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Common fixed point under contractive condition of Ćirić's type on cone metric type spaces

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Abstract

The purpose of this article is to generalize common fixed point theorems under contractive condition of Ćirić's type on a cone metric type space. We give basic facts about cone metric type spaces, and we prove common fixed point theorems under contractive condition of Ćirić's type on a cone metric type space without assumption of normality for cone. As special cases we get the corresponding fixed point theorems on a cone metric space with respect to a solid cone. Obtained results in this article extend, generalize, and improve, well-known comparable results in the literature.

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1 Introduction

Replacing the real numbers, as the co-domain of a metric, by an ordered Banach space we obtain a generalization of metric space (see, e.g., [1-3]). Huang and Zhang [4] introduced such spaces under the name of cone metric spaces. They described the convergence in cone metric space, introduced their completeness and proved some fixed point theorems for contractive mappings. Cones and ordered normed spaces have some applications in optimization theory (see [5,6]). The initial study of Huang and Zhang [4] inspired many authors to prove fixed point theorems, as well as common fixed point theorems for two or more mappings on cone metric space, e.g., [7-18].

In [19], a generalization of a cone metric space, called a cone metric type space was considered, and some common fixed point theorems for four mappings in such space were proved. Common fixed point theorem under contractive condition of Ćirić's type (see [20]) on cone metric space in settings of a normal cone was proved in [21]. In this article, we extend that result proving common fixed point theorems under contractive condition of Ćirić's type on a cone metric type space without assumption of normality for cone. As special cases we get the corresponding fixed point theorems in a cone metric space with respect to a solid cone.

The article is organized as follows. In Section 2, we repeat some definitions and well known results which will be needed in the sequel. In Section 3, we prove common fixed point theorems on a cone metric type space and present some corollaries.

2 Definitions and notation

Let E be a real Banach space and P be a subset of E . By θ we denote zero element of E and by $\text{int } P$ the interior of P . The subset P is called a *cone* if and only if:

- (i) P is closed, nonempty and $P \neq \{\theta\}$;
- (ii) $a, b \in \mathbb{R}$, $a, b \geq 0$, and $x, y \in P$ imply $ax + by \in P$;
- (iii) $P \cap (-P) = \{\theta\}$.

For a given cone P , a partial ordering \preceq with respect to P is introduced in the following way: $x \preceq y$ if and only if $y - x \in P$. In order to indicate that $x \preceq y$, but $x \neq y$, we write $x \prec y$. If $y - x \in \text{int } P$, we write $x \ll y$.

The cone P is called *normal* if there is a number $k > 0$, such that, for all $x, y \in E$, $\theta \preceq x \preceq y$ implies $\|x\| \leq k\|y\|$. If a cone is not normal, it is called *non-normal*.

If $\text{int } P \neq \emptyset$, the cone P is called *solid*.

In the sequel, we always suppose that E is a real Banach space, P is a solid cone in E , and \preceq is partial ordering with respect to P .

Definition 2.1. ([19]) Let X be a nonempty set and E be a real Banach space with cone P . A vector-valued function $d : X \times X \rightarrow E$ is said to be a *cone metric type function* on X with constant $K \geq 1$, if the following conditions are satisfied:

- (d₁) $\theta \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$;
- (d₂) $d(x, y) = d(y, x)$ for all $x, y \in X$;
- (d₃) $d(x, y) \preceq K(d(x, z) + d(z, y))$ for all $x, y, z \in X$.

The pair (X, d) is called a *cone metric type space* (in brief *CMTS*).

Remark 2.1. For $K = 1$ in Definition 2.1 we obtain a cone metric space introduced in [4].

Definition 2.2. Let (X, d) be a CMTS and $\{x_n\}$ be a sequence in X .

(c₁) $\{x_n\}$ converges to $x \in X$ if for every $c \in E$ with $\theta \ll c$ there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x) \ll c$ for all $n > n_0$. We write $\lim_{n \rightarrow \infty} x_n = x$, or $x_n \rightarrow x, n \rightarrow \infty$.

