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Large strong convergence theorems for total asymptotically strict pseudocontractive semigroup in banach spaces

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Abstract

The purpose of this is to introduce and study total asymptotically strict pseudocontractive semigroup, asymptotically strict pseudocontractive semigroup etc. the strong convergence theorems of the explicit iteration process for the new semigroups in arbitrary Banach spaces are established. The results presented in the paper extend and improve some recent results announced by many authors.
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1 Introduction and preliminaries

Let E be a real Banach space, E^* be the dual space of E, C is a nonempty closed convex subset of E, $\Re^+$ is the set of nonnegative real numbers and $J : E \to 2^{E^*}$ is the normalized duality mapping defined by

$$J(x) = \{f \in E^* \ : \ \langle x, f\rangle = ||x|| \cdot ||f||, \ ||x|| = ||f||\}, \quad x \in E. \tag{1.1}$$

Let $T : C \to C$ be a mapping, We use $F(T)$ to denote the set of fixed points of T. We also use "$\to$" to stand for strong convergence and "$\rightharpoonup$" for weak convergence. We first recall some definitions:

A one parameter family $\Im := \{T(t) : t \geq 0\}$ of self mappings of C is said a *nonexpansive semigroup*, if the following conditions are satisfied:

(i) $T(t_1 + t_2)x = T(t_1)T(t_2)x$, for any $t_1, t_2 \in \Re^+$ and $x \in C$;
(ii) $T(0)x = x$, for each $x \in C$;
(iii) for each $x \in C$, $t \mapsto T(t)x$ is continuous;
(iv) for any $t \geq 0$, $T(t)$ is nonexpansive mapping on C, that is for any $x, y \in C$,

$$||T(t)x - T(t)y|| \leq ||x - y|| \tag{1.2}$$

for any $t \geq 0$.

If the family $\Im := \{T(t) : t \geq 0\}$ satisfies conditions (i)-(iii), then it is said

(a) *pseudocontractive semigroup*, if for any x, $y \in C$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle T(t)x - T(t)y,\ j(x - y)\rangle \ \leq\ ||x - y||^2 \tag{1.3}$$

(b) *uniformly Lipschitzian semigroup*, if there exists a bounded measurable function $L : [0, \infty) \to (0, \infty)$ such that, for any x, $y \in C$ and $t \geq 0$,

$$||T^n(t)x - T^n(t)y|| \ \leq\ L(t)||x - y|| \quad \forall n \geq 1 \tag{1.4}$$

(c) *strict pseudocontractive semigroup*, if there exists a bounded function $\lambda : [0, \infty) \to (0, \infty)$ and for any given x, $y \in C$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle T(t)x - T(t)y,\ j(x - y)\rangle \leq\ ||x - y||^2 - \lambda(t)||(I - T(t))x - (I - T(t))y||^2 \tag{1.5}$$

for any $t \geq 0$.

It is easy to see that such semigroup is $((1 + \lambda(t))/\lambda(t))$-Lipschitzian and pseudocontractive semigroup.

(d) *demicontractive semigroup*, if $F(T(t)) \neq \varphi$ for all $t \geq 0$, there exists bounded function λ: $[0, \infty) \to (0, \infty)$ and for any $t \geq 0$, $x \in C$ and $y \in F(T(t))$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle T(t)x - y,\ j(x - y)\rangle \leq\ ||x - y||^2 - \lambda(t)||(I\ - T(t))x||^2 \tag{1.6}$$

In this article, we introduce the following semigroups.

