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Coupled fixed point of generalized contractive mappings on partially ordered G-metric spaces

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Abstract

Coupled fixed point results for nonlinear contraction mappings having a mixed monotone property in a partially ordered G-metric space due to Choudhury and Maity are extended and unified. We also provide example to validate the main results in this article.

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1. Introduction

One of the simplest and the most useful result in the fixed point theory is the Banach-Caccioppoli contraction [1] mapping principle, a power tool in analysis. This principle has been generalized in different directions in different spaces by mathematicians over the years (see [2-10] and references mentioned therein). On the other hand, fixed point theory has received much attention in metric spaces endowed with a partial ordering. The first result in this direction was given by Ran and Reurings [11] and they presented applications of their results to matrix equations. Subsequently, Nieto and Rodríguez-López [12] extended the results in [11] for nondecreasing mappings and obtained a unique solution for a first order ordinary differential equation with periodic boundary conditions (see also, [13-19]).

Bhaskar and Lakshmikantham [20] introduced the concept of a coupled fixed point and the mixed monotone property. Furthermore, they proved some coupled fixed point theorems for mappings which satisfy the mixed monotone property and gave some applications in the existence and uniqueness of a solution for a periodic boundary value problem. A number of articles in this topic have been dedicated to the improvement and generalization see in [21-24] and reference therein.

Mustafa and Sims [25,26] introduced a new concept of generalized metric spaces, called G-metric spaces. In such spaces every triplet of elements is assigned to a non-negative real number. Based on the notion of G-metric spaces, Mustafa et al. [27] established fixed point theorems in G-metric spaces. Afterward, many fixed point results were proved in this space (see [28-34]).

Recently, Choudhury and Maity [35] studied necessary conditions for existence of coupled fixed point in partially ordered G-metric spaces. They obtained the following interesting result.

Theorem 1.1 ([35]). *Let* $(X, \preccurlyeq)$ *be a partially ordered set such that X is a complete G-metric space and F*: $X \times X \to X$ *be a mapping having the mixed monotone property on X. Suppose there exists* $k \in [0,1)$ *such that*

$$G(F(x,y),F(u,v),F(w,z)) \le \frac{k}{2}(G(x,u,w)+G(y,v,z))$$

for all $x, y, z, u, v, w \in X$ *for which* $x \succcurlyeq u \succcurlyeq w$ *and* $y \preccurlyeq v \preccurlyeq z$, *where either* $u \neq w$ *or* $v \neq z$. *If there exists* $x_0, y_0 \in X$ *such that*

$$x_0 \preccurlyeq F(x_0,y_0) \text{ and } y_0 \succcurlyeq F(y_0,x_0)$$

and either

(*a*) *F is continuous or*
(*b*) *X has the following property:*
(i) *if a non-decreasing sequence* $\{x_n\} \to x$, *then* $x_n \preccurlyeq x$ *for all* $n \in \mathbb{N}$,
(ii) *if a non-increasing sequence* $\{y_n\} \to y$, *then* $y_n \succcurlyeq y$ *for all* $n \in \mathbb{N}$,

then F has a coupled fixed point.

The aim of this article is to extend and unify coupled fixed point results in [35] and to study necessary conditions to guarantee the uniqueness of coupled fixed point. We also provide illustrative example in support of our results.

2. Preliminaries

Throughout this article, $(X, \preccurlyeq)$ denotes a partially ordered set with the partial order $\preccurlyeq$.

By $x \prec y$, we mean $x \preccurlyeq y$ but $x \neq y$. If $(X, \preccurlyeq)$ is a partially ordered set. A mapping $f: X \to X$ is said to be non-decreasing (non-increasing) if for all $x, y \in X$, $x \preccurlyeq y$ implies $f(x) \preccurlyeq f(y)$ ($f(y) \preccurlyeq f(x)$, respectively).

Definition 2.1 ([20]). Let $(X, \preccurlyeq)$ be a partial ordered set. A mapping $F: X \times X \to X$ is said to has the a *mixed monotone property* if F is monotone non-decreasing in its first argument and is monotone non-increasing in its second argument, that is, for any $x, y \in X$

$$x_1, x_2 \in X, x_1 \preccurlyeq x_2 \Rightarrow F(x_1,y) \preccurlyeq F(x_2,y) \tag{2.1}$$

and

$$y_1, y_2 \in X, y_1 \preccurlyeq y_2 \Rightarrow F(x,y_1) \succcurlyeq F(x,y_2). \tag{2.2}$$

Definition 2.2 ([20]). An element $(x, y) \in X \times X$ is called a *coupled fixed point* of mapping $F: X \times X \to X$ if

$$x = F(x,y) \text{ and } y = F(y,x).$$

Consistent with Mustafa and Sims [25,26], the following definitions and results will be needed in the sequel.

Definition 2.3 ([26]). Let X be a nonempty set. Suppose that a mapping $G: X \times X \times X \to \mathbb{R}_+$ satisfies:

(G_1) $G(x,y,z) = 0$ if $x = y = z$;
(G_2) $G(x, x, y) > 0$ for all $x, y \in X$ with $x \neq y$;

(G_3) $G(x,x,y) \leq G(x,y,z)$ for all $x,y,z \in X$ with $z \neq y$;
(G_4) $G(x,y,z) = G(x,z,y) = G(y,z,x) = \ldots$, (symmetry in all three variables);
(G_5) $G(x,y,z) \leq G(x,a,a) + G(a,y,z)$ for all $x,y,z,a \in X$ (rectangle inequality).

Then G is called a *G-metric* on X and (X, G) is called a *G-metric space*.

Definition 2.4 ([26]). Let X be a G-metric space and let $\{x_n\}$ be a sequence of points of X, a point $x \in X$ is said to be the *limit of a sequence* $\{x_n\}$ if $G(x,x_n,x_m) \to 0$ as $n, m \to \infty$ and sequence $\{x_n\}$ is said to be G-convergent to x.

