$. Let $u \in \omega_\Delta(T_s x)$. Then there exists a subnet $\{T_{s_\alpha} x\}$ of $\{T_s x\}$ such that $A(C, \{T_{s_\alpha} x\}) = \{u\}$. By Lemma 2.4, there exists a subnet $\{T_{s_{\alpha_\beta}} x\}$

of $\{T_{s_\alpha}x\}$ such that $\Delta\text{-}\lim_\beta T_{s_{\alpha_\beta}}x = y \in C$. We will show that $y \in F(\mathfrak{T})$. Let $\varepsilon > 0$. Since $\mathfrak{T}$ is a generalized asymptotically nonexpansive semigroup, there exists $t_0 \in S$ such that

$$\begin{aligned} d(T_t T_{s_{\alpha_\beta}}x, T_t y) &\le \limsup_a d(T_a T_{s_{\alpha_\beta}}x, T_a y) + \varepsilon \\ &\le d(T_{s_{\alpha_\beta}}x, y) + \varepsilon, \end{aligned}$$

for each $t \succeq t_0$ and each β. It follows that

$$\begin{aligned} d(T_{s_{\alpha_\beta}}x, T_t y) &\le d(T_{s_{\alpha_\beta}}x, T_{ts_{\alpha_\beta}}x) + d(T_{ts_{\alpha_\beta}}x, T_t y) \\ &\le d(T_{s_{\alpha_\beta}}x, T_{ts_{\alpha_\beta}}x) + d(T_{s_{\alpha_\beta}}x, y) + \varepsilon \end{aligned}$$

for each $t \succeq t_0$ and each β. By $\lim_s d(T_s x, T_{ts}x) = 0$ for all $t \in S$, we have

$$\limsup_\beta d(T_{s_{\alpha_\beta}}x, T_t y) \le \limsup_\beta d(T_{s_{\alpha_\beta}}x, y) + \varepsilon$$

for all $t \succeq t_0$. Since ε is arbitrary, we get

$$\limsup_\beta d(T_{s_{\alpha_\beta}}x, T_t y) \le \limsup_\beta d(T_{s_{\alpha_\beta}}x, y)$$

for all $t \succeq t_0$. Since $\{T_{s_{\alpha_\beta}}x\}$ Δ-converges to y, it follows by the uniqueness of asymptotic centers that $T_t y = y$ for all $t \succeq t_0$. So, $d(T_t y, y) \to 0$. This implies $y \in F(\mathfrak{T})$. By Lemma 4.1, $\lim_s d(T_s x, y)$ exists. Suppose that $u \neq y$. By the uniqueness of asymptotic centers,

$$\begin{aligned} \limsup_\beta d(T_{s_{\alpha_\beta}}x, y) &< \limsup_\beta d(T_{s_{\alpha_\beta}}x, u) \\ &\le \limsup_\alpha d(T_{s_\alpha}x, u) \\ &< \limsup_\alpha d(T_{s_\alpha}x, y) \\ &= \limsup_s d(T_s x, y) \\ &= \limsup_\beta d(T_{s_{\alpha_\beta}}x, y). \end{aligned}$$

This is a contradiction, hence $u = y \in F(\mathfrak{T})$. This shows that $\omega_\Delta(T_s x) \subset F(\mathfrak{T})$.

Next, we show that $\omega_\Delta(T_s x)$ consists of exactly one point. Let $\{T_{s_\alpha}x\}$ be a subnet of $\{T_s x\}$ with $A(C, \{T_{s_\alpha}x\}) = \{u\}$ and let $A(C, \{T_s\}) = \{z\}$. Since $u \in \omega_\Delta(T_s x) \subset F(\mathfrak{T})$, it follows by Lemma 4.1 that $\lim_s d(T_s x, u)$ exists. We can complete the proof by showing that $z = u$. To show this, suppose not. By the uniqueness of asymptotic centers,

$$\begin{aligned} \limsup_\alpha d(T_{s_\alpha}x, u) &< \limsup_\alpha d(T_{s_\alpha}x, z) \\ &\le \limsup_s d(T_s x, z) \\ &< \limsup_s d(T_s x, u) \\ &= \limsup_s d(T_s x, y) \\ &= \limsup_\alpha d(T_{s_\alpha}x, u), \end{aligned}$$

which is a contradiction, and so $z = u$. Hence, $\{T_s x\}$ Δ-converges to a common fixed point of the semigroup $\mathfrak{T}$. $\square$

The following result is a strong convergence theorem for a right reversible semitopological semigroup.