(c₂) If for every $c \in E$ with $\theta \ll c$, there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x_m) \ll c$ for all $n, m > n_0$, then $\{x_n\}$ is called a *Cauchy sequence* in X .

If every Cauchy sequence is convergent in X , then X is called a *complete CMTS*.

Remark 2.2. If (X, d) is a cone metric space (i.e., CMTS with $K = 1$) relative to a normal cone P , then a sequence $\{x_n\}$ in X converges to $x \in X$ if and only if $d(x_n, x) \rightarrow \theta, n \rightarrow \infty$, i.e., if and only if $\|d(x_n, x)\| \rightarrow 0, n \rightarrow \infty$ (see [[4], Lemma 4]). Further, $\{x_n\}$ in X is a Cauchy sequence if and only if $d(x_n, x_m) \rightarrow \theta, n, m \rightarrow \infty$, i.e., if and only if $\|d(x_n, x_m)\| \rightarrow 0, n, m \rightarrow \infty$ (see [[4], Lemma 4]).

In the case of a non-normal cone equivalences in previous statements do not hold. For a non-normal cone $d(x_n, x) \rightarrow \theta, n \rightarrow \infty$ implies $x_n \rightarrow x, n \rightarrow \infty$, and $d(x_n, x_m) \rightarrow \theta, n, m \rightarrow \infty$ implies that $\{x_n\}$ is a Cauchy sequence.

Example 2.1. ([19]) Let $B = \{e_i \mid i = 1, \dots, n\}$ be orthonormal basis of \mathbb{R}^n with inner product (\cdot, \cdot) . Let $p > 0$ and

$$X_p = \left\{ [x] \mid x : [0, 1] \rightarrow \mathbb{R}^n, \int_0^1 |(x(t), e_k)|^p dt \in \mathbb{R}, k = 1, \dots, n \right\},$$

where $[x]$ represents class of element x with respect to equivalence relation of functions equal almost everywhere. If we choose $E = \mathbb{R}^n$ and

$$P_B = \{y \in \mathbb{R}^n | (y, e_i) \geq 0, i = 1, \dots, n\}$$

then P_B is a solid cone. For $d : X_p \times X_p \rightarrow P_B$ defined by

$$d(f, g) = \sum_{i=1}^n e_i \int_0^1 |((f - g)(t), e_i)|^p dt, f, g \in X_p,$$

(X_p, d) is CMTS with $K = 2^{p-1}$.

The following properties hold in the case of a CMTS.

Lemma 2.1. Let (X, d) be a CMTS over ordered real Banach space E with a cone P . The following properties hold $(a, b, c \in E)$:

- (p₁) If $a \preceq b$ and $b \ll c$, then $a \ll c$.
- (p₂) If $\theta \preceq a \ll c$ for all $c \in \text{int } P$, then $a = \theta$.
- (p₃) If $a \preceq \lambda a$, where $a \in P$ and $0 \leq \lambda < 1$, then $a = \theta$.
- (p₄) Let $x_n \rightarrow \theta$ in E and let $\theta \ll c$. Then there exists positive integer n_0 such that $x_n \ll c$ for each $n > n_0$.

3 Fixed point theorems

Theorem 3.1. Let (X, d) be a complete CMTS with constant $K \in [1, 2]$ relative to a solid cone P . Let $\{F, T\}$ be a pair of self-mappings on X such that for some constant $\lambda \in (0, 1/(2K))$ for all $x, y \in X$ there exists

$$u(x, y) \in \{d(x, y), d(x, Fx), d(y, Ty), d(x, Ty), d(y, Fx)\}, \tag{3.1}$$

such that the following inequality

$$d(Fx, Ty) \preceq \lambda u(x, y) \tag{3.2}$$

holds. Then F and T have a unique common fixed point.