From this definition, we obtain that if $x_n \to x$ in a G-metric space X, then for any $\epsilon > 0$ there exists a positive integer N such that $G(x,x_n,x_m) < \epsilon$, for all $n,m \geq N$.

It has been shown in [26] that the G-metric induces a Hausdorff topology and the convergence described in the above definition is relative to this topology. So, a sequence can converge at the most to one point.

Definition 2.5 ([26]). Let X be a G-metric space, a sequence $\{x_n\}$ is called *G-Cauchy* if for every $\epsilon > 0$ there is a positive integer N such that $G(x_n,x_m,x_l) < \epsilon$ for all $n, m, l \geq N$, that is, if $G(x_n, x_m, x_l) \to 0$, as $n, m, l \to \infty$.

We next state the following lemmas.

Lemma 2.6 ([26]). *If X is a G-metric space, then the following are equivalent:*

(1) $\{x_n\}$ *is G-convergent to x.*
(2) $G(x_n,x_n,x) \to 0$ *as* $n \to \infty$.
(3) $G(x_n,x,x) \to 0$ *as* $n \to \infty$.
(4) $G(x_m,x_n,x) \to 0$ *as* $n,m \to \infty$.

Lemma 2.7 ([26]). *If X is a G-metric space, then the following are equivalent:*

(*a*) *The sequence* $\{x_n\}$ *is G-Cauchy.*
(*b*) *For every* $\epsilon > 0$, *there exists a positive integer N such that* $G(x_n,x_m,x_m) < \epsilon$, *for all* $n,m \geq N$.

Lemma 2.8 ([26]). *If X is a G-metric space then* $G(x,y,y) \leq 2G(y,x,x)$ *for all* $x,y \in X$.

Definition 2.9 ([26]). Let (X, G), (X', G') be two generalized metric spaces. A mapping $f: X \to X'$ is *G-continuous at a point* $x \in X$ if and only if it is G sequentially continuous at x, that is, whenever $\{x_n\}$ is G-convergent to x, $\{f(x_n)\}$ is G'-convergent to $f(x)$.

Definition 2.10 ([26]). A G-metric space X is called a *symmetric G-metric space* if

$$G(x,y,y) = G(y,x,x)$$

for all $x,y \in X$.

Definition 2.11 ([26]). A G-metric space X is said to be *G-complete* (or a *complete G-metric space*) if every G-Cauchy sequence in X is convergent in X.

Definition 2.12 ([26]). Let X be a G-metric space. A mapping $F: X \times X \to X$ is said to be *continuous* if for any two G-convergent sequences $\{x_n\}$ and $\{y_n\}$ converging to x and y, respectively, $\{F(x_n,y_n)\}$ is G-convergent to $F(x,y)$.

3. Coupled fixed point in *G*-metric spaces

Let Θ denotes the class of all functions θ: $[0, \infty) \times [0, \infty) \to [0,1)$ which satisfies following condition:

For any two sequences $\{t_n\}$ and $\{s_n\}$ of nonnegative real numbers,

$$\theta(t_n, s_n) \to 1 \text{ implies that } t_n, s_n \to 0.$$

Following are examples of some function in Θ.

(1) $\theta_1(s,t) = k$ for $s,t \in [0,\infty)$, where $k \in [0,1)$.

(2) $\theta_2(s,t) = \begin{cases} \dfrac{\ln(1+ks+lt)}{ks+lt}; & s > 0 \text{ or } t > 0, \\ r \in [0,1) & ; s = 0, t = 0, \end{cases}$

where $k, l \in (0,1)$

(3) $\theta_3(s,t) = \begin{cases} \dfrac{\ln(1+\max\{s,t\})}{\max\{s,t\}}; & s > 0 \text{ or } t > 0, \\ r \in [0,1) & ; \quad s = 0, t = 0, \end{cases}$

Now, we prove our main result.

Theorem 3.1. *Let* $(X, \preccurlyeq)$ *be a partially ordered set such that there exists a complete G-metric on X and F*: $X \times X \to X$ *be a continuous mapping having the mixed monotone property. Suppose that there exists* $\theta \in \Theta$ *such that*

$$G(F(x,y), F(u,v), F(w,z)) + G(F(y,x), F(v,u), F(z,w)) \leq \theta(G(x,u,w), G(y,v,z))(G(x,u,w) + G(y,v,z)) \tag{3.1}$$

for all $x, y, z, u, v, w \in X$ *for which* $x \succcurlyeq u \succcurlyeq w$ *and* $y \preccurlyeq v \preccurlyeq z$ *where either* $u \neq w$ *or* $v \neq z$. *If there exists* $x_0, y_0 \in X$ *such that*

$$x_0 \preccurlyeq F(x_0, y_0) \text{ and } y_0 \succcurlyeq F(y_0, x_0),$$

then F has a coupled fixed point.

Proof. As $F(X \times X) \subseteq X$, we can construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$x_{n+1} = F(x_n, y_n) \text{ and } y_{n+1} = F(y_n, x_n) \text{ for all } n \geq 0. \tag{3.2}$$

Next, we show that

$$x_n \preccurlyeq x_{n+1} \text{ and } y_n \succcurlyeq y_{n+1} \text{ for all } n \geq 0. \tag{3.3}$$

Since $x_0 \preccurlyeq F(x_0,y_0) = x_1$ and $y_0 \succcurlyeq F(y_0,x_0) = y_1$, therefore (3.3) holds for $n = 0$. Suppose that (3.3) holds for some fixed $n \geq 0$, that is,

$$x_n \preccurlyeq x_{n+1} \text{ and } y_n \succcurlyeq y_{n+1}. \tag{3.4}$$

Since F has a mixed monotone property, from (3.4) and (2.1), we have

$$F(x_n, y) \preccurlyeq F(x_{n+1}, y) \text{ and } F(y_{n+1}, x) \preccurlyeq F(y_n, x) \tag{3.5}$$

for all $x, y \in X$ and from (3.4) and (2.2), we have

$$F(y, x_n) \succcurlyeq F(y, x_{n+1}) \text{ and } F(x, y_{n+1}) \succcurlyeq F(x, y_n), \tag{3.6}$$

for all $x,y \in X$. If we take $y = y_n$ and $x = x_n$ in (3.5), then we obtain

$$x_{n+1} = F(x_n, y_n) \preccurlyeq F(x_{n+1}, y_n) \text{ and } F(y_{n+1}, x_n) \preccurlyeq F(y_n, x_n) = y_{n+1}. \tag{3.7}$$

