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A fixed point approach to the stability of an AQ-functional equation on β-Banach modules

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Abstract

Using the fixed point method, we prove the Hyers-Ulam stability of the following mixed additive and quadratic functional equation $f(kx + y) + f(kx - y) = f(x + y) + f(x - y) + (k - 1)[(k + 2) f(x) + kf(-x)]$ ($k \in \mathbb{N}$, $k \neq 1$) in β-Banach modules on a Banach algebra.

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1 Introduction

The study of stability problems for functional equations is related to a question of Ulam [1] concerning the stability of group homomorphisms and affirmatively answered for Banach spaces by Hyers [2]. The result of Hyers was generalized by Aoki [3] for approximate additive mappings and by Rassias [4] for approximate linear mappings by allowing the Cauchy difference operator $CDf(x, y) = f(x + y) - [f(x) + f(y)]$ to be controlled by $\epsilon(\|x\|^p + \|y\|^p)$. In 1994, a further generalization was obtained by Găvruța [5], who replaced $\epsilon(\|x\|^p + \|y\|^p)$ by a general control function $\phi(x,y)$. Rassias [6,7] treated the Ulam-Gavruta-Rassias stability on linear and nonlinear mappings and generalized Hyers result. The reader is referred to the following books and research articles which provide an extensive account of progress made on Ulam's problem during the last seventy years (cf. [8-33]).

The functional equation

$$f(x + y) + f(x - y) = 2f(x) + 2f(y) \tag{1.1}$$

is related to a symmetric biadditive function [15]. It is natural that such equation is called a quadratic functional equation. In particular, every solution of the quadratic Equation (1.1) is said to be a quadratic function. It is well known that a function f between real vector spaces is quadratic if and only if there exists a unique symmetric biadditive function B such that $f(x) = B(x,x)$ for all x (see [15]). The biadditive function B is given by $B(x, y) = \frac{1}{4}\left(f(x + y) + f(x - y)\right)$. In [34], Czerwik proved the Hyers-Ulam stability of the quadratic functional Equation (1.1). A Hyers-Ulam stability problem for the quadratic functional Equation (1.1) was proved by Skof for functions $f : E_1 \to E_2$, where E_1 is a normed space and E_2 a Banach space (see [35]). Cholewa [36] noticed that the theorem of Skof is still true if the relevant domain E_1 is replaced by an Abelian group. Grabiec in [37] has generalized the above mentioned results. Park

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and Rassias proved the Hyers-Ulam stability of generalized Apollonius type quadratic functional equation (see [18]). The quadratic functional equation and several other functional equations are useful to characterize inner product spaces (cf. [8,24,28,29,38]).

Now we consider a mapping $f : X \to Y$ satisfies the following additive-quadratic (AQ) functional equation, which is introduced by Eskandani et al. (see [11]),

$$f(kx+y)+f(kx-y)=f(x+y)+f(x-y)+(k-1)[(k+2)f(x)+kf(-x)] \tag{1.2}$$

for a fixed integer with $k \geq 2$. It is easy to see that the function $f(x) = ax^2 + bx$ is a solution of the functional Equation (1.2). The main purpose of this article is to prove the Hyers-Ulam stability of an AQ-functional Equation (1.2) in β-normed left Banach modules on Banach algebras using the fixed point method.

2 Preliminaries

Let β be a real number with $0 < \beta \leq 1$ and let $\mathbb{K}$ denotes either $\mathbb{R}$ or $\mathbb{C}$. Let X be a linear space over $\mathbb{K}$. A real-valued function $\|\cdot\|_\beta$ is called a β-norm on X if and only if it satisfies

($\beta N1$) $\|x\|_\beta = 0$ if and only if $x = 0$;

($\beta N2$) $\|\lambda x\|_\beta = |\lambda|^\beta \cdot \|x\|_\beta$ for all $\lambda \in \mathbb{K}$ and all $x \in X$;

($\beta N3$) $\|x + y\|_\beta \leq \|x\|_\beta + \|y\|_\beta$ for all $x, y \in X$.

The pair $(X, \|\cdot\|_\beta)$ is called a β-normed space (see [39]). A β-Banach space is a complete β-normed space.

For explicitly later use, we recall the following result by Diaz and Margolis [40].

Theorem 2.1 *Let* (Ω, d) *be a complete generalized metric space and* $J : \Omega \to \Omega$ *be a strictly contractive mapping with Lipschitz constant* $L < 1$, *that is*

$$d(Jx, Jy) \leq Ld(x, y), \quad \forall x, y \in \Omega.$$

Then, for each given $x \in \Omega$, *either*

$$d(J^n x, J^{n+1} x) = \infty, \quad \forall n \geq 0,$$

or there exists a non-negative integer n_0 *such that*

(1) $d(J^n x, J^{n+1} x) < \infty$ *for all* $n \geq n_0$;

(2) *the sequence* $\{J^n x\}$ *is converges to a fixed point* y^* *of* J;

(3) y^* *is the unique fixed point of* J *in the set* $\Omega^* = \left\{y \in \Omega | d\left(J^{n_0} x, y\right) < \infty\right\}$;

(4) $d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$*for all* $y \in \Omega^*$.

The following Lemma 2.2 and Theorem 2.3 about solutions of Equation (1.2) have been proved in [11].