Definition 1.1 A one parameter family $\Im := \{T(t) : t \geq 0\}$ of self mapping of C satisfies conditions (i)-(iii), then it is said

(e) *total asymptotically strict pseudocontractive semigroup*, if there exists bounded function $\lambda : [0, \infty) \to (0, \infty)$ and sequences $\{\mu_n\} \subset [0,\ \infty)$and $\{\xi_n\} \subset [0,\ \infty)$ with $\mu_n \to 0$ and $\xi_n \to 0$ as $n \to \infty$. for any given x, $y \in C$, there exists $j(x - y) \in J(x - y)$, such that

$$\begin{aligned}\langle T^n(t)x - T^n(t)y,\ j(x - y)\rangle \leq ||x - y||^2 - \lambda(t)||(I\ - T^n(t))x - (I\ - T^n(t))y||^2 \\ +\mu_n\phi(||x - y||) + \xi_n, \quad \forall n \geq 1\end{aligned} \tag{1.7}$$

for any $t \geq 0$.

where $\phi : [0,\ \infty) \to [0,\ \infty)$ is a continuous and strictly increasing function with $\varphi(0) = 0$.

(f) *asymptotically strict pseudocontractive semigroup*, if there exists a bounded function $\lambda : [0, \infty) \to (0, \infty)$ and a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$, for any given x, $y \in C$, there exists $j(x - y) \in J(x - y)$ such that

$$\begin{aligned}&\langle T^n(t)x - T^n(t)y,\ j(x - y)\rangle \\ &\quad \leq k_n||x - y||^2 - \lambda(t)||(I\ - T^n(t))x - (I\ - T^n(t))y||^2, \quad \forall n \geq 1\end{aligned} \tag{1.8}$$

for any $t \geq 0$.

(g) *asymptotically demicontractive semigroup*, if $F(T(t)) \neq \varphi$ for all $t \geq 0$ and there exists a bounded function $\lambda : [0, \infty) \to (0, \infty)$and a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$, for any $t \geq 0$, $x \in C$ and $y \in F(T(t))$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle T^n(t)x - y,\ j(x - y)\rangle \leq k_n||x - y||^2 - \lambda(t)||(I\ - T^n(t))x||^2, \quad \forall n \geq 1 \tag{1.9}$$

for any $t \geq 0$.

Remark 1.2 If $\varphi(\lambda) = \lambda^2$ and $\xi_n = 0$, a total asymptotically strict pseudocontractive semigroup is a asymptotically strict pseudocontractive semigroup. Every asymptotically strict pseudocontractive semigroup with $\bigcap_{t>0} F(T(t)) \neq \phi$ is asymptotically demicontractive semigroup. If $k_n = 1$, $n = 1$, a asymptotically strict pseudocontractive semigroup is a strict pseudocontractive semigroups a asymptotically demicontractive semigroup is a demicontractive semigroup.

It is easy to see that the condition (1.7) is equivalent to following condition: for any $t \geq 0$, $x \in C$ and $y \in F(T(t))$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle x - T^n(t)x,\ j(x-y)\rangle \geq \lambda(t)||x - T^n(t)x||^2 - \mu_n\phi(||x-y||) - \xi_n \tag{1.10}$$

The convergence problems of implicit and explicit iterative sequences for nonexpansive semigroups to common fixed points has been considered by some authors in various spaces. see, for example [1-11].

In 1998, Shioji and Takahashi [1] introduced in a Hilbert space the implicit iteration

$$x_n = \alpha_n u + (1-\alpha_n)\sigma_{t_n}(x_n), \quad n \geq 1, \tag{1.11}$$

where $\{\alpha_n\}$ is a sequence in (0, 1), $\{t_n\}$ a sequence of positive real number divergent to ∞ and for each $t > 0$ and $x \in C$, $\sigma_t(x)$ is the average given by

$$\sigma_t(x) = \frac{1}{t}\int_0^t T(s)x ds.$$

Under certain restrictions to the sequence $\{\alpha_n\}$, they proved the strong convergence of $\{x_n\}$ to a point $p \in F := \bigcap_{t\geq 0} F(T(t))$.

In 2003, Suzuki [2] first introduced the following implicit iteration process:

$$x_n = \alpha_n u + (1-\alpha_n)T(t_n)(x_n), \quad n \geq 1, \tag{1.12}$$

for the nonexpansive semigroup in a Hilbert space. He proved strong convergence of his process (1.12) with appropriate assumptions imposed upon the parameter sequences $\{\alpha_n\}$ and $\{t_n\}$. Xu [3] proved that Suzuki's result holds in a uniformly convex Banach space with a weakly continuous duality mapping.

