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Feasibility-solvability theorems for generalized vector equilibrium problem in reflexive Banach spaces

Gang Wang[1*], Hai-tao Che[2] and Hai-bin Chen[1]

* Correspondence: wgglj1977@163.com
[1]School of Management Science, Qufu Normal University, Shandong Rizhao 276826, China
Full list of author information is available at the end of the article

Abstract

In this article, the solvability of generalized vector equilibrium problem (GVEP) with set-valued mapping in reflexive Banach spaces is considered. Under suitable conditions, we establish a link between S_K and $S^D_{K,\text{loc}}$ for the GVEP. Furthermore, new sufficient conditions are provided for the nonemptiness and boundedness of the solution set of the GVEP if it is strictly feasible in the strong sense. The new results extend and improve some existence theorems for vector equilibrium problem in some sense.

Mathematical subject classification: 49K30; 90C29.

Keywords: generalized vector equilibrium problem, C-pseudomonotone, strict feasibility, asymptotic cone

1 Introduction

Let X be a real reflexive Banach space and U be a metric space, and $K \subseteq X$, $D \subseteq U$ be two nonempty and closed sets. Let T: $K \to 2^D$ be a nonempty-compact-valued mapping, i.e., $T(x)$ is a nonempty compact subset for any $x \in K$, and upper semicontinuous on K. Let $F{:}D \times K \times K \times Y$ bea vector-valued map, where Y is a real normed space with an ordered cone C, that is, a proper, closed and convex cone such that $\text{int}\, C \neq \emptyset$. The generalized vector equilibrium problem [1], abbreviated by GVEP, is to find $\bar{x} \in K$ and $\bar{u} \in T(\bar{x})$ such that

$$\text{(GVEP)}\quad F(\bar{u}, \bar{x}, y) \notin -\text{int}\, C, \quad \forall y \in K.$$

The GVEP includes vector optimization problem, vector variational inequality problem, vector complementarity problem, vector Nash equilibrium problem, and fixed point problem, which has made notable influence in several branches of pure and applied sciences.

For the GVEP, its dual problem is to find $\bar{x} \in K$ such that

$$\text{(DGVEP)}\quad F(\upsilon, y, \bar{x}) \notin \text{int}\, C, \quad \forall y \in K, \quad \upsilon \in T(y).$$

Throughout this article, we denote the solution set of the GVEP and the solution set of the DGVEP by S_K and S^D_K, respectively.

According to the definition of local solution of the dual problem for the equilibrium problem introduced in [2], we use

$$S^D_{K,\mathrm{loc}} = \{x \in K : \exists r > 0, \quad F(\upsilon, y, x) \notin \mathrm{int}\, C, \quad \forall y \in K, \quad \upsilon \in T(y), \quad \|y - x\| < r\}$$

to denote the local solutions of the DGVEP. Obviously,

$$S^D_K \subseteq S^D_{K,\mathrm{loc}}.$$

It is well known that the solvability of the (vector) equilibrium problem is an important issue. Based on the coercivity assumption, the existence of solution for (vector) equilibrium problem received much attention of researchers, see e.g., [1-8]. For the vector equilibrium problem, Bianchi discussed the existence of solution under the condition that F is pseu-domonotone or quasimonotone [3]. Later, Gong established the existence of the solution for strong vector equilibrium problem via the separation theorem for convex sets [6]. Recently, Long considered the existence, connectedness, and compactness of the solutions for vector equilibrium problem [8-10]. In this article, motivated by the work of Bianchi for equilibrium problem [2], we establish the relation between S_K and $S^D_{K,\mathrm{loc}}$ for the GVEP.

It should be noted that for the vector equilibrium problem, the existence of solution can also be established on the strict feasibility condition which was originally used in scalar variational inequality and vector variational inequality [11-14], and this can be extended to the scalar equilibrium problem by establishing solvability of a scalar monotone equilibrium problem when it is strictly feasible [15]. On other way, Hu and Fang [16] generalized the concept of strict feasibility to the vector equilibrium problem and established the nonemptiness and boundedness of the solution set for C-pseudomonotone vector equilibrium problem under suitable conditions. In this article, we establish the nonemptiness and boundedness of the solution set for the GVEP under a relatively weaker condition.