If we take $y = y_{n+1}$ and $x = x_{n+1}$ in (3.6), then

$$F(y_{n+1}, x_n) \succcurlyeq F(y_{n+1}, x_{n+1}) = y_{n+2} \text{ and } x_{n+2} = F(x_{n+1}, y_{n+1}) \succcurlyeq F(x_{n+1}, y_n). \tag{3.8}$$

Now, from (3.7) and (3.8), we have

$$x_{n+1} \preccurlyeq x_{n+2} \text{ and } y_{n+1} \succcurlyeq y_{n+2}. \tag{3.9}$$

Therefore, by the mathematical induction, we conclude that (3.3) holds for all $n \geq 0$, that is,

$$x_0 \preccurlyeq x_1 \preccurlyeq x_2 \preccurlyeq \cdots \preccurlyeq x_n \preccurlyeq_{n+1} \preccurlyeq \cdots \tag{3.10}$$

and

$$y_0 \succcurlyeq y_1 \succcurlyeq y_2 \succcurlyeq \cdots \succcurlyeq y_n \succcurlyeq y_{n+1} \succcurlyeq \cdots . \tag{3.11}$$

If there exists some integer $k \geq 0$ such that

$$G(x_{k+1}, x_{k+1}, x_k) + G(y_{k+1}, y_{k+1}, y_k) = 0,$$

then $G(x_{k+1},x_{k+1},x_k) = G(y_{k+1},y_{k+1},y_k) = 0$ implies that $x_k = x_{k+1}$ and $y_k = y_{k+1}$. Therefore, $x_k = F(x_k,y_k)$ and $y_k = F(y_k,x_k)$ gives that (x_k,y_k) is a coupled fixed point of F.

Now, we assume that $G(x_{n+1},x_{n+1},x_n) + G(y_{n+1},y_{n+1},y_n) \neq 0$ for all $n \geq 0$. Since $x_n \preccurlyeq x_{n+1}$ and $y_n \succcurlyeq y_{n+1}$ for all $n \geq 0$ so from (3.1) and (3.2), we have

$$\begin{aligned} &G(x_{n+1}, x_{n+1}, x_n) + G(y_{n+1}, y_{n+1}, y_n) \\ &\quad = G(F(x_n, y_n), F(x_n, y_n), F(x_{n-1}, y_{n-1})) + G(F(y_n, x_n), F(y_n, x_n), F(y_{n-1}, x_{n-1})) \\ &\quad \leq \theta(G(x_n, x_n, x_{n-1},), G(y_n, y_n, y_{n-1}))\left[G(x_n, x_n, x_{n-1},) + G(y_n, y_n, y_{n-1})\right] \end{aligned} \tag{3.12}$$

which implies that

$$G(x_{n+1}, x_{n+1}, x_n) + G(y_{n+1}, y_{n+1}, y_n) < G(x_n, x_n, x_{n-1},) + G(y_n, y_n, y_{n-1}). \tag{3.13}$$

Thus the sequence $\{G_{n+1} := G(x_{n+1}, x_{n+1}, x_n) + G(y_{n+1}, y_{n+1}, y_n)\}$ is monotone decreasing. It follows that $G_n \to g$ as $n \to \infty$ for some $g \geq 0$. Next, we claim that $g = 0$. Assume on contrary that $g > 0$, then from (3.12), we obtain

$$\frac{G(x_{n+1}, x_{n+1}, x_n) + G(y_{n+1}, y_{n+1}, y_n)}{G(x_n, x_n, x_{n-1}) + G(y_n, y_n, y_{n-1})} \leq \theta(G(x_n, x_n, x_{n-1}), G(y_n, y_n, y_{n-1})) < 1.$$

On taking limit as $n \to \infty$, we obtain

$$\theta(G(x_n, x_n, x_{n-1}), G(y_n, y_n, y_{n-1})) \to 1.$$

By property of function θ, we have $G(x_n, x_n, x_{n-1}) \to 0$, $G(y_n, y_n, y_{n-1}) \to 0$ as $n \to \infty$ and we have

$$G(x_n, x_n, x_{n-1}) + G(y_n, y_n, y_{n-1}) \to 0, \tag{3.14}$$

a contradiction. Therefore,

$$G(x_{n+1}, x_{n+1}, x_n) + G(y_{n+1}, y_{n+1}, y_n) \to 0.$$

Similarly, we can prove that

$$G'_{n+1} := G(x_{n+1}, x_n, x_n) + G(y_{n+1}, y_n, y_n) \to 0. \tag{3.15}$$

Next, we show that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences. On contrary, assume that at least one of $\{x_n\}$ or $\{y_n\}$ is not a Cauchy sequence. By Lemma 2.7, there is an $\epsilon > 0$ for which we can find subsequences $\{x_{n(k)}\}$, $\{x_{m(k)}\}$ of $\{x_n\}$ and $\{y_{n(k)}\}$, $\{y_{m(k)}\}$ of $\{y_n\}$ with $m(k) > n(k) \geq k$ such that

$$G(x_{n(k)}, x_{m(k)}, x_{m(k)}) + G(y_{n(k)}, y_{m(k)}, y_{m(k)}) \geq \varepsilon. \tag{3.16}$$

and

$$G(x_{n(k)-1}, x_{m(k)}, x_{m(k)}) + G(y_{n(k)-1}, y_{m(k)}, y_{m(k)}) < \varepsilon. \tag{3.17}$$