In 2005, Aleyner and Reich [4] first introduced the following explicit iteration sequence

$$x_{n+1} = \alpha_n u + (1-\alpha_n)T(t_n)x_n, \quad n \geq 0 \tag{1.13}$$

in a reflexive Banach space with a uniformly Gâteaux differentiable norm such that each nonempty, bounded, closed and convex subset of E has the common fixed point property for nonexpansive mappings. Under appropriate assumptions imposed upon the parameter sequences $\{\alpha_n\}$ and $\{t_n\}$, they proved that the sequence $\{x_n\}$ defined by (1.13) converges strongly to a common sixed point of the semigroup $\{T(t) : t \geq 0\}$.

More recently, Chang et al. [11] introduced the following explicit iteration process:

$$x_1 \in C,$$

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T(t_n)x_n, \quad n \geq 1 \tag{1.14}$$

for the Lipschitzian and demicontractive semigroup $\Im := \{T(t) : t \geq 0\}$ in general Banach spaces. Also under appropriate assumptions imposed upon the parameter sequences $\{\alpha_n\}$ and $\{t_n\}$, they proved the sequence $\{x_n\}$ defined by (1.14) converges strongly to some point in $F =: \bigcap_{t\geq 0} F(T(t))$.

Inspired and motivated by the above works of Shioji and Takahashi [1], Suzuki [2], Xu [3], Aleyner and Reich [4] and Chang et al. [11], the purpose of this article is to introduce and study the strong convergence problem of the following explicit iteration process:

$$x_1 \in C,$$

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n(t_n)x_n, \quad n \geq 1 \tag{1.15}$$

For the uniformly Lipschitzian and total asymptotically strict pseudocontractive semigroup $\Im := \{T(t) : t \geq 0\}$ in general Banach spaces. The results presented in the article extend and improve some recent results given in [4,5,7,9].

The following Lemmas will be needed in proving our main results.

Lemma 1.3 Let $\{a_n\}$, $\{b_n\}$ and $\{\delta_n\}$ be sequences of nonnegative real numbers satisfying

$$a_{n+1} \leq (1 + \delta_n)a_n + b_n, \quad \forall n \geq n_0, \tag{1.16}$$

where n_0 is some nonnegative integer. If $\sum_{i=1}^{\infty} \delta_n < \infty$ and $\sum_{i=1}^{\infty} b_n < \infty$, then the limit $\lim_{n\to\infty} a_n$ exists.

Lemma 1.4 [12] Let E be any real Banach space, E^* be the dual space of E and $J : E \to 2^{E^*}$ be the normalized duality mapping. Then for any $x, y \in E$ we have

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y)\rangle, \quad \forall j(x + y) \in J(x + y) \tag{1.17}$$

2 Main results

Now, we are ready to give our main results.

Theorem 2.1 Let C be a nonempty closed convex subset of a real Banach space E, and let $\Im := \{T(t) : t \geq 0\}$ be a uniformly Lipschitzian with bounded measurable function $L(t) : [0, \infty) \to (0, \infty)$ and total asymptotically strict pseudocontractive semigroup as defined in (1.7), such that

$$L := \sup_{t\geq 0} L(t) < \infty, \quad \lambda := \inf_{t\geq 0} \lambda(t) > 0, \quad F := \bigcap_{t\geq 0} F(T(t)) \neq \emptyset \tag{2.1}$$

There exist positive constants M and M^* such that $\varphi(\lambda) \leq M^* \lambda^2$ for all, $\lambda \geq M$. Let $\{x_n\}$ be the sequence defined by (1.15), where $\{\alpha_n\}$ is a sequence in (0, 1) and $\{t_n\}$ be an increasing sequence in $[0, \infty)$. If the following conditions are satisfied:

(1) $\sum_{n=1}^{\infty} \alpha_n^2 < \infty, \quad \sum_{n=1}^{\infty} \alpha_n = \infty, \quad \sum_{n=1}^{\infty} \alpha_n \mu_n < \infty, \quad \sum_{n=1}^{\infty} \alpha_n \xi_n < \infty.$

(2) for any bounded subset $D \subset C$

$$\lim_{n\to\infty} \sup_{x\in D, s\in R^+} ||T^n(s + t_n)x - T^n(t_n)x|| = 0 \tag{2.2}$$

(3) There exist a compact subset G of E such that $\bigcap_{t \geq 0} T(t)(C) \subset G$.