Proof. Let us choose $x_0 \in X$ arbitrary and define sequence $\{x_n\}$ as follows: $x_{2n+1} = Fx_{2n}$, $x_{2n+2} = Tx_{2n+1}$, $n = 0, 1, 2, \dots$. We shall show that

$$d(x_{k+1}, x_k) \preceq \alpha d(x_k, x_{k-1}), k \geq 1, \tag{3.3}$$

where $\alpha = \lambda K / (1 - \lambda K)$ (since $\lambda K < 1/2$, it is easy to see that $\alpha \in (0, 1)$). In order to prove this, we consider the cases of an odd integer k and of an even k .

For $k = 2n + 1$, from (3.2) we have

$$d(x_{2n+2}, x_{2n+1}) = d(Fx_{2n}, Tx_{2n+1}) \preceq \lambda u(x_{2n}, x_{2n+1}),$$

where, according to (3.1),

$$\begin{aligned} u(x_{2n}, x_{2n+1}) \in & \{d(x_{2n}, x_{2n+1}), d(x_{2n}, Fx_{2n}), d(x_{2n+1}, Tx_{2n+1}), \\ & d(x_{2n}, Tx_{2n+1}), d(x_{2n+1}, Fx_{2n})\} \\ = & \{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}), d(x_{2n}, x_{2n+2}), \theta\} \end{aligned}$$

Thus, we get the following cases:

- $d(x_{2n+2}, x_{2n+1}) \leq \lambda d(x_{2n+1}, x_{2n+2})$, which, according to (p_3) , implies $d(x_{2n+1}, x_{2n+2}) = \theta$;
- $d(x_{2n+2}, x_{2n+1}) \leq \lambda d(x_{2n}, x_{2n+1})$;
- $d(x_{2n+2}, x_{2n+1}) \leq \lambda d(x_{2n}, x_{2n+2})$, that is, because of (d_3) ,

$$d(x_{2n+2}, x_{2n+1}) \leq \lambda K(d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})),$$

which implies

$$d(x_{2n+2}, x_{2n+1}) \leq \frac{\lambda K}{1 - \lambda K} d(x_{2n}, x_{2n+1}).$$

Hence, (3.3) is satisfied, where $\alpha = \max\{\lambda, \lambda K/(1 - \lambda K)\} = \lambda K/(1 - \lambda K)$.
 Now, for $k = 2n + 2$, we have

$$d(x_{2n+3}, x_{2n+2}) = d(Fx_{2n+2}, Tx_{2n+1}) \leq \lambda u(x_{2n+2}, x_{2n+1}),$$

where

$$\begin{aligned} u(x_{2n+2}, x_{2n+1}) &\in \{d(x_{2n+2}, x_{2n+1}), d(x_{2n+2}, Fx_{2n+2}), d(x_{2n+1}, Tx_{2n+1}), \\ &\quad d(x_{2n+2}, Tx_{2n+1}), d(x_{2n+1}, Fx_{2n+2})\} \\ &= \{d(x_{2n+2}, x_{2n+1}), d(x_{2n+2}, x_{2n+3}), \theta, d(x_{2n+1}, x_{2n+3})\}, \end{aligned}$$

and we get the following cases:

- $d(x_{2n+3}, x_{2n+2}) \leq \lambda d(x_{2n+2}, x_{2n+1})$;
- $d(x_{2n+3}, x_{2n+2}) \leq \lambda d(x_{2n+3}, x_{2n+2})$, which gives $d(x_{2n+3}, x_{2n+2}) = \theta$;
- $d(x_{2n+3}, x_{2n+2}) \leq \lambda d(x_{2n+3}, x_{2n+1}) \leq \lambda K(d(x_{2n+3}, x_{2n+2}) + d(x_{2n+2}, x_{2n+1}))$, which implies

$$d(x_{2n+3}, x_{2n+2}) \leq \frac{\lambda K}{1 - \lambda K} d(x_{2n+2}, x_{2n+1}).$$

So, inequality (3.3) is satisfied in this case, too.