Lemma 2.2 (1) *If an odd mapping* $f : X \to Y$ *satisfies* (1.2) *for all* $x, y \in X$, *then* f *is additive.*

(2) *If an even mapping* $f : X \to Y$ *satisfies* (1.2) *for all* $x, y \in X$, *then* f *is quadratic.*

Theorem 2.3 *A mapping* $f : X \to Y$ *satisfies* (1.2) *for all* $x, y \in X$ *if and only if there exist a symmetric bi-additive mapping* $B : X \times X \to Y$ *and an additive mapping* $A: X \to Y$ *such that* $f(x) = B(x, x) + A(x)$ *for all* $x \in X$.

3 Main results

Throughout this section, let B be a unital Banach algebra with norm $|\cdot|$, $B_1 := \{b \in B | |b| = 1\}$, X be a β-normed left B-module and Y be a β-normed left Banach B-module, and let $k \in \mathbb{N}$, $k \neq 1$ be a fixed integer. For a given mapping $f : X \to Y$, we define the difference operators

$$D_b f(x, y) := f(kbx+by)+f(kbx-by)-bf(x+y)-bf(x-y)-(k-1)b[(k+2)f(x)+kf(-x)]$$

and

$$\tilde{D}_b f(x, y) := f(kbx+by)+f(kbx-by)-b^2 f(x+y)-b^2 f(x-y)-(k-1)b^2[(k+2)f(x)+kf(-x)]$$

for *all* $x,y \in X$ *and* $b \in B_1$.

Theorem 3.1 *Let* $\phi : X^2 \to [0, \infty)$ *be a function such that*

$$\lim_{n\to\infty} \frac{1}{k^{n\beta}} \varphi(k^n x, k^n y) = 0 \tag{3.1}$$

for all $x, y \in X$. *Let* $f : X \to Y$ *be an odd mapping such that*

$$\left\| D_b f(x, y) \right\|_\beta \leq \varphi(x, y) \tag{3.2}$$

for all $x,y \in X$ *and all* $b \in B_1$. *If there exists a Lipschitz constant* $0 < L < 1$ *such that*

$$\varphi(kx, 0) \leq k^\beta L \varphi(x, 0) \tag{3.3}$$

for all $x \in X$, *then there exists a unique additive mapping* $A: X \to Y$ *such that*

$$\left\| f(x) - A(x) \right\|_\beta \leq \frac{1}{(2k)^\beta (1 - L)} \varphi(x, 0) \tag{3.4}$$

for all $x \in X$. *Moreover, if* $f(tx)$ *is continuous in* $t \in \mathbb{R}$ *for each fixed* $x \in X$, *then* A *is B-linear, i.e.,* $A(bx) = bA(x)$ *for all* $x \in X$ *and all* $b \in B$.

Proof Letting $b = 1$ and $y = 0$ in (3.2), we get

$$\left\| f(kx) - kf(x) \right\|_\beta \leq \frac{1}{2^\beta} \varphi(x, 0) \tag{3.5}$$

for all $x \in X$. Consider the set $\Omega := \{g \mid g : X \to Y, g(0) = 0\}$ and introduce the generalized metric on Ω:

$$d(g, h) = \inf\{C \in (0, \infty) | \quad \left\| g(x) - h(x) \right\|_\beta \leq C\varphi(x, 0), \quad \forall x \in X\}. \tag{3.6}$$

It is easy to show that (Ω, d) is a complete generalized metric space (see [10, Theorem 2.5]). We now define a function $J : \Omega \to \Omega$ by

$$(Jg)(x) = \frac{1}{k} g(kx), \quad \forall g \in \Omega, x \in X. \tag{3.7}$$

Let $g, h \in \Omega$ and $C \in [0, \infty]$ be an arbitrary constant with $d(g, h) < C$, by the definition of d, it follows

$$\left\| g(x) - h(x) \right\|_\beta \leq C\varphi(x, 0), \quad \forall x \in X. \tag{3.8}$$

By the given hypothesis and the last inequality, one has

$$\left\| \frac{1}{k} g(kx) - \frac{1}{k} h(kx) \right\|_\beta \leq CL\varphi(x, 0), \quad \forall x \in X. \tag{3.9}$$

Hence, it holds that $d(Jg, Jh) < Ld(g, h)$. It follows from (3.5) that $d(Jf, f) < 1/(2k)^\beta < \infty$. Therefore, by Theorem 2.1, J has a unique fixed point $A : X \to Y$ in the set $\Omega^* = \{g \in$

$\Omega \mid d\ (f, g) < \infty\}$ such that

$$A(x) := \lim_{n\to\infty} (J^n f)(x) = \lim_{n\to\infty} \frac{1}{k^n} f(k^n x) \tag{3.10}$$

and $A(kx) = kA(x)$ for all $x \in X$:Also,

$$d(A, f) \le \frac{1}{1-L} d(Jf, f) \le \frac{1}{(2k)^\beta (1-L)}. \tag{3.11}$$

This means that (3.4) holds for all $x \in X$.