Then the sequence $\{x_n\}$ converges strongly to a common fixed point of the semigroup $\Im := \{T(t) : t \geq 0\}$.

Proof The proof of Theorem 2.1 is divided into four steps:

Step 1. First we prove that $\lim_{n \to \infty} ||x_n - p||$ exist for all $p \in F$.

For any $p \in F$, by (1.4) we have

$$||T^n(t_n)x_n - p|| = ||T^n(t_n)x_n - T^n(t_n)p|| \leq L(t_n)||x_n - p|| \leq L||x_n - p|| \tag{2.3}$$

This follows from (1.15) and (2.3) that

$$\begin{aligned} ||x_{n+1} - p|| &= ||(1 - \alpha_n)x_n + \alpha_n T^n(t_n)x_n - (1 - \alpha_n)p - \alpha_n p|| \\ &\leq (1 - \alpha_n)||x_n - p|| + \alpha_n ||T^n(t_n)x_n - p|| \\ &\leq (1 - \alpha_n)||x_n - p|| + \alpha_n L||x_n - p|| \\ &\leq (1 + L)||x_n - p|| \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} ||x_{n+1} - x_n|| &= \alpha_n ||T^n(t_n)x_n - x_n|| \\ &\leq \alpha_n (||T^n(t_n)x_n - p|| + ||x_n - p||) \\ &\leq \alpha_n (1 + L)||x_n - p|| \end{aligned} \tag{2.5}$$

Since $\Im := \{T(t) : t \geq 0\}$ is total asymptotically strict pseudocontractive semigroup with $\lambda := \inf_{t \geq 0} \lambda(t) > 0$, for the point x_{n+1} and p there exists $j(x_{n+1} - p) \in J(x_{n+1} - p)$ such that

$$\begin{aligned} \langle T^n(t_n)x_{n+1} - x_{n+1}, \ j(x_{n+1} - p)\rangle \leq -\lambda ||T^n(t_n)x_{n+1} - x_{n+1}||^2 \\ + \mu_n \phi(||x_{n+1} - p||) + \xi_n, \quad \forall n \geq 1 \end{aligned} \tag{2.6}$$

Again since φ is an increasing function, it results that $\varphi(\lambda) \leq \varphi(M)$ if $\lambda \leq M$ and $\varphi(\lambda) \leq M^* \lambda^2$, if $\lambda \leq M$. In either case, we can obtain that