Theorem 4.3 *Let S be a right reversible semitopological semigroup, C be a nonempty closed convex subset of a complete CAT(0) space X, and $x \in C$. Assume that $\mathfrak{T} = \{T_s : s \in S\}$ is a generalized asymptotically nonexpansive semigroup of C into itself with $F(\mathfrak{T}) \neq \emptyset$. Then $\{\pi T_s x\}$ converges strongly to a point of $F(\mathfrak{T})$, where $\pi : C \to F(\mathfrak{T})$ is the nearest point projection.*

Moreover, if S is reversible, then $Px := \lim_s \pi T_s x$ is the unique asymptotic center of the net $\{T_s x : s \in S\}$.

Proof By Lemma 3.3, $F(\mathfrak{T})$ is closed and convex. So, the mapping π is well defined. Put $R = \inf_s d(T_s x, \pi T_s x)$. As in the proof of Lemma 4.1, we have

$$R = \inf_s d(T_s x, \pi T_s x) = \limsup_s d(T_s x, \pi T_s x).$$

We will show that $\{\pi T_s x\}$ is a Cauchy net. To show this, we divide into two cases.

Case 1: $R = 0$. For $\varepsilon > 0$, there exists $s_0 \in S$ such that

$$d(T_s x, \pi T_s x) < \frac{\varepsilon}{4} \quad \text{for each } s \succeq s_0.$$

Since $\mathfrak{T}$ is a generalized asymptotically nonexpansive semigroup, there exists $t_0 \in S$ such that

$$\begin{aligned} d(T_{ts_0} x, \pi T_{s_0} x) &\le \limsup_u d(T_u T_{s_0} x, T_u \pi T_{s_0} x) + \frac{\varepsilon}{4} \\ &\le d(T_{s_0} x, \pi T_{s_0} x) + \frac{\varepsilon}{4} \end{aligned}$$

for each $t \succeq t_0$. Let $a, b \succeq t_0 s_0$. Since S is right reversible, $a, b \in \{t_0 s_0\} \cup \overline{St_0 s_0}$. Then we may assume $a, b \in \overline{St_0 s_0}$. So, there exist $\{t_\alpha\}$ and $\{s_\beta\}$ in S such that $t_\alpha t_0 s_0 \to a$ and $s_\beta t_0 s_0 \to b$. Therefore, we have

$$\begin{aligned} d(T_{t_\alpha t_0 s_0} x, T_{s_\beta t_0 s_0} x) &\le d(T_{t_\alpha t_0 s_0} x, \pi T_{s_0} x) + d(T_{s_\beta t_0 s_0} x, \pi T_{s_0} x) \\ &\le 2d(T_{s_0} x, \pi T_{s_0} x) + \frac{\varepsilon}{2}. \end{aligned}$$

This implies

$$\begin{aligned} d(\pi T_a x, \pi T_b x) &\le 2d(T_{s_0} x, \pi T_{s_0} x) + \frac{\varepsilon}{2} \\ &< 2\left(\frac{\varepsilon}{4}\right) + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Hence, $\{\pi T_s x\}$ is a Cauchy net.

Case 2: $R > 0$. Suppose that $\{\pi T_s x\}$ is not a Cauchy net. Then, there exists $\varepsilon > 0$ such that for any $s \in S$, there are $a_s, b_s \in S$ with $a_s, b_s \succeq s$ and $d(\pi T_{a_s} x, \pi T_{b_s} x) \geq \varepsilon$.

We choose a positive number η such that

$$(R + \eta)^2 - \frac{\varepsilon^2}{4} < R^2.$$

So, there exists $u_0 \in S$ such that

$$d(T_t x, \pi T_t x) \leq R + \frac{\eta}{2} \quad \text{for each } t \succeq u_0. \tag{4.1}$$

Then $d(\pi T_{a_{u_0}} x, \pi T_{b_{u_0}} x) \geq \varepsilon$. Since $\mathfrak{T}$ is a generalized asymptotically nonexpansive semigroup, there exists $v_0 \in S$ such that

$$\begin{aligned} d(T_t T_s x, \pi T_s x) &\leq \limsup_u d(T_u T_s x, \pi T_s x) + \frac{\eta}{2} \\ &\leq d(T_s x, \pi T_s x) + \frac{\eta}{2} \end{aligned} \tag{4.2}$$

for each $t \succeq v_0$ and each $s \in S$.