Now we show that A is additive. By (3.1), (3.2), and (3.10), we have

$$\begin{aligned} \left\|D_1 A(x, y)\right\|_\beta &= \lim_{n\to\infty} \left\| \frac{1}{k^n} D_1 f(k^n x, k^n y) \right\|_\beta \\ &= \lim_{n\to\infty} \frac{1}{k^{n\beta}} \left\|D_1 f(k^n x, k^n y)\right\|_\beta \\ &\le \lim_{n\to\infty} \frac{1}{k^{n\beta}} \varphi(k^n x, k^n y) = 0 \end{aligned}$$

that is,

$$A(kx + y) + A(kx - y) = A(x + y) + A(x - y) + (k - 1)[(k + 2)A(x) + kA(-x)]$$

for all $x, y \in X$. Therefore by Lemma 2.2, we get that the mapping A is additive.

Moreover, if $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then by the same reasoning as in the proof of [4]A is $\mathbb{R}$-linear. Letting $y = 0$ in (3.2), we get

$$\left\|2f(kbx) - 2kbf(x)\right\|_\beta \le \varphi(x, 0) \tag{3.12}$$

for all $x \in X$ and all $b \in B_1$. By definition of A, (3.1) and (3.12), we obtain

$$\begin{aligned} \left\|2A(kbx) - 2kbA(x)\right\|_\beta &= \lim_{n\to\infty} \frac{1}{k^{n\beta}} \left\|2f(k^{n+1} bx) - 2kbf(k^n x)\right\|_\beta \\ &\le \lim_{n\to\infty} \frac{1}{k^{n\beta}} \varphi(k^n x, 0) = 0 \end{aligned}$$

for all $x \in X$ and all $b \in B_1$. So $A\ (kbx)$ - $kbA\ (x) = 0$ for all $x \in X$ and all $b \in B_1$. Since A is additive, we get $A(bx) = bA(x)$ for all $x \in X$ and all $b \in B_1 \cup \{0\}$. Now, let $a \in B\backslash\{0\}$. Since A is $\mathbb{R}$-linear,

$$A(bx) = A\left(|b| \cdot \frac{b}{|b|} x\right) = |b|\, A\left(\frac{b}{|b|} x\right) = |b| \cdot \frac{b}{|b|} A(x) = bA(x)$$

for all $x \in X$ and all $b \in B$. This proves that A is B-linear.

Corollary 3.2 *Let* $0 < r < 1$ *and* δ, θ *be non-negative real numbers, and let* $f : X \to Y$ *be an odd mapping for which*

$$\left\|D_b f(x, y)\right\|_\beta \le \delta + \theta(\|x\|_\beta^r + \left\|y\right\|_\beta^r) \tag{3.13}$$

for all $x, y \in X$ *and* $b \in B_1$. *Then there exists a unique additive mapping* $A : X \to Y$ *such that*

$$||f(x) - A(x)||_\beta \le \frac{1}{2^\beta(k^\beta - k^{\beta r})}\delta + \frac{1}{2^\beta(k^\beta - k^{\beta r})}\theta||x||_\beta^r$$

for all $x \in X$. Moreover, if $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then A is B-linear.

Proof The proof follows from Theorem 3.1 by taking $\varphi(x,y) = \delta + \theta\left(\|x\|_\beta^r + \|y\|_\beta^r\right)$ for all $x, y \in X$. We can choose $L = k^{\beta(r-1)}$ to get the desired result.

The Hyers-Ulam stability for the case of $r = 1$ was excluded in Corollary 3.2. In fact, the functional Equation (1.2) is not stable for $r = 1$ in (3.13) as we shall see in the following example, which is a modification of the example of Gajda [41] for the additive functional inequality (see also [20]).

Example 3.3 Let $\varphi : \mathbb{C} \to \mathbb{C}$ be defined by

$$\phi(x) = \begin{cases} x, \text{for } |x| < 1, \\ 1, \text{ for } |x| \geq 1. \end{cases}$$

Consider the function $f : \mathbb{C} \to \mathbb{C}$ be defined by

$$f(x) = \sum_{m=0}^{\infty} \alpha^{-m} \phi(\alpha^m x)$$

for all $x \in \mathbb{C}$, where $\alpha > k$. Let

$$D_\mu f(x,y) := f(k\mu x + \mu y) + f(k\mu x - \mu y) - \mu f(x+y) - \mu f(x-y) - (k-1)\mu[(k+2)f(x) + kf(-x)]$$

for all $x, y \in \mathbb{C}$ and $\mu \in \mathbb{T} := \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$. Then f satisfies the functional inequality

$$|D_\mu f(x,y)| \leq \frac{2\alpha^2(k^2+1)}{\alpha - 1}(|x| + |y|) \tag{3.14}$$

for all $x, y \in \mathbb{C}$, but there do not exist an additive function $A : \mathbb{C} \to \mathbb{C}$ and a constant $d > 0$ such that $|f(x) - A(x)| < d\,|x|$ for all $x \in \mathbb{C}$.