$$\phi(\lambda) \leq \phi(M) + M^* \lambda^2 \tag{2.7}$$

Thus, by Lemma 1.4, (2.4)-(2.7), we have

$$\begin{aligned} ||x_{n+1} - p||^2 \quad &\leq ||x_n - p + \alpha_n (T^n(t_n)x_n - x_n)||^2 \\ &\leq ||x_n - p||^2 + 2\alpha_n \langle T^n(t_n)x_n - x_n, \ j(x_{n+1} - p)\rangle \\ &= ||x_n - p||^2 + 2\alpha_n \langle T^n(t_n)x_n - T^n(t_n)x_{n+1}, \ j(x_{n+1} - p)\rangle \\ &\quad + 2\alpha_n \langle T^n(t_n)x_{n+1} - x_{n+1}, \ j(x_{n+1} - p)\rangle \\ &\quad + 2\alpha_n \langle x_{n+1} - x_n, \ j(x_{n+1} - p)\rangle \\ &\leq ||x_n - p||^2 + 2\alpha_n L||x_{n+1} - x_n||||x_{n+1} - p|| \\ &\quad - 2\alpha_n \lambda ||T^n(t_n)x_{n+1} - x_{n+1}||^2 + 2\alpha_n \mu_n \phi(||x_{n+1} - p||) \\ &\quad + 2\alpha_n \xi_n + 2\alpha_n ||x_{n+1} - x_n||||x_{n+1} - p|| \\ &\leq (1 + 2\alpha_n^2 (1 + L)^3 + 2\alpha_n \mu_n M^* (1 + L)^2)||x_n - p||^2 \\ &\quad - 2\alpha_n \lambda ||T^n(t_n)x_{n+1} - x_{n+1}||^2 \\ &\quad + 2\alpha_n \mu_n \phi(M) + 2\alpha_n \xi_n \\ &\leq (1 + 2\alpha_n^2 (1 + L)^3 + 2\alpha_n \mu_n M^* (1 + L)^2)||x_n - p||^2 \\ &\quad + 2\alpha_n \mu_n \phi(M) + 2\alpha_n \xi_n \end{aligned} \tag{2.8}$$

By the condition (1), it follows from Lemma 1.3 that the limit $\lim_{n\to\infty}||x_n - p||$ exist and so the sequence $\{x_n\}$ is bounded in C.

Step 2. Now we prove that

$$\liminf_{n\to\infty}||x_n - T^n(t_n)x_n|| = 0 \tag{2.9}$$

In fact, it follows from (2.8) that

$$\begin{aligned} ||x_{n+1} - p||^2 \quad &\le ||x_n - p||^2 - 2\alpha_n\lambda||T^n(t_n)x_{n+1} - x_{n+1}||^2 \\ &+ (2\alpha_n^2(1+L)^3 + 2\alpha_n\mu_n M^*(1+L)^2)||x_n - p||^2 \\ &+ 2\alpha_n\mu_n\phi(M) + 2\alpha_n\xi_n \end{aligned} \tag{2.10}$$

This implies that

$$\begin{aligned} 2\alpha_n\lambda||T^n(t_n)x_{n+1} - x_{n+1}||^2 \le\ &||x_n - p||^2 - ||x_{n+1} - p||^2 + 2\alpha_n^2(1+L)^3K^2 \\ &+ (2M^*(1+L)^2K^2 + 2\phi(M))\alpha_n\mu_n + 2\alpha_n\xi_n \end{aligned} \tag{2.11}$$

Where $K = \sup_{n\ge 1}||x_n - p||$. Hence, for some $m \ge 1$,

$$\begin{aligned} &2\lambda\sum_{n=1}^{m}\alpha_n||T^n(t_n)x_{n+1} - x_{n+1}||^2 \\ &\quad\le \sum_{n=1}^{m}(||x_n - p||^2 - ||x_{n+1} - p||^2) + 2(1+L)^3K^2\sum_{n=1}^{m}\alpha_n^2 \\ &\quad\quad +(2M^*(1+L)^2K^2 + 2\phi(M))\sum_{n=1}^{m}\alpha_n\mu_n + 2\sum_{n=1}^{m}\alpha_n\xi_n \\ &\quad\le ||x_1 - p||^2 + 2(1+L)^3K^2\sum_{n=1}^{m}\alpha_n^2 \\ &\quad\quad +(2M^*(1+L)^2K^2 + 2\phi(M))\sum_{n=1}^{m}\alpha_n\mu_n + 2\sum_{n=1}^{m}\alpha_n\xi_n \end{aligned} \tag{2.12}$$

Letting $m \to \infty$, we have

$$\begin{aligned} &2\lambda\sum_{n=1}^{\infty}\alpha_n||T^n(t_n)x_{n+1} - x_{n+1}||^2 \\ &\quad\le ||x_1 - p||^2 + 2(1+L)^3K^2\sum_{n=1}^{\infty}\alpha_n^2 \\ &\quad\quad + (2M^*(1+L)^2K^2 + 2\phi(M))\sum_{n=1}^{\infty}\alpha_n\mu_n + 2\sum_{n=1}^{\infty}\alpha_n\xi_n \end{aligned} \tag{2.13}$$