Therefore, (3.3) is satisfied for all $k \in \mathbb{N}_0$, and by iterating we get

$$d(x_k, x_{k+1}) \leq \alpha^k d(x_0, x_1). \tag{3.4}$$

Since $K \geq 1$, for $m > k$ we have

$$\begin{aligned} d(x_k, x_m) &\leq Kd(x_k, x_{k+1}) + K^2 d(x_{k+1}, x_{k+2}) + \dots + K^{m-k-1} d(x_{m-1}, x_m) \\ &\leq (K\alpha^k + K^2\alpha^{k+1} + \dots + K^{m-k}\alpha^{m-1})d(x_0, x_1) \\ &\leq \frac{K\alpha^k}{1 - K\alpha} d(x_0, x_1) \rightarrow \theta, \text{ as } k \rightarrow \infty. \end{aligned}$$

Hence, $\{x_k\}$ is a Cauchy sequence in X (it follows, by (p_4) and (p_1) , that for every $c \in \text{int } P$ there exists positive integer k_0 such that $d(x_k, x_m) \ll c$ for every $m > k > k_0$).

Since X is complete CMTS, there exists $v \in X$ such that $x_k \rightarrow v$, as $k \rightarrow \infty$. Let us show that $Fv = Tv = v$. We have $d(Fx_{2n}, Tv) \leq \lambda u(x_{2n}, v)$, where

$$u(x_{2n}, v) \in \{d(x_{2n}, v), d(x_{2n}, Fx_{2n}), d(v, Tv), d(x_{2n}, Tv), d(v, Fx_{2n})\}.$$

Thus, for any $\theta \ll c$ and sufficiently large n , at least one of the following cases hold:

- $d(Fx_{2n}, Tv) \leq \lambda d(x_{2n}, v) \ll \lambda \cdot c/\lambda = c$;
- $d(Fx_{2n}, Tv) \leq \lambda d(x_{2n}, Fx_{2n})$, i.e.,

$$d(Fx_{2n}, Tv) \leq \lambda Kd(x_{2n}, v) + \lambda Kd(v, x_{2n+1}) \ll \lambda K \frac{c}{2\lambda K} + \lambda K \frac{c}{2\lambda K} = c;$$

- $d(Fx_{2n}, Tv) \leq \lambda d(v, Tv) \leq \lambda K(d(v, Fx_{2n}) + d(Fx_{2n}, Tv))$, i.e.,

$$d(Fx_{2n}, Tv) \leq \frac{\lambda K}{1 - \lambda K} d(v, x_{2n+1}) \ll \frac{\lambda K}{1 - \lambda K} \frac{c(1 - \lambda K)}{\lambda K} = c;$$

- $d(Fx_{2n}, Tv) \leq \lambda d(x_{2n}, Tv) \leq \lambda K(d(x_{2n}, v) + Kd(v, Fx_{2n}) + Kd(Fx_{2n}, Tv))$, i.e.,

$$\begin{aligned} d(Fx_{2n}, Tv) &\leq \frac{\lambda K}{1 - \lambda K^2} d(x_{2n}, v) + \frac{\lambda K^2}{1 - \lambda K^2} d(v, x_{2n+1}) \\ &\ll \frac{\lambda K}{1 - \lambda K^2} \frac{c(1 - \lambda K^2)}{2\lambda K} + \frac{\lambda K^2}{1 - \lambda K^2} \frac{c(1 - \lambda K^2)}{2\lambda K^2} = c \end{aligned}$$

(since $1 \leq K \leq 2$, we have $0 \leq \lambda \leq 1/(2K) \leq 1/K^2$, i.e., $1 - \lambda K^2 > 0$);

- $d(Fx_{2n}, Tv) \leq \lambda d(v, Fx_{2n}) = \lambda d(v, x_{2n+1}) \ll \lambda \cdot c/\lambda = c$.