It is clear that f is bounded by $\frac{\alpha}{\alpha - 1}$ on $\mathbb{C}$. If $|x| + |y| = 0$ or $|x| + |y| \geq \frac{1}{\alpha}$, then

$$|D_\mu f(x,y)| \leq \frac{2\alpha^2(k^2+1)}{\alpha - 1}\left(|x| + |y|\right).$$

Now suppose that $0 < |x| + |y| < \frac{1}{\alpha}$. Then there exists an integer $n \geq 1$ such that

$$\frac{1}{\alpha^{n+1}} \leq |x| + |y| < \frac{1}{\alpha^n}. \tag{3.15}$$

Hence

$$\alpha^m |k\mu x \pm \mu y| < 1, \quad \alpha^m |x \pm y| < 1, \quad \alpha^m |x| < 1$$

for all $m = 0,1,..., n - 1$. From the definition of f and (3.15), we obtain that

$$\begin{aligned} |D_\mu f(x,y)| = \Bigg| & \sum_{m=n}^{\infty} \alpha^{-m}\phi(\alpha^m(k\mu x + \mu y)) + \sum_{m=n}^{\infty} \alpha^{-m}\phi(\alpha^m(k\mu x - \mu y)) \\ & - \mu \sum_{m=n}^{\infty} \alpha^{-m}\phi(\alpha^m(x+y)) - \mu \sum_{m=n}^{\infty} \alpha^{-m}\phi(\alpha^m(x-y)) \\ & -(k-1)\mu\left[(k+2)\sum_{m=n}^{\infty} \alpha^{-m}\phi(\alpha^m x) + k\sum_{m=n}^{\infty} \alpha^{-m}\phi(-\alpha^m x)\right]\Bigg| \\ \leq & \frac{2\alpha^2(k^2+1)}{\alpha - 1}(|x| + |y|) \end{aligned}$$

Therefore, f satisfies (3.14). Now, we claim that the functional Equation (1.2) is not stable for $r = 1$ in Corollary 3.2. Suppose on the contrary that there exist an additive function $A : \mathbb{C} \to \mathbb{C}$ and a constant $d > 0$ such that $|f(x) - A(x)| \le d\,|x|$ for all $x \in \mathbb{C}$. Then there exists a constant $c \in \mathbb{C}$ such that $A(x) = cx$ for all rational numbers x. So we obtain that

$$\left|f(x)\right| \le (d + |c|)\,|x| \tag{3.16}$$

for all rational numbers x. Let $s \in \mathbb{N}$ with $s + 1 > d + |c|$. If x is a rational number in $(0, \alpha^{-s})$, then $\alpha^m x \in (0,1)$ for all $m = 0,1,\ldots, s$, and for this x we get

$$f(x) = \sum_{m=0}^{\infty} \frac{\phi(\alpha^m x)}{\alpha^m} \ge \sum_{m=0}^{s} \frac{\phi(\alpha^m x)}{\alpha^m} = (s+1)x > (d + |c|)\,x,$$

which contradicts (3.16).

Corollary 3.4 *Let $t, s > 0$ such that $\lambda := t + s < 1$ and δ, θ be non-negative real numbers, and let $f : X \to Y$ be an odd mapping for which*

$$\left\|D_b f(x,y)\right\|_\beta \le \delta + \theta\left[\|x\|_\beta^t \left\|y\right\|_\beta^s + \left(\|x\|_\beta^\lambda + \left\|y\right\|_\beta^\lambda\right)\right]$$

for all $x, y \in X$ and $b \in B_1$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$||f(x) - A(x)||_\beta \le \frac{1}{2^\beta(k^\beta - k^{\beta r})}\delta + \frac{1}{2^\beta(k^\beta - k^{\beta r})}\theta||x||_\beta^r$$

for all $x \in X$. Moreover, if $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then A is B-linear.

Proof The proof follows from Theorem 3.1 by taking $\varphi(x,y) = \delta + \theta\left[\|x\|_\beta^r \left\|y\right\|_\beta^s + \left(\|x\|_\beta^\lambda + \left\|y\right\|_\beta^\lambda\right)\right]$ for all $x, y \in X$. We can choose $L = k^{\beta(\lambda-1)}$ to get the desired result.

The Hyers-Ulam stability for the case of $\lambda = 1$ was excluded in Corollary 3.4. Similar to Theorem 3.1, one can obtain the following theorem.

Theorem 3.5 *Let $\phi : X^2 \to [0, \infty)$ be a function such that*

$$\lim_{n\to\infty} k^{n\beta}\varphi\left(\frac{x}{k^n}, \frac{y}{k^n}\right) = 0$$

for all $x,y \in X$. Let $f : X \to Y$ be an odd mapping such that

$$\left\|D_b f(x,y)\right\|_\beta \le \varphi(x,y)$$

for all $x,y \in X$ and all $b \in B_1$. If there exists a Lipschitz constant $0 < L < 1$ such that $\phi(x, 0) \le k^{-\beta} L\, \phi(kx, 0)$ for all $x \in X$, then there exists a unique additive mapping $A: X \to Y$ such that

$$\left\|f(x) - A(x)\right\|_\beta \le \frac{L}{(2k)^\beta(1-L)}\varphi(x,0)$$

for all $x \in X$. Moreover, if $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then A is B-linear.

As applications for Theorems 3.5, one can get the following Corollaries 3.6 and 3.7.

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Corollary 3.6 *Let $r > 1$ and θ be a non-negative real number, and let $f : X \to Y$ be an odd mapping for which*

$$\left\| D_b f(x, y) \right\|_\beta \le \theta \left(\|x\|_\beta^r + \|y\|_\beta^r \right)$$

for all $x,y \in X$ and $b \in B_1$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\left\| f(x) - A(x) \right\|_\beta \le \frac{1}{2^\beta (k^{\beta r} - k^\beta)} \theta \|x\|_\beta^r$$

for all $x \in X$. Moreover, if $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then A is B-linear.