By the condition (1), we obtain

$$\sum_{n=1}^{\infty}\alpha_n||T^n(t_n)x_{n+1} - x_{n+1}||^2 < \infty \tag{2.14}$$

Which implies

$$\liminf_{n\to\infty} ||x_{n+1} - T^n(t_n)x_{n+1}|| = 0 \tag{2.15}$$

Since $\lim_{n\to\infty} ||x_n - p||$ exist for all $p \in F$ and $\lim_{n\to\infty} \alpha_n = 0$, using (2.5), we have

$$\lim_{n\to\infty} ||x_{n+1} - x_n|| = 0 \tag{2.16}$$

it follows from (2.15) and (2.16) that

$$\begin{aligned}
&\liminf_{n\to\infty} ||x_n - T^n(t_n)x_n|| \\
&\quad \le \liminf_{n\to\infty} \{||x_n - x_{n+1}|| + ||x_{n+1} - T^n(t_n)x_{n+1}|| + ||T^n(t_n)x_{n+1} - T^n(t_n)x_n||\} \\
&\quad \le \liminf_{n\to\infty} \{(1+L)||x_n - x_{n+1}|| + ||x_{n+1} - T^n(t_n)x_{n+1}||\} = 0
\end{aligned} \tag{2.17}$$

Therefore the conclusion (2.9) is proved.

Step 3. Now we prove that

$$\liminf_{n\to\infty} ||x_n - T(t_n)x_n|| = 0 \tag{2.18}$$

Letting $d_n = ||x_n - T^n(t_n)x_n||$, it follows from (1.4) that

$$\begin{aligned}
&||x_{n+1} - T(t_{n+1})x_{n+1}|| \\
&\quad \le ||x_{n+1} - T^{n+1}(t_{n+1})x_{n+1}|| + ||T^{n+1}(t_{n+1})x_{n+1} - T(t_{n+1})x_{n+1}|| \\
&\quad \le d_{n+1} + L||T^n(t_{n+1})x_{n+1} - x_{n+1}|| \\
&\quad \le d_{n+1} + L\{||T^n(t_{n+1})x_{n+1} - T^n(t_n)x_{n+1}|| \\
&\qquad + ||T^n(t_n)x_{n+1} - T^n(t_n)x_n|| \\
&\qquad + ||T^n(t_n)x_n - x_n|| + ||x_n - x_{n+1}||\} \\
&\quad \le d_{n+1} + Ld_n + L(1+L)||x_{n+1} - x_n|| \\
&\qquad + L||T^n((t_{n+1} - t_n) + t_n)x_{n+1} - T^n(t_n)x_{n+1}|| \\
&\quad \le d_{n+1} + Ld_n + L(1+L)||x_{n+1} - x_n|| \\
&\qquad + L \sup_{z\in\{x_n\}, s\in R^+} ||T^n(s+t_n)z - T^n(t_n)z||
\end{aligned} \tag{2.19}$$

By the condition (2), (2.9), and (2.16), we have

$$\begin{aligned}
&\liminf_{n\to\infty} ||x_{n+1} - T(t_{n+1})x_{n+1}|| \le \liminf_{n\to\infty} \{d_{n+1} + Ld_n + L(1+L)||x_{n+1} - x_n|| \\
&\quad + L \sup_{z\in\{x_n\}, s\in R^+} ||T^n(s+t_n)z - T^n(t_n)z||\} = 0
\end{aligned} \tag{2.20}$$

Therefore the conclusion (2.18) is proved.

Step 4. Finally we prove the sequence $\{x_n\}$ converges strongly to a common fixed point of the semigroup $\Im := \{T(t) : t \ge 0\}$.