Since S is right reversible, there exists $c \in S$ such that $c \succeq v_0 a_{u_0}$ and $c \succeq v_0 b_{u_0}$. Then, there exist $\{t_\alpha\}$ and $\{s_\beta\}$ in S such that $t_\alpha v_0 a_{u_0} \to c$ and $s_\beta v_0 b_{u_0} \to c$. So, by (4.1) and (4.2), we have

$$d(T_{t_\alpha v_0 a_{u_0}} x, \pi T_{a_{u_0}} x) \leq d(T_{a_{u_0}} x, \pi T_{a_{u_0}} x) + \frac{\eta}{2} \leq R + \eta$$

and

$$d(T_{s_\beta v_0 b_{u_0}} x, \pi T_{b_{u_0}} x) \leq d(T_{b_{u_0}} x, \pi T_{b_{u_0}} x) + \frac{\eta}{2} \leq R + \eta.$$

This implies

$$d(T_c x, \pi T_{a_{u_0}} x) \leq R + \eta \quad \text{and} \quad d(T_c x, \pi T_{b_{u_0}} x) \leq R + \eta.$$

By the (CN) inequality, we get

$$\begin{aligned} & d^2\left(T_c x, \frac{\pi T_{a_{u_0}} x \oplus \pi T_{b_{u_0}} x}{2}\right) \\ & \quad \leq \frac{1}{2} d^2(T_c x, \pi T_{a_{u_0}} x) + \frac{1}{2} d^2(T_c x, \pi T_{b_{u_0}} x) - \frac{1}{4} d^2(\pi T_{a_{u_0}} x, \pi T_{b_{u_0}} x) \\ & \quad \leq (R + \eta)^2 - \frac{\varepsilon^2}{4} \\ & \quad < R^2, \end{aligned}$$

and so $d(T_c x, \frac{\pi T_{a_{u_0}} x \oplus \pi T_{b_{u_0}} x}{2}) < R$. Since π is the nearest point projection of C onto $F(\mathfrak{T})$, we have

$$d(T_c x, \pi T_c x) \leq d\left(T_c x, \frac{\pi T_{a_{u_0}} x \oplus \pi T_{b_{u_0}} x}{2}\right) < R.$$

This contradicts with $R = \inf_s d(T_s x, \pi T_s x)$.

So, $\{\pi T_s x\}$ is Cauchy in a closed subset $F(\mathfrak{T})$ of a complete CAT(0) space X, hence it converges to some point in $F(\mathfrak{T})$, say Px.

Finally, by Lemma 4.1, we have $\{T_s x : s \in S\}$ is bounded. So, let $z \in A(C, \{T_s x\})$. Since S is reversible, it implies by Theorem 3.1 that $z \in F(\mathfrak{T})$. Thus, by the property of π, we obtain

$$\begin{aligned}\limsup_s d(T_s x, Px) &\le \limsup_s \bigl(d(T_s x, \pi T_s x) + d(\pi T_s x, Px)\bigr)\\ &= \limsup_s d(T_s x, \pi T_s x)\\ &\le \limsup_s d(T_s x, z).\end{aligned}$$

This implies, by the uniqueness of asymptotic centers, that $Px = z$. □

Taking $S = \mathbb{N}$ in Theorem 4.2, we obtain the following Δ-convergence theorem of a generalized asymptotically nonexpansive mapping in CAT(0) spaces.

Theorem 4.4 *Let C be a nonempty closed convex subset of a complete CAT(0) space X and $x \in C$. Assume that $T : C \to C$ is a continuous generalized asymptotically nonexpansive mapping with $F(T) \neq \emptyset$. If $\lim_{n\to\infty} d(T^n x, T^{n+1} x) = 0$, then $\{T^n x : n \in \mathbb{N}\}$ Δ-converges to a fixed point of T.*

Taking $S = \mathbb{N}$ in Theorem 4.3, we obtain the following strong convergence theorem of a generalized asymptotically nonexpansive mapping in CAT(0) spaces.

Theorem 4.5 *Let C be a nonempty closed convex subset of a complete CAT(0) space X and $x \in C$. Assume that $T : C \to C$ is a continuous generalized asymptotically nonexpansive mapping with $F(T) \neq \emptyset$. Then $\{\pi T^n x\}$ converges strongly to a point of $F(T)$, where $\pi : C \to F(T)$ is the nearest point projection. Moreover, $Px := \lim_{n\to\infty} \pi T^n x$ is the unique asymptotic center of the sequence $\{T^n x : n \in \mathbb{N}\}$.*

Remark 4.6

(i) It is well known that every commutative semigroup is both left and right reversible and every discrete amenable semigroup is reversible. Then Theorems 3.1, 3.2, 3.3, 4.2, and 4.3 are also obtained for a class of commutative and discrete amenable semigroups.

(ii) Theorem 4.2 extends and generalizes the results of [15, 20] to generalized asymptotically nonexpansive semigroups and to CAT(0) spaces.

(iii) Theorem 4.3 extends and generalizes the results of [16] from amenable semigroups to right reversible semigroups and from nonexpansive semigroups to generalized asymptotically nonexpansive semigroups.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contribute equally and significantly in this research work. All authors read and approved the final manuscript.

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