Corollary 3.7 *Let t, $s > 0$ such that $\lambda := t + s > 1$ and θ be a non-negative real number, and let $f :X \to Y$ be an odd mapping for which*

$$\left\| D_b f(x, y) \right\|_\beta \le \theta \left[\|x\|_\beta^t \|y\|_\beta^s + \left(\|x\|_\beta^\lambda + \|y\|_\beta^\lambda \right) \right]$$

for all $x,y \in X$ and $b \in B_1$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\left\| f(x) - A(x) \right\|_\beta \le \frac{1}{2^\beta (k^{\beta\lambda} - k^\beta)} \theta \|x\|_\beta^\lambda$$

for all $x \in X$. Moreover, if $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then A is B-linear.

Theorem 3.8 *Let $\phi : X^2 \to [0, \infty)$ be a function such that*

$$\lim_{n\to\infty} \frac{1}{k^{2n\beta}} \varphi(k^n x, k^n y) = 0 \tag{3.17}$$

for all $x,y \in X$. Let $f : X \to Y$ be an even mapping such that

$$\left\| \tilde{D}_b f(x, y) \right\|_\beta \le \varphi(x, y) \tag{3.18}$$

for all $x,y \in X$ and all $b \in B_1$. If there exists a Lipschitz constant $0 < L < 1$ such that

$$\varphi(kx, 0) \le k^{2\beta} L \varphi(x, 0) \tag{3.19}$$

for all $x \in X$, then there exists a unique quadratic mapping $Q:X \to Y$ such that

$$\left\| f(x) - Q(x) \right\|_\beta \le \frac{1}{(2k^2)^\beta (1 - L)} \varphi(x, 0) \tag{3.20}$$

for all $x \in X$. Moreover, if $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then Q is B-quadratic, i.e., $Q(bx) = b^2 Q(x)$ for all $x \in X$ and all $b \in B$.

Proof Letting $b = 1$ and $y = 0$ in (3.18), we get

$$\left\| f(kx) - k^2 f(x) \right\|_\beta \le \frac{1}{2^\beta} \varphi(x, 0) \tag{3.21}$$

for all $x \in X$. Consider the set $\Omega := \{g \mid g : X \to Y, g(0) = 0\}$ and introduce the generalized metric on Ω:

$$d(g,h) = \inf\left\{C \in (0,\infty) | \left\|g(x) - h(x)\right\|_\beta \le C\varphi(x,0), \quad \forall x \in X.\right\}$$

It is easy to show that (Ω, *d*) is a complete generalized metric space. We now define a function $J : \Omega \to \Omega$ by

$$(Jg)(x) = \frac{1}{k^2}g(kx), \quad \forall g \in \Omega, x \in X.$$

Let $g, h \in \Omega$ and $C \in [0, \infty]$ be an arbitrary constant with $d(g, h) < C$, by the definition of *d*, it follows

$$\left\|g(x) - h(x)\right\|_\beta \le C\varphi(x,0), \quad \forall x \in X.$$

By the given hypothesis and the last inequality, one has

$$\left\|\frac{1}{k^2}g(kx) - \frac{1}{k^2}h(kx)\right\|_\beta \le CL\varphi(x,0), \quad \forall x \in X.$$

Hence, it holds that $d(Jg, Jh) \le Ld(g, h)$. It follows from (3.21) that $d(Jf, f) \le 1/(2k^2)^\beta < \infty$. Therefore, by Theorem 2.1, *J* has a unique fixed point $Q : X \to Y$ in the set $\Omega^* = \{g \in \Omega \mid d(f, g) < \infty\}$ such that

$$Q(x) := \lim_{n\to\infty}(J^n f)(x) = \lim_{n\to\infty}\frac{1}{k^{2n}}f(k^n x) \tag{3.22}$$

and $Q(kx) = k^2Q(x)$ for all $x \in X$. Also,

$$d(Q,f) \le \frac{1}{1-L}d(Jf,f) \le \frac{1}{(2k^2)^\beta(1-L)}.$$

This means that (3.20) holds for all $x \in X$.

The mapping *Q* is quadratic because as follows it satisfies in Equation (1.2):

$$\begin{aligned}\left\|\tilde{D}_1 Q(x,y)\right\|_\beta &= \lim_{n\to\infty}\left\|\frac{1}{k^{2n}}\tilde{D}_1 f(k^n x, k^n y)\right\|_\beta \\ &= \lim_{n\to\infty}\frac{1}{k^{2n\beta}}\left\|\tilde{D}_1 f(k^n x, k^n y)\right\|_\beta \\ &\le \lim_{n\to\infty}\frac{1}{k^{2n\beta}}\varphi(k^n x, k^n y) = 0,\end{aligned}$$

for all $x,y \in X$, therefore by Lemma 2.2, it is quadratic.