By (2.9) and (2.18), we have

$$\liminf_{n\to\infty} ||x_n - T^n(t_n)x_n|| = 0, \quad \liminf_{n\to\infty} ||x_n - T(t_n)x_n|| = 0$$

Again by the condition (3), there exists a compact subset G of E such that $\bigcap_{t\ge 0} T(t)(C) \subset G$ and so there exists subsequence $\{x_{n_i}\}$ of $\{x_n\}$, for some point $q \in G$ such that

$$\lim_{i\to\infty} T(t_{n_i})x_{n_i} = q, \quad \lim_{i\to\infty} ||x_{n_i} - T(t_{n_i})x_{n_i}|| = 0 \tag{2.21}$$

and

$$\lim_{i\to\infty} ||x_{n_i} - T^{n_i}(t_{n_i})x_{n_i}|| = 0 \tag{2.22}$$

Hence it follows from (2.21) that $x_{n_i} \to q$ as $i \to \infty$.

Next, we prove that

$$\lim_{i\to\infty} ||x_{n_i} - T^{n_i}(t)x_{n_i}|| = 0, \tag{2.23}$$

for all $t \geq 0$. In fact, it follows from the condition (2) and (2.22) that, for any $t \geq 0$

$$\begin{aligned} ||x_{n_i} - T^{n_i}(t)x_{n_i}|| &\leq ||x_{n_i} - T^{n_i}(t_{n_i})x_{n_i}|| \\ &\quad + ||T^{n_i}(t + t_{n_i})x_{n_i} - T^{n_i}(t_{n_i})x_{n_i}|| + ||T^{n_i}(t)x_{n_i} - T^{n_i}(t + t_{n_i})x_{n_i}|| \\ &\leq (1 + L)||x_{n_i} - T^{n_i}(t_{n_i})x_{n_i}|| + ||T^{n_i}(t + t_{n_i})x_{n_i} - T^{n_i}(t_{n_i})x_{n_i}|| \\ &\leq (1 + L)||x_{n_i} - T^{n_i}(t_{n_i})x_{n_i}|| \\ &\quad + \sup_{z\in\{x_n\}, s\in R^+} ||T^{n_i}(s + t_{n_i})z - T^{n_i}(t_{n_i})z|| \to 0 \end{aligned} \tag{2.24}$$

as $i \to \infty$. Letting $e_{n_i} = ||x_{n_i} - T^{n_i}(t)x_{n_i}||$, it follows from (2.16) and (2.23) that

$$\begin{aligned} ||x_{n_i} - T(t)x_{n_i}|| &\leq ||x_{n_i} - T^{n_i}(t)x_{n_i}|| + ||T^{n_i}(t)x_{n_i} - T(t)x_{n_i}|| \\ &\leq e_{n_i} + L||T^{n_i-1}(t)x_{n_i} - x_{n_i}|| \\ &\leq e_{n_i} + L(||x_{n_i} - x_{n_i-1}|| + ||x_{n_i-1} - T^{n_i-1}(t)x_{n_i-1}|| \\ &\quad + ||T^{n_i-1}(t)x_{n_i-1} - T^{n_i-1}(t)x_{n_i}||) \\ &\leq e_{n_i} + L(||x_{n_i} - x_{n_i-1}|| + e_{n_i-1} + L||x_{n_i-1} - x_{n_i}||) \\ &\leq e_{n_i} + L(1 + L)||x_{n_i} - x_{n_i-1}|| + Le_{n_i-1} \to 0 \end{aligned} \tag{2.25}$$

as $i \to \infty$. Since $x_{n_i} \to q$ as $i \to \infty$ and the semigroup $\Im := \{T(t) : t \geq 0\}$ is Lipschitzian, it follows from (2.25) that $q = T(t)q$ for all $t \geq 0$, that is

$$q \in F := \bigcap_{t\geq 0} F(T(t)) \tag{2.26}$$

Since $x_{n_i} \to q$ as $i \to \infty$ and the limit $\lim_{n\to\infty} ||x_n - q||$ exist, which implies that $x_{n_i} \to q \in F$ as $n \to \infty$. This completes the proof.