In all these cases, we obtain that $Fx_{2n} \rightarrow Tv$, as $n \rightarrow \infty$, that is $x_n \rightarrow Tv, n \rightarrow \infty$. Since the limit of a convergent sequence in a CMTS is unique, we have that $v = Tv$. Now, we have to prove that $Fv = Tv$. Since

$$d(Fv, v) = d(Fv, Tv) \leq \lambda u(v, v),$$

where

$$\begin{aligned} u(v, v) &\in \{d(v, v), d(v, Fv), d(v, Tv), d(v, Tv), d(v, Fv)\} \\ &= \{\theta, d(v, Fv)\}. \end{aligned}$$

Hence, we get the following cases: $d(Fv, v) \leq \lambda \theta$ and $d(Fv, v) \leq \lambda d(Fv, v)$. According to (p_3) , it follows that $Fv = v$, that is, v is a common fixed point of F and T . It can be easily verified that v is the unique common fixed point of F and T .

By using the same steps as in proof of Theorem 3.1, one can prove the following theorem.

Theorem 3.2. *Let (X, d) be a complete CMTS with constant $K > 2$ relative to a solid cone P . Let $\{F, T\}$ be a pair of self-mappings on X such that for some constant $\lambda \in (0, 1/K^2)$ for all $x, y \in X$ there exists*

$$u(x, y) \in \{d(x, y), d(x, Fx), d(y, Ty), d(x, Ty), d(y, Fx)\},$$

such that the inequality $d(Fx, Ty) \leq \lambda u(x, y)$ holds. Then F and T have a unique common fixed point.

In the case of CMTS with constant $K = 1$ we get the following corollary, which extends [[21], Theorem 2.1].

Corollary 3.1. *Let (X, d) be a complete cone metric space relative to a solid cone P . Let $\{F, T\}$ be a pair of self-mappings on X such that for some constant $\lambda \in (0, 1/2)$ for all $x, y \in X$ there exists*

$$u(x, y) \in \{d(x, y), d(x, Fx), d(y, Ty), d(x, Ty), d(y, Fx)\},$$

such that the inequality $d(Fx, Ty) \leq \lambda u(x, y)$ holds. Then F and T have a unique common fixed point.

Theorem 3.3. *Let (X, d) be a complete CMTS with constant $K \geq 1$ relative to a solid cone P . Let $\{S, T\}$ be a pair of self-mappings on X such that there exist nonnegative constants $a_i, i = 1, \dots, 5$, satisfying*

$$a_1 + a_2 + a_3 + 2K \max\{a_4, a_5\} < 1, \quad a_3K + a_4K^2 < 1, \quad a_2K + a_5K^2 < 1,$$

such that for all $x, y \in X$ inequality

$$d(Sx, Ty) \leq a_1d(x, y) + a_2d(x, Sx) + a_3d(y, Ty) + a_4d(x, Ty) + a_5d(y, Sx)$$

holds. Then S and T have a unique common fixed point.

Proof. Setting $F = G = I_X$ from [[19], Theorem 3.8] (I_X is the identity mapping on X) we get what is stated. \square

In the case of CMTS with constant $K = 1$ we get the following corollary.

Corollary 3.2. *Let (X, d) be a complete cone metric space relative to a solid cone P . Let $\{S, T\}$ be a pair of self-mappings on X such that there exist nonnegative constants $a_i, i = 1, \dots, 5$, satisfying $a_1 + a_2 + a_3 + 2 \max\{a_4, a_5\} < 1$, such that for all $x, y \in X$ inequality*

$$d(Sx, Ty) \leq a_1d(x, y) + a_2d(x, Sx) + a_3d(y, Ty) + a_4d(x, Ty) + a_5d(y, Sx)$$

holds. Then S and T have a unique common fixed point.

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Authors' contributions

All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

Competing interests

The authors declare that they have no competing interests.

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