Using (3.16), (3.17) and the rectangle inequality, we have

$$\begin{aligned}
\varepsilon \leq r_k := G(x_{n(k)}, x_{m(k)}, x_{m(k)}) + G(y_{n(k)}, y_{m(k)}, y_{m(k)}) \\
&\leq G(x_{n(k)}, x_{n(k)-1}, x_{n(k)-1}) + G(x_{n(k)-1}, x_{m(k)}, x_{m(k)}) \\
&\quad + G(y_{n(k)}, y_{n(k)-1}, y_{n(k)-1}) + G(y_{n(k)-1}, y_{m(k)}, y_{m(k)}) \\
&< G(x_{n(k)}, x_{n(k)-1}, x_{n(k)-1}) + G(y_{n(k)-1}, y_{n(k)-1}, y_{n(k)-1}) + \varepsilon.
\end{aligned}$$

On taking limit as $k \to \infty$, we have

$$r_k = G(x_{n(k)}, x_{m(k)}, x_{m(k)}) + G(y_{n(k)}, y_{m(k)}, y_{m(k)}) \to \varepsilon. \tag{3.18}$$

By the rectangle inequality, we get

$$\begin{aligned}
r_k &= G(x_{n(k)}, x_{m(k)}, x_{m(k)}) + G(y_{n(k)}, y_{m(k)}, y_{m(k)}) \\
&\leq G(x_{n(k)}, x_{n(k)+1}, x_{n(k)+1}) + G(x_{n(k)+1}, x_{m(k)+1}, x_{m(k)+1}) + G(x_{m(k)+1}, x_{m(k)}, x_{m(k)}) \\
&\quad + G(y_{n(k)}, y_{n(k)+1}, y_{n(k)+1}) + G(y_{n(k)+1}, y_{m(k)+1}, y_{m(k)+1}) + G(y_{m(k)+1}, y_{m(k)}, y_{m(k)}) \\
&= \left[G(x_{n(k)+1}, x_{m(k)+1}, x_{m(k)+1}) + G(y_{n(k)+1}, y_{m(k)+1}, y_{m(k)+1})\right] \\
&\quad + \left[G(x_{n(k)}, x_{n(k)+1}, x_{n(k)+1}) + G(y_{n(k)}, y_{n(k)+1}, y_{n(k)+1})\right] \\
&\quad + \left[G(x_{m(k)+1}, x_{m(k)}, x_{m(k)}) + G(y_{m(k)+1}, y_{m(k)}, y_{m(k)})\right] \\
&= \left[G(x_{n(k)+1}, x_{m(k)+1}, x_{m(k)+1}) + G(y_{n(k)+1}, y_{m(k)+1}, y_{m(k)+1})\right] + G_{n(k)+1} + G'_{m(k)+1} \\
&= \left[G(x_{m(k)+1}, x_{m(k)+1}, x_{n(k)+1}) + G(y_{m(k)+1}, y_{m(k)+1}, y_{n(k)+1})\right] + G_{n(k)+1} + G'_{m(k)+1} \\
&= \left[G(F(x_{m(k)}, y_{m(k)}), F(x_{m(k)}, y_{m(k)}), F(x_{n(k)}, y_{n(k)}))\right. \\
&\quad \left. + G(F(y_{m(k)}, x_{m(k)}), F(y_{m(k)}, x_{m(k)}), F(y_{n(k)}, x_{n(k)}))\right] + G_{n(k)+1} + G'_{m(k)+1} \\
&\leq \theta(G(x_{m(k)}, x_{m(k)}, x_{n(k)}), G(y_{m(k)}, y_{m(k)}, y_{n(k)}))(G(x_{m(k)}, x_{m(k)}, x_{n(k)}) + G(y_{m(k)}, y_{m(k)}, y_{n(k)})) \\
&\quad + G_{n(k)+1} + G'_{m(k)+1} \\
&= \theta(G(x_{n(k)}, x_{m(k)}, x_{m(k)}), G(y_{n(k)}, y_{m(k)}, y_{m(k)}))r_k + G_{n(k)+1} + G'_{m(k)+1}.
\end{aligned}$$

Therefore, we have

$$r_k \leq \theta(G(x_{n(k)}, x_{m(k)}, x_{m(k)}), G(y_{n(k)}, y_{m(k)}, y_{m(k)}))r_k + G_{n(k)+1} + G'_{m(k)+1}.$$

This further implies that

$$\frac{r_k - G_{n(k)+1} - G'_{m(k)+1}}{r_k} \leq \theta(G(x_{n(k)}, x_{m(k)}, x_{m(k)}), G(y_{n(k)}, y_{m(k)}, y_{m(k)})) < 1.$$

On taking limit as $k \to \infty$ and using (3.14), (3.15) and (3.18), we obtain

$$\theta(G(x_{n(k)}, x_{m(k)}, x_{m(k)}), G(y_{n(k)}, y_{m(k)}, y_{m(k)})) \to 1.$$

Since $\theta \in \Theta$, we have $G(x_{n(k)}, x_{m(k)}, x_{m(k)}) \to 0$ and $G(y_{n(k)}, y_{m(k)}, y_{m(k)}) \to 0$, that is

$$G(x_{n(k)}, x_{m(k)}, x_{m(k)}) + G(y_{n(k)}, y_{m(k)}, y_{m(k)}) \to 0,$$

a contradiction. Therefore, $\{x_n\}$ and $\{y_n\}$ are G-Cauchy sequence. By G-completeness of X, there exists $x,y \in X$ such that $\{x_n\}$ and $\{y_n\}$ G-converges to x and y, respectively. Now, we show that F has a coupled fixed point. Since F is a continuous, taking $n \to \infty$ in (3.2), we get

$$x = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} F(x_n, y_n) = F\left(\lim_{n\to\infty} x_n, \lim_{n\to\infty} y_n\right) = F(x, y)$$

and

$$y = \lim_{n\to\infty} y_{n+1} = \lim_{n\to\infty} F(y_n, x_n) = F\left(\lim_{n\to\infty} y_n, \lim_{n\to\infty} x_n\right) = F(y, x).$$

Therefore, $x = F(x, y)$ and $y = F(y, x)$, that is, F has a coupled fixed point.