Moreover, if $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then by the same reasoning as in the proof of [4]*Q* is $\mathbb{R}$-quadratic. Letting $y = 0$ in (3.18), we get

$$\left\|2f(kbx) - 2k^2b^2f(x)\right\|_\beta \le \varphi(x,0) \tag{3.23}$$

for all $x \in X$ and all $b \in B_1$. By definition of *Q*, (3.17) and (3.23), we obtain

$$\begin{aligned}\left\|2Q(kbx) - 2k^2b^2Q(x)\right\|_\beta &= \lim_{n\to\infty}\frac{1}{k^{2n\beta}}\left\|2f(k^{n+1}bx) - 2k^2b^2f(k^n x)\right\|_\beta \\ &\le \lim_{n\to\infty}\frac{1}{k^{2n\beta}}\varphi(k^n x, 0) = 0\end{aligned}$$

for all $x \in X$ and all $b \in B_1$. So $Q(kbx) - k^2b^2Q(x) = 0$ for all $x \in X$ and all $b \in B_1$. Since $Q(kx) = k^2Q(x)$, we get $Q(bx) = b^2Q(x)$ for all $x \in X$ and all $b \in B_1 \cup \{0\}$. Now, let $b \in B\backslash\{0\}$. Since Q is $\mathbb{R}$-quadratic,

$$Q(bx) = Q\left(|b| \cdot \frac{1}{|b|}x\right) = |b|^2 Q\left(\frac{b}{|b|}x\right) = |b|^2 \cdot \left(\frac{b}{|b|}\right)^2 Q(x) = b^2 Q(x)$$

for all $x \in X$ and all $b \in B$. This proves that Q is B-quadratic.

Corollary 3.9 *Let* $0 < r < 2$ *and* δ, θ *be non-negative real numbers, and let* $f : X \to Y$ *be an even mapping for which*

$$\left\|\tilde{D}_b f(x, y)\right\|_\beta \le \delta + \theta\left(\|x\|_\beta^r + \|y\|_\beta^r\right)$$

for all $x,y \in X$ *and be* B_1. *Then there exists a unique quadratic mapping* $Q{:}X \to Y$ *such that*

$$\|f(x) - Q(x)\|_\beta \le \frac{1}{2^\beta(k^{2\beta} - k^{\beta r})}\delta + \frac{1}{2^\beta(k^{2\beta} - k^{\beta r})}\theta\,\|x\|_\beta^r$$

for all $x \in X$. *Moreover, if* $f(tx)$ *is continuous in* $t \in \mathbb{R}$ *for each fixed* $x \in X$, *then* Q *is* B*-quadratic.*

The following example shows that the Hyers-Ulam stability for the case of $r = 2$ was excluded in Corollary 3.9.

Example 3.10 Let $\varphi : \mathbb{C} \to \mathbb{C}$ be defined by

$$\phi(x) = \begin{cases} x^2, \text{for } |x| < 1, \\ 1, \quad \text{for } |x| \ge 1. \end{cases}$$

Consider the function $f : \mathbb{C} \to \mathbb{C}$ be defined by

$$f(x) = \sum_{m=0}^{\infty} \alpha^{-2m}\phi(\alpha^m x)$$

for all $x \in \mathbb{C}$, where $\alpha > k$. Let

$$\begin{aligned}\tilde{D}_\mu f(x, y) := &f(k\mu x + \mu y) + f(k\mu x - \mu y) - \mu^2 f(x + y) - \mu^2 f(x - y) \\ &- (k - 1)\mu^2\left[(k + 2)f(x) + kf(-x)\right]\end{aligned}$$

for all x, $y \in \mathbb{C}$ and $\mu \in \mathbb{T} := \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$. Then f satisfies the functional inequality

$$\left|\tilde{D}_\mu f(x, y)\right| \le \frac{2(k^2 + 1)\alpha^4}{\alpha^2 - 1}\left(|x|^2 + |y|^2\right) \tag{3.24}$$

for all x, $y \in \mathbb{C}$, but there do not exist a quadratic function $Q : \mathbb{C} \to \mathbb{C}$ and a constant $d > 0$ such that $|f(x) - Q(x)| \le d\, |x|^2$ for all $x \in \mathbb{C}$.

It is clear that f is bounded by $\frac{\alpha^2}{\alpha^2 - 1}$ on $\mathbb{C}$. If $|x|^2 + |y|^2 = 0$ or $|x|^2 + |y|^2 \ge \frac{1}{\alpha^2}$, then

$$\left|\tilde{D}_\mu f(x, y)\right| \le \frac{2\alpha^4(k^2 + 1)}{\alpha^2 - 1}\left(|x|^2 + |y|^2\right).$$

Now suppose that $0 < |x|^2 + |y|^2 < \frac{1}{\alpha^2}$. Then there exists an integer $n \ge 1$ such that

$$\frac{1}{\alpha^{2(n+2)}} \le |x|^2 + |y|^2 < \frac{1}{\alpha^{2(n+1)}}. \tag{3.25}$$

Hence

$$\alpha^m \left|k\mu x \pm \mu y\right| < 1, \quad \alpha^m \left|x \pm y\right| < 1, \quad \alpha^m |x| < 1$$

for all $m = 0,1,..., n - 1$. From the definition of f and the inequality (3.25), we obtain that

$$\begin{aligned}\left|\tilde{D}_\mu f(x,y)\right| &= \Bigg|\sum_{m=n}^{\infty}\alpha^{-2m}\phi(\alpha^m(k\mu x+\mu y)) + \sum_{m=n}^{\infty}\alpha^{-2m}\phi(\alpha^m(k\mu x-\mu y)) \\ &\quad - \mu^2\sum_{m=n}^{\infty}\alpha^{-2m}\phi(\alpha^m(x+y)) - \mu^2\sum_{m=n}^{\infty}\alpha^{-2m}\phi(\alpha^m(x-y)) \\ &\quad -(k-1)\mu^2\left[(k+2)\sum_{m=n}^{\infty}\alpha^{-2m}\phi(\alpha^m x) + k\sum_{m=n}^{\infty}\alpha^{-2m}\phi(-\alpha^m x)\right]\Bigg| \\ &\leq \frac{2(k^2+1)\alpha^{2(1-n)}}{\alpha^2-1} \leq \frac{2(k^2+1)\alpha^4}{\alpha^2-1}\left(|x|^2+|y|^2\right)\end{aligned}$$