The following theorem can be obtained from Theorem 2.1 immediately.

Theorem 2.2 Let C be a nonempty closed convex subset of a real Banach space E, and let $\Im := \{T(t) : t \geq 0\}$ be a uniformly Lipschitzian with bounded measurable function $L(t) : [0, \infty) \to (0, \infty)$ and asymptotically strict pseudocontractive semigroup as defined in (1.8), such that

$$L := \sup_{t\geq 0} L(t) < \infty, \ \lambda := \inf_{t\geq 0} \lambda(t) > 0, \ F := \bigcap_{t\geq 0} F(T(t)) \neq \emptyset$$

Let $\{x_n\}$ be the sequence defined by (1.15), where $\{\alpha_n\}$ is a sequence in (0, 1) and $\{t_n\}$ be an increasing sequence in $[0, \infty)$. If the following conditions are satisfied:

(1) $\sum_{n=1}^{\infty} \alpha_n^2 < \infty$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\sum_{n=1}^{\infty} \alpha_n(k_n - 1) < \infty$.

(2) for any bounded subset $D \subset C$

$$\lim_{n\to\infty} \sup_{x\in D, s\in R^+} ||T^n(s + t_n)x - T^n(t_n)x|| = 0$$

(3) There exist a compact subset G of E such that $\bigcap_{t\geq 0} T(t)(C) \subset G$.

Then the sequence $\{x_n\}$ converges strongly to a common fixed point of the semigroup $\Im := \{T(t) : t \geq 0\}$.

Proof Taking $\varphi(\lambda) = \lambda^2$, $\xi_n = 0$, $\mu_n = k_n - 1$ in Theorem 2.1, Since all conditions in Theorem 2.1 are satisfied. It follows from Theorem 2.1 that the sequence $x_n \to q \in F := \bigcap_{t\geq 0} F(T(t))$ as $n \to \infty$. This completes the proof of Theorem 2.2.

The following theorem can be obtained from Theorem 2.2 immediately.

Theorem 2.3 Let C be a nonempty closed convex subset of a real Banach space E, and let $\Im := \{T(t) : t \geq 0\}$ be a uniformly Lipschitzian with bounded measurable function $L(t) : [0, \infty) \to (0, \infty)$ and asymptotically demicontractive semigroup as defined in (1.9), such that

$$L := \sup_{t\geq 0} L(t) < \infty, \quad \lambda := \inf_{t\geq 0} \lambda(t) > 0, \quad F := \bigcap_{t\geq 0} F(T(t)) \neq \emptyset$$

Let $\{x_n\}$ be the sequence defined by (1.15), where $\{\alpha_n\}$ is a sequence in (0, 1) and $\{t_n\}$ be an increasing sequence in $[0, \infty)$. If the following conditions are satisfied:

(1) $\sum_{n=1}^{\infty} \alpha_n^2 < \infty$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\sum_{n=1}^{\infty} \alpha_n(k_n - 1) < \infty$.

(2) for any bounded subset $D \subset C$

$$\lim_{n\to\infty} \sup_{x\in D, s\in R^+} ||T^n(s + t_n)x - T^n(t_n)x|| = 0$$

(3) There exist a compact subset G of E such that $\bigcap_{t\geq 0} T(t)(C) \subset G$.

Then the sequence $\{x_n\}$ converges strongly to a common fixed point of the semigroup $\Im := \{T(t) : t \geq 0\}$.

Proof Taking $y \in F(T(t))$, for any $t \geq 0$ in Theorem 2.2, Since all conditions in Theorem 2.2 are satisfied. It follows from Theorem 2.2 that the sequence $x_n \to q \in F := \bigcap_{t\geq 0} F(T(t))$ as $n \to \infty$. This completes the proof of Theorem 2.3.

Remark 2.4 Theorem 2.3 extend and improved the corresponding results of Chang et al. [11], Shioji and Takahashi [1], Suzuki [2], Xu [3], Aleyner and Reich [4] and others.

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Authors' contributions
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