Theorem 3.2. *Let $(X, \preccurlyeq)$ be a partially ordered set such that there exists a complete G-metric on X and F: $X \times X \to X$ be a mapping having the mixed monotone property. Suppose that there exists $\theta \in \Theta$ such that*

$$\begin{aligned} &G(F(x,y), F(u,v), F(w,z)) + G(F(y,x), F(v,u), F(z,w)) \\ &\quad \le \theta(G(x,u,w), G(y,v,z))(G(x,u,w) + G(y,v,z)) \end{aligned} \tag{3.19}$$

for all x, y, z, u, v, $w \in X$ for which $x \succcurlyeq u \succcurlyeq w$ and $y \preccurlyeq v \preccurlyeq z$ where either $u \ne w$ or $v \ne z$. If there exists $x_0,y_0 \in X$ such that

$$x_0 \preccurlyeq F(x_0, y_0) \ \text{ and } y_0 \succcurlyeq F(y_0, x_0)$$

and X has the following property:

(i) *if a non-decreasing sequence $\{x_n\} \to x$, then $x_n \preccurlyeq x$ for all $n \in \mathbb{N}$,*
(ii) *if a non-increasing sequence $\{y_n\} \to y$, then $y_n \succcurlyeq y$ for all $n \in \mathbb{N}$,*

then F has a coupled fixed point.

Proof. Following arguments similar to those given in Theorem 3.1, we obtain a non-decreasing sequence $\{x_n\}$ converges to x and a non-increasing sequence $\{y_n\}$ converges to y for some $x,y \in X$. By using (i) and (ii), we have $x_n \preccurlyeq x$ and $y_n \succcurlyeq y$ for all n.

If $x_n = x$ and $y_n = y$ for some $n \ge 0$, then, by construction, $x_{n+1} = x$ and $y_{n+1} = y$. Thus (x, y) is a coupled fixed point of F. So we may assume either $x_n \ne x$ or $y_n \ne y$, for all $n \ge 0$. Then by the rectangle inequality, we obtain

$$\begin{aligned} &G(F(x,y), x, x) + G(F(y,x), y, y) \\ &\le G(F(x,y), F(x_n,y_n), F(x_n,y_n)) + G(F(x_n,y_n), x, x) \\ &\quad + G(F(y,x), F(y_n,x_n), F(y_n,x_n)) + G(F(y_n,x_n), y, y) \\ &= G(F(x_n,y_n), F(x_n,y_n), F(x,y)) + G(x_{n+1}, x, x) \\ &\quad + G(F(y_n,x_n), F(y_n,x_n), F(y,x)) + G(y_{n+1}, y, y) \\ &= G(F(y_n,x_n), F(y_n,x_n), F(y,x)) + G(F(x_n,y_n), F(x_n,y_n), F(x,y)) \\ &\quad + G(x_{n+1}, x, x) + G(y_{n+1}, y, y) \\ &\le \theta(G(y_n,y_n,y) + G(x_n,x_n,x))(G(y_n,y_n,y) + G(x_n,x_n,x)) \\ &\quad + G(x_{n+1}, x, x) + G(y_{n+1}, y, y) \\ &\le (G(y_n,y_n,y) + G(x_n,x_n,x)) + G(x_{n+1}, x, x) + G(y_{n+1}, y, y). \end{aligned}$$

On taking limit as $n \to \infty$, we have $G(F(x,y),x,x) + G(F(y,x),y,y) = 0$. Thus $x = F(x,y)$ and $y = F(x, y)$ and so (x, y) is a coupled fixed point of F.

Corollary 3.3. *Let $(X, \preccurlyeq)$ be a partially ordered set such that there exists a complete G-metric on X and F: $X \times X \to X$ be a mapping having the mixed monotone property. Suppose that there exists $\eta \in \Theta$ such that*

$$G(F(x,y),F(u,v),F(w,z)) \leq \frac{1}{2}\eta(G(x,u,w),G(y,v,z))(G(x,u,w)+G(y,v,z)) \quad (3.20)$$

for all $x, y, z, u, v, w \in X$ for which $x \succcurlyeq u \succcurlyeq w$ and $y \preccurlyeq v \preccurlyeq z$, where either $u \neq w$ or $v \neq z$. If there exists $x_0, y_0 \in X$ such that

$$x_0 \preccurlyeq F(x_0,y_0) \text{ and } y_0 \succcurlyeq F(y_0,x_0)$$

and either

(*a*) *F is continuous or*
(*b*) *X has the following property:*
  (i) *if a non-decreasing sequence $\{x_n\} \to x$, then $x_n \preccurlyeq x$ for all $n \in \mathbb{N}$,*
  (ii) *if a non-increasing sequence $\{y_n\} \to y$, then $y_n \succcurlyeq y$ for all $n \in \mathbb{N}$,*

then F has a coupled fixed point.