Therefore, f satisfies (3.24). Now, we claim that the functional Equation (1.2) is not stable for $r = 2$ in Corollary 3.9. Suppose on the contrary that there exist a quadratic function $Q : \mathbb{C} \to \mathbb{C}$ and a constant $d > 0$ such that $|f(x) - Q(x)| \leq d\, |x|^2$ for all $x \in \mathbb{C}$. Then there exists a constant $c \in \mathbb{C}$ such that $Q(x) = cx^2$ for all rational numbers x. So we obtain that

$$\left|f(x)\right| \leq (d+|c|)\,|x|^2 \tag{3.26}$$

for all rational numbers x. Let $s \in \mathbb{N}$ with $s + 1 > d + |c|$. If x is a rational number in $(0, \alpha^{-s})$, then $\alpha^m x \in (0,1)$ for all $m = 0,1,..., s$, and for this x we get

$$f(x) = \sum_{m=0}^{\infty}\frac{\phi(\alpha^m x)}{\alpha^{2m}} \geq \sum_{m=0}^{s}\frac{\phi(\alpha^m x)}{\alpha^{2m}} = (s+1)x^2 > (d+|c|)\,x^2,$$

which contradicts (3.26).

Similar to Corollary 3.9, one can obtain the following corollary.

Corollary 3.11 *Lett, $s > 0$ such that $\lambda := t + s < 2$ and δ, θ be non-negative real numbers, and let $f : X \to Y$ be an even mapping for which*

$$\left\|\tilde{D}_b f(x,y)\right\|_\beta \leq \delta + \theta\left[\|x\|_\beta^t\,\|y\|_\beta^s + \left(\|x\|_\beta^\lambda + \|y\|_\beta^\lambda\right)\right]$$

for all $x,y \in X$ and $b \in B_1$. Then there exists a unique quadratic mapping $Q:X \to Y$ such that

$$\|f(x) - Q(x)\|_\beta \leq \frac{1}{2^\beta(k^{2\beta}-k^{\beta\lambda})}\delta + \frac{1}{2^\beta(k^{2\beta}-k^{\beta\lambda})}\theta\,\|x\|_\beta^\lambda$$

for all $x \in X$. Moreover, if $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then Q is B-quadratic.

Similar to Theorem 3.8, one can obtain the following theorem.

Theorem 3.12 *Let $\phi : X^2 \to [0, \infty)$ be a function such that*

$$\lim_{n\to\infty} k^{2n\beta}\varphi\left(\frac{x}{k^n}, \frac{y}{k^n}\right) = 0$$

for all x, y ∈ X. Let f : X → Y be an even mapping such that

$$\left\|\tilde{D}_b f(x,y)\right\|_\beta \le \varphi(x,y)$$

for all x, y ∈ X and all b ∈ B_1. If there exists a Lipschitz constant 0 <L < 1 such that $\phi(x, 0) \le k^{-2\beta}L\ \phi(kx, 0)$ for all x ∈ X, then there exists a unique quadratic mapping Q: X → Y such that

$$\|f(x) - Q(x)\|_\beta \le \frac{L}{(2k^2)^\beta(1-L)}\varphi(x,0)$$

for all x ∈ X. Moreover, if f(tx) is continuous in t ∈ $\mathbb{R}$ for each fixed x ∈ X, then Q is B-quadratic.

We now prove our main theorem in this section.

Theorem 3.13 *Let $\phi : X^2 \to [0, \infty)$ be a function such that*

$$\lim_{n\to\infty}\frac{1}{k^{n\beta}}\varphi(k^n x, k^n y) = 0 \tag{3.27}$$

for all x, y ∈ X. Let f : X → Y be a mapping such that

$$\|D_b f(x,y)\|_\beta \le \varphi(x,y) \quad \text{and} \quad \left\|\tilde{D}_b f(x,y)\right\|_\beta \le \varphi(x,y) \tag{3.28}$$

for all x, y ∈ X and all b ∈ B_1. If there exists a Lipschitz constant 0 <L < 1 such that

$$\varphi(kx,0) \le k^\beta L\varphi(x,0) \tag{3.29}$$

for all x ∈ X, then there exist a unique additive mapping A: X → Y and a unique quadratic mapping Q:X → Y such that

$$\|f(x) - A(x) - Q(x)\|_\beta \le \frac{2^{1-2\beta}}{k^\beta(1-L)}\left[\varphi(x,0) + \varphi(-x,0)\right] \tag{3.30}$$

for all x ∈ X. Moreover, if f(tx) is continuous in t ∈ $\mathbb{R}$ for each fixed x ∈ X, then A is B-linear and Q is B-quadratic.

Proof If we decompose *f* into the even and the odd parts by putting

$$f_e(x) = \frac{f(x)+f(-x)}{2} \quad \text{and} \quad f_o(x) = \frac{f(x)-f(-x)}{2} \tag{3.31}$$

for all $x \in X$, then $f(x) = f_e(x) + f_o(x)$. Let $\psi(x,y) = [\phi(x,y) + \phi(-x, -y)]/2^\beta$, then by (3.27)-(3.29) and (3.31) we have

$$\lim_{n\to\infty}\frac{1}{k^{n\beta}}\psi\left(k^n x, k^n y\right) = 0, \quad \psi(kx,0) \le k^\beta L\psi(x,0),$$
$$\|D_b f_o(x,y)\|_\beta \le \psi(x,y), \quad \left\|\tilde{D}_b f_e(x,y)\right\|_\beta \le \psi(x,y).$$

Hence by Theorems 3.1 and 3.8, there exist a unique additive mapping $A : X \to Y$ and a unique quadratic mapping $Q : X \to Y$ such that

$$\|f_o(x) - A(x)\|_\beta \le \frac{1}{(2k)^\beta(1-L)}\psi(x,0), \quad \|f_e(x) - Q(x)\|_\beta \le \frac{1}{(2k^2)^\beta(1-L)}\psi(x,0)$$

for all $x \in X$. Therefore

$$\begin{aligned}\left\|f(x) - A(x) - Q(x)\right\|_\beta &\le \left\|f_o(x) - A(x)\right\|_\beta + \left\|f_e(x) - Q(x)\right\|_\beta \\ &\le \frac{1}{(2k)^\beta(1-L)}\psi(x,0) + \frac{1}{(2k^2)^\beta(1-L)}\psi(x,0) \\ &\le \frac{2}{(2k)^\beta(1-L)}\psi(x,0) \\ &= \frac{2^{1-2\beta}}{k^\beta(1-L)}\left[\varphi(x,0) + \varphi(-x,0)\right]\end{aligned}$$

for all $x \in X$.

Corollary 3.14 *Let* $0 < r < 1$ *and* δ, θ *be non-negative real numbers, and let* $f : X \to Y$ *be a mapping for which*

$$\left\|D_b f(x,y)\right\|_\beta \le \delta + \theta\left(\|x\|_\beta^r + \|y\|_\beta^r\right) \quad and \quad \left\|\tilde{D}_b f(x,y)\right\|_\beta \le \delta + \theta\left(\|x\|_\beta^r + \|y\|_\beta^r\right)$$

for all $x,y \in X$ *and be* B_1. *Then there exist a unique additive mapping* $A : X \to Y$ *and a unique quadratic mapping* $Q: X \to Y$ *such that*

$$\left\|f(x) - A(x) - Q(x)\right\|_\beta \le \frac{2^{2(1-\beta)}}{k^\beta - k^{\beta r}}\left[\delta + \theta\|x\|_\beta^r\right]$$

for all $x \in X$. *Moreover, if* $f(tx)$ *is continuous in* $t \in \mathbb{R}$ *for each fixed* $x \in X$, *then* A *is* B*-linear and* Q *is* B*-quadratic.*

Similar to Theorem 3.13, one can obtain the following theorem.

Theorem 3.15 *Let* $\phi: X^2 \to [0, \infty)$ *be a function such that*

$$\lim_{n\to\infty} k^{2n\beta}\varphi\left(\frac{x}{k^n}, \frac{y}{k^n}\right) = 0$$

for all $x,y \in X$. *Let* $f : X \to Y$ *be a mapping such that*

$$\left\|D_b f(x,y)\right\|_\beta \le \varphi(x,y) \quad and \quad \left\|\tilde{D}_b f(x,y)\right\|_\beta \le \varphi(x,y)$$

for all $x, y \in X$ *and all* $b \in B_1$. *If there exists a Lipschitz constant* $0 < L < 1$ *such that*

$$\varphi(x,0) \le k^{-2\beta}L\varphi(kx,0)$$

for all $x \in X$, *then there exist a unique additive mapping* $A: X \to Y$ *and a unique quadratic mapping* $Q: X \to Y$ *such that*

$$\left\|f(x) - A(x) - Q(x)\right\|_\beta \le \frac{2^{1-2\beta}L}{k^\beta(1-L)}\left[\varphi(x,0) + \varphi(-x,0)\right]$$

for all $x \in X$. *Moreover, if* $f(tx)$ *is continuous in* $t \in \mathbb{R}$ *for each fixed* $x \in X$, *then* A *is* B*-linear and* Q *is* B*-quadratic.*

Corollary 3.16 *Let* $r > 2$ *and* θ *be a non-negative real number, and let* $f : X \to Y$ *be a mapping for which*

$$\left\|D_b f(x,y)\right\|_\beta \le \theta\left(\|x\|_\beta^r + \|y\|_\beta^r\right) \quad and \quad \left\|\tilde{D}_b f(x,y)\right\|_\beta \le \theta\left(\|x\|_\beta^r + \|y\|_\beta^r\right)$$

for all $x,y \in X$ *and* $b \in B_1$. *Then there exist a unique additive mapping* $A : X \to Y$ *and a unique quadratic mapping* $Q:X \to Y$ *such that*

$$\left\|f(x) - A(x) - Q(x)\right\|_{\beta} \le \frac{2^{2(1-\beta)}k^{\beta}}{k^{r\beta} - k^{2\beta}}\theta \left\|x\right\|_{\beta}^{r}$$

for all $x \in X$. Moreover, if $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then A is B-linear and Q is B-quadratic.

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Authors' contributions
All authors carried out the proof. All authors conceived of the study, and participated in its design and coordination. All authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

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