Proof. For $x,y,z,u,v,w \in X$ with $x \succcurlyeq u \succcurlyeq w$ and $y \preccurlyeq v \preccurlyeq z$, where either $u \neq w$ or $v \neq z$, from (3.20), we have

$$G(F(x,y),F(u,v),F(w,z)) \leq \frac{1}{2}\eta(G(x,u,w),G(y,v,z))(G(x,u,w)+G(y,v,z)) \quad (3.21)$$

and

$$\begin{aligned} G(F(y,x),F(v,u),F(z,w)) &= G(F(z,w),F(v,u),F(y,x)) \\ &\leq \frac{1}{2}\eta(G(z,v,y),G(w,u,x))(G(z,v,y)+G(w,u,x)) \\ &= \frac{1}{2}\eta(G(y,v,z),G(x,u,w))(G(x,u,w)+G(y,v,z)). \end{aligned} \quad (3.22)$$

From (3.21) and (3.22), we have

$$\begin{aligned} &G(F(x,y),F(u,v),F(w,z)) + G(F(y,x),F(v,u),F(z,w)) \\ &\quad \leq \frac{1}{2}\left[\eta(G(x,u,w),G(y,v,z)) + \eta(G(y,v,z),G(x,u,w))\right](G(x,u,w)+G(y,v,z)) \end{aligned} \quad (3.23)$$

$$= \theta(G(x,u,w),G(y,v,z)))(G(x,u,w)+G(y,v,z)) \quad (3.24)$$

for $x,y,z,u,v,w \in X$ with $x \succcurlyeq u \succcurlyeq w$ and $y \preccurlyeq v \preccurlyeq z$ where either $u \neq w$ or $v \neq z$, where

$$\theta(t_1,t_2) = \frac{1}{2}\left[\eta(t_1,t_2)+\eta(t_2,t_1)\right]$$

for all $t_1,t_2 \in [0, \infty)$. It is easy to verify that $\theta \in \Theta$ and we can apply Theorems 3.1 and 3.2. Hence F has a coupled fixed point.

Corollary 3.4. [[35], Theorems 3.1 and 3.2] *Let $(X,\preccurlyeq)$ be a partially ordered set such that there exists a complete G-metric on X and F: $X \times X \to X$ be a mapping having the mixed monotone property. Suppose that there exists a $k \in [0,1)$ such that*

$$G(F(x,y),F(u,v),F(w,z)) \le \frac{k}{2}(G(x,u,w)+G(y,v,z))$$

for all x, y, z, u, v, w ∈ *X for which* $x \succcurlyeq u \succcurlyeq w$ *and* $y \preccurlyeq v \preccurlyeq z$, *where either* $u \neq w$ *or* $v \neq z$. *If there exists* x_0, $y_0 \in X$ *such that*

$$x_0 \preccurlyeq F(x_0,y_0) \ \text{and}\ y_0 \succcurlyeq F(y_0,x_0)$$

and either

(*a*) *F is continuous or*
(*b*) *X has the following property:*
(i) *if a non-decreasing sequence* $\{x_n\} \to x$, *then* $x_n \preccurlyeq x$ *for all* $n \in \mathbb{N}$,
(ii) *if a non-increasing sequence* $\{y_n\} \to y$, *then* $y_n \succcurlyeq y$ *for all* $n \in \mathbb{N}$,

then F has a coupled fixed point.

Proof. Taking $\eta(t_1, t_2) = k$ with $k \in [0,1)$ for all $t_1, t_2 \in [0, \infty)$ in Theorems 3.1 and 3.2, result follows immediately.

Let Ω denotes the class of those functions ω: $[0, \infty) \to [0,1)$ which satisfies the condition: For any sequences $\{t_n\}$ of nonnegative real numbers, $\omega(t_n) \to 1$ implies $t_n \to 0$.

Theorem 3.5. *Let* $(X, \preccurlyeq)$ *be a partially ordered set such that there exists a complete G-metric on X and F*: $X \times X \to X$ *be a mapping having the mixed monotone property. Suppose that there exists* $\omega \in \Omega$ *such that*

$$\begin{aligned} &G(F(x,y),F(u,v),F(w,z)) + G(F(y,x),F(v,u),F(z,w)) \\ &\quad \le \omega(G(x,u,w)+G(y,v,z))(G(x,u,w)+G(y,v,z)) \end{aligned} \tag{3.25}$$

for all x, y, z, u, v, w ∈ *X for which* $x \succcurlyeq u \succcurlyeq w$ *and* $y \preccurlyeq v \preccurlyeq z$ *where either* $u \neq w$ *or* $v \neq z$. *If there exists* $x_0,y_0 \in X$ *such that*

$$x_0 \preccurlyeq F(x_0,y_0) \ \text{and}\ y_0 \succcurlyeq F(y_0,x_0)$$

and either

(*a*) *F is continuous or*
(*b*) *X has the following property:*
(i) *if a non-decreasing sequence* $\{x_n\} \to x$, *then* $x_n \preccurlyeq x$ *for all* $n \in \mathbb{N}$,
(ii) *if a non-increasing sequence* $\{y_n\} \to y$, *then* $y_n \succcurlyeq y$ *for all* $n \in \mathbb{N}$,

then F has a coupled fixed point.

Proof. Taking $\theta(t_1,t_2) = \omega(t_1 + t_2)$ for all $t_1,t_2 \in [0,\infty)$ in Theorems 3.1 and 3.2, result follows.

Taking $\omega(t) = k$ with $k \in [0,1)$ for all $t \in [0, \infty)$ in Theorem 3.5, we obtain the following corollary.

Corollary 3.6. *Let* $(X, \preccurlyeq)$ *be a partially ordered set such that there exists a complete G-metric on X and F*: $X \times X \to X$ *be a mapping having the mixed monotone property. Suppose that there exists* ∈ [0,1) *such that*

$$\begin{aligned} &G(F(x,y),F(u,v),F(w,z)) + G(F(y,x),F(v,u),F(z,w)) \\ &\quad \le k(G(x,u,w)+G(y,v,z)) \end{aligned}$$

for all x, y, z, u, v, w $\in X$ *for which* $x \succcurlyeq u \succcurlyeq w$ *and* $y \preccurlyeq v \preccurlyeq z$ *where either* $u \neq w$ *or* $v \neq z$. *If there exists* $x_0, y_0 \in X$ *such that*

$$x_0 \preccurlyeq F(x_0, y_0) \text{ and } y_0 \succcurlyeq F(y_0, x_0)$$

and either

(a) *F is continuous or*
(b) *X has the following property:*
(i) *if a non-decreasing sequence* $\{x_n\} \to x$, *then* $x_n \preccurlyeq x$ *for all* $n \in \mathbb{N}$,
(ii) *if a non-increasing sequence* $\{y_n\} \to y$, *then* $y_n \succcurlyeq y$ *for all* $n \in \mathbb{N}$,

then F has a coupled fixed point.

Theorem 3.7. *Let* $(X, \preccurlyeq)$ *be a partially ordered set such that there exists a complete G-metric on X and F*: $X \times X \to X$ *be a mapping having the mixed monotone property and* $F(x,y) \preccurlyeq F(y,x)$, *whenever* $x \preccurlyeq y$. *Suppose that there exists* $\theta \in \Theta$ *such that*

$$\begin{aligned} &G(F(x,y), F(u,v), F(w,z)) + G(F(y,x), F(v,u), F(z,w)) \\ &\quad \leq \theta(G(x,u,w), G(y,v,z))(G(x,u,w) + G(y,v,z)) \end{aligned} \tag{3.26}$$

for all x, y, z, u, v, w $\in X$ *for which* $w \preccurlyeq u \preccurlyeq x \preccurlyeq y \preccurlyeq v \preccurlyeq z$, *where either* $u \neq w$ *or* $v \neq z$. *If there exists* $x_0, y_0 \in X$ *such that*

$$x_0 \preccurlyeq y_0, \quad x_0 \preccurlyeq F(x_0, y_0) \text{ and } y_0 \succcurlyeq F(y_0, x_0)$$

and either

(a) *F is continuous or*
(b) *X has the following property:*
(i) *if a non-decreasing sequence* $\{x_n\} \to x$, *then* $x_n \preccurlyeq x$ *for all* $n \in \mathbb{N}$,
(ii) *if a non-increasing sequence* $\{y_n\} \to y$, *then* $y_n \succcurlyeq y$ *for all* $n \in \mathbb{N}$,

then F has a coupled fixed point.

Proof. By given hypothesis, there exist $x_0, y_0 \in X$ such that

$$x_0 \preccurlyeq F(x_0, y_0) \text{ and } y_0 \succcurlyeq F(y_0, x_0).$$

We define $x_1, y_1 \in X$ by

$$x_1 = F(x_0, y_0) \succcurlyeq x_0 \text{ and } y_1 = F(y_0, x_0) \preccurlyeq y_0.$$

Since $x_0 \preccurlyeq y_0$, by given assumptions, we have $F(x_0,y_0) \preccurlyeq F(y_0,x_0)$. Hence

$$x_0 \preccurlyeq x_1 = F(x_0, y_0) \preccurlyeq F(y_0, x_0) = y_1 \preccurlyeq y_0.$$

Continuing the above process, we have two sequences $\{x_n\}$ and $\{y_n\}$ such that

$$x_{n+1} = F(x_n, y_n), \quad y_{n+1} = F(y_n, x_n)$$

and

$$x_n \preccurlyeq x_{n+1} = F(x_n, y_n) \preccurlyeq F(y_n, x_n) = y_{n+1} \preccurlyeq y_n$$

for all $n \geq 0$. If there is $k \in \mathbb{N}$ such that $x_k = y_k = \alpha$ (say), then we have

$$\alpha \preccurlyeq F(\alpha, \alpha) \preccurlyeq F(\alpha, \alpha) \preccurlyeq \alpha,$$

that is, $\alpha = F(\alpha, \alpha)$. Therefore, (α, α) is a coupled fixed point of F. Next, assume that

$$x_n \prec y_n \tag{3.27}$$

for all $n \in \mathbb{N}$. Further, using similar arguments as stated in Theorem 3.1, we may assume that $(x_n,y_n) \neq (x_{n+1},y_{n+1})$. Then, in view of (3.27), for all $n \geq 0$, the inequality (3.26) holds with

$$x = x_{n+2}, u = x_{n+1}, w = x_n, y = y_n, v = y_{n+1} \text{ and } z = y_{n+2}.$$

The rest of the proof follows by following the same steps as given in Theorem 3.1 for case (a). For case (b), we follow the same steps as given in Theorem 3.2.

Example 3.8. Let $X = \mathbb{N} \cup \{0\}$ and G: $X \times X \times X \rightarrow X$ be define by

$$G(x, y, z) = \begin{cases} x + y + z \text{ ; if } x, y, z \text{ are all distinct and different from zero,} \\ x + z \quad \text{; if } x = y \neq z \text{ and are all different from zero,} \\ y + z + 1 \text{ ; if } x = 0, y \neq z \text{ and } y, z \text{ different from zero,} \\ y + 2 \quad \text{; if } x = 0, y = z \neq 0, \\ z + 1 \quad \text{; if } x = y = 0, z \neq 0, \\ 0 \quad \text{; if } x = y = z. \end{cases}$$

Then X is a complete G-metric space. Let partial order $\preccurlyeq$ on X be defined as follows: For $x,y \in X$,

$$x \preccurlyeq y \text{ holds if } x > y \text{ and 3 divides } (x - y) \text{ and } 3 \preccurlyeq 1 \text{ and } 0 \preccurlyeq 1 \text{ hold.}$$

Let F: $X \times X \rightarrow X$ be defined by

$$F(x, y) = \begin{cases} 1 \text{ ; if } x \prec y, \\ 0 \text{ ; if otherwise.} \end{cases}$$

If $w \preccurlyeq u \preccurlyeq x \preccurlyeq y \preccurlyeq v \preccurlyeq z$ holds, then we have $w \geq u \geq x > y \geq v \geq z$. Therefore $F(x,y) = F(u,v) = F(w,z) = 1$ and $F(y,x) = F(v,u) = F(z,w) = 0$. So the left side of (3.26) becomes

$$G(1, 1, 1) + G(0, 0, 0) = 0$$

and (3.26) is satisfied for all $\theta \in \Theta$. Thus Theorem 3.7 is applicable to this example with $x_0 = 0$ and $y_0 = 81$. Moreover, F has coupled fixed points (0,0) and (1,0).

Remark 3.9. A G-metric naturally induces a metric d_G given by $d_G(x,y) = G(x,y,y) + G(x,x,y)$ [25]. From the condition that either $u \neq w$ or $v \neq z$, the inequality (3.1), (3.19), (3.25) and (3.26) do not reduce to any metric inequality with the metric d_G. Therefore, the corresponding metric space (X, d_G) results are not applicable to Example 3.8.

Remark 3.10. Example 3.8 is not supported by Theorems 3.1, 3.2 and 3.5. This is evident by the fact that the inequality (3.1), (3.19) and (3.25) are not satisfied when $w = u = x = y = 3$, $v = 0$ and $z = 1$. Moreover, the coupled fixed point is not unique.

4. Uniqueness of coupled fixed point in *G*-metric spaces

In this section, we study necessary conditions to obtain the uniqueness of a coupled fixed point in the setting of partially ordered G-metric spaces. If $(X, \preccurlyeq)$ is a partially ordered set, then we endow the product of $X \times X$ with the following partial order:

For $(x, y), (u, v) \in X \times X$, $(x, y) \trianglelefteq (u, v)$ if and only if $x \preccurlyeq u$ and $y \succcurlyeq v$.

Theorem 4.1. *In addition to the hypotheses in Theorem 3.1, suppose that for every $(x, y), (z, t) \in X \times X$, there exists a point $(u,v) \in X \times X$ that is comparable to (x,y) and (z, t). Then F has a unique coupled fixed point.*

Proof. From Theorem 3.1, F has a coupled fixed points. Suppose (x, y) and (z, t) are coupled fixed points of F, that is, $x = F(x,y), y = F(y,x), z = F(z,t)$ and $t = F(t,z)$. Next, we claim that $x = z$ and $y = t$. By given hypothesis, there exists $(u,v) \in X \times X$ that is comparable to (x,y) and (z,t). We put $u_0 = u$ and $v_0 = v$ and construct sequences $\{u_n\}$ and $\{v_n\}$ by

$$u_n = F(u_{n-1}, v_{n-1}) \text{ and } v_n = F(v_{n-1}, u_{n-1}) \text{ for all } n \in \mathbb{N}.$$

Since (u,v) is comparable with (x,y), we assume that $(u_0,v_0) = (u,v) \trianglelefteq (x,y)$. Using the mathematical induction, it is straight forward to prove that

$$(u_n, v_n) \trianglelefteq (x, y) \text{ for all } n \in \mathbb{N}.$$

From (3.1), we have

$$\begin{aligned} G(x, x, u_n) + G(y, y, v_n) &= G(F(x, y), F(x, y), F(u_{n-1}, v_{n-1})) + G(F(y, x), F(y, x), F(v_{n-1}, u_{n-1})) \\ &\le \theta(G(x, x, u_{n-1}), G(v_{n-1}, y, y))[G(x, x, u_{n-1}) + G(v_{n-1}, y, y)] \\ &< G(x, x, u_{n-1}) + G(v_{n-1}, y, y). \end{aligned} \tag{4.1}$$

Consequently, sequence $\{G(x,x,u_n) + G(y,y,v_n)\}$ is non-negative and decreasing, so

$$G(x, x, u_n) + G(y, y, v_n) \to g,$$

for some $g \geq 0$. We claim that $g = 0$. Indeed, if $g > 0$ then following similar arguments to those given in the proof of Theorem 3.1, we conclude that

$$\theta(G(x, x, u_{n-1}), G(u_{n-1}, y, y)) \to 1.$$

Since $\theta \in \Theta$, we obtain $G(x,x,u_{n-1}) \to 0$ and $G(v_{n-1},y,y) \to 0$. Therefore,

$$G(x, x, u_{n-1}) + G(v_{n-1}, y, y) \to 0$$

which is a contradiction. Hence

$$G(x, x, u_n) + G(v_n, y, y) \to 0. \tag{4.2}$$

Similarly, one can prove that

$$G(x, u_n, u_n) + G(v_n, v_n, y) \to 0, \tag{4.3}$$

$$G(z, z, u_n) + G(v_n, t, t) \to 0, \tag{4.4}$$

and

$$G(z, u_n, u_n) + G(t, v_n, v_n) \to 0. \tag{4.5}$$

From rectangular inequality, we have

$$G(z, x, x) \le G(z, u_n, u_n) + G(u_n, x, x) \tag{4.6}$$

and

$$G(y, t, t) \le G(y, v_n, v_n) + G(v_n, t, t). \tag{4.7}$$

Combine (4.6) and (4.7), we have

$$\begin{aligned} G(z,x,x)+G(y,t,t) &\le (G(z,u_n,u_n)+G(u_n,x,x))+(G(y,v_n,v_n)+G(v_n,t,t)) \\ &\le (G(x,x,u_n)+G(v_n,y,y))+(G(x,u_n,u_n)+G(v_n,v_n,y)) \\ &\quad +(G(z,z,u_n)+G(v_n,t,t))+(G(z,u_n,u_n)+G(t,v_n,v_n)) \end{aligned}$$

Taking $n \to \infty$, by (4.2), (4.3), (4.4) and (4.5), we have $G(z,x,x) + G(y,t,t) \le 0$. So $G(z,x,x) = 0$ and $G(y,t,t) = 0$, that is, $z = x$ and $y = t$. Therefore, F has a unique coupled fixed point. This completes the proof.

Theorem 4.2. *In addition to the hypotheses in Theorem 3.2, suppose that for every* (x, y), $(z, t) \in X \times X$, *there exists a point* $(u,v) \in X \times X$ *that is comparable to* (x,y) and (z, t). *Then F has a unique coupled fixed point*.

Proof. Proof is similar to the one given in Theorems 4.1 and 3.2.

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Authors' contributions

All authors read and approved the final manuscript.

Competing interests

The authors declare that they have no competing interests.

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