In summary, in this article, we first establish the relation S_K and $S^D_{K,\mathrm{loc}}$ for the GVEP under suitable conditions, and then establish the equivalence between solution set of the GVEP and solution set of the DGVEP. In addition, some new sufficient conditions are presented for the nonemptiness and boundedness of the solution set for the GVEP are proposed under the condition that it is strictly feasible in the strong sense.

2 Notations and preliminaries

In this section, we mainly give some notations and some preliminary results needed in the following.

Definition 2.1 *Let K be a nonempty convex subset of X and C be a closed convex cone in real normed space Y.*

(i) The mapping F: $K \to Y$ is said to be C-convex if

$$\alpha F(x) + (1 - \alpha)F(y) \in F(\alpha x + (1 - \alpha)y) + C, \quad \forall x, y \in K, \quad \alpha \in [0, 1].$$

(ii) The mapping F: $K \to Y$ *is said to be C-quasiconvex if the set* $\{x \in K \mid F(x) \in a - C\}$ *is convex for any* $a \in Y$.

The mapping F: $K \to Y$ *is said to be C-explicitly quasiconvex if F is quasiconvex and*

$$F(y) - F(x) \in \operatorname{int} C \Rightarrow F(y) - F(tx + (1-t)y) \in \operatorname{int} C, \quad \forall x, y \in K, \quad t \in (0,1).$$

(iii) [3]*The mapping F*: $K \to Y$ *is said to be C-lower semicontinuous if the set* $\{x \in K \mid F(x) - a \notin \operatorname{int} C\}$ *is closed on K for any* $a \in Y$. *F is said to be weakly C-lower semicontinuous if F is C-lower semicontinuous with respect to the weak topology of X. The map F is said to be weakly lower semicontinuous on K if it is weakly lower semicontinuous onK.*

(iv) The mapping F: $D \times K \times K \to Y$ *is said to be generalized hemicontinuous if the map* $t \to F(u_t, x + t(y-x), y)$ *is continuous at* 0^+ *for any* $x,y \in K$ *and* $u_t \lfloor T(x + t(y-x))$.

(v) The mapping F: $D \times K \times K \to Y$ *is said to be C-pseudomonotone if for all* $x,y \in K$, $u \in T(x), v \in T(y)$,

$$\exists u \in T(x) \text{ such that } F(u,x,y) \notin -\operatorname{int} C \Rightarrow \forall \upsilon \in T(y) \text{ such that } F(\upsilon,y,x) \notin \operatorname{int} C,$$

or equivalently,

$$\exists \upsilon \in T(y) \text{ such that } F(\upsilon,y,x) \in \operatorname{int} C \Rightarrow \forall u \in T(x) \text{ such that } F(u,x,y) \in -\operatorname{int} C.$$

(vi) The asymptotic cone K_∞ *and barrier cone barr*(K) *of K are, respectively defined by*

$$K_\infty = \left\{ d \in X \mid \exists t_k \to +\infty, \quad \exists x_k \in K \text{ with} \frac{x_k}{t_k} \rightharpoonup d \right\}$$

and

$$\operatorname{barr}(K) = \left\{ x^* \in X^* \mid \sup_{x \in K} \langle x^*, x \rangle < +\infty \right\},$$

where X^* *denotes the dual space of X and* $\rightharpoonup$ *stands for the weak convergence.*

Remark 2.1 *The explicit quasiconvexity of the function F(.) implies that* [3]

(a) for all $c \notin C$, *the set* $\{y \in K \mid F(y) \leq_C c\}$ *is convex;*

(b) if $F(z) - F(y) \in \operatorname{int} C$ *and* $F(z) \notin \operatorname{int} C$, *then* $F(z) - F(z_t) \in \operatorname{int} C$, *for* $z = ty + (1-t)z, t \in (0,1)$.

The asymptotic cone K_∞ has the following useful properties [3,17].

Lemma 2.1 *Let* $K \subset X$ *be nonempty and closed. Then the followings hold:*

(i) K_∞ *is a closed cone;*

(ii) if K is convex, then $K_\infty = \{d \in X \mid K + d \subset K\} = \{d \in X \mid x + td \in K, \forall t > 0\}$, *where* $x \in K$ *is arbitrary point;*

(iii) if K is convex cone, then $K_\infty = K$.

Lemma 2.2 *Let a,b∈ Y be such that $a \in$ -int C and $b \notin C$. Then, the set of upper bounds of a and b is nonempty and intersects with Y\C.*

Definition 2.2 *The GVEP is said to be strictly feasible in the strong sense if $\Psi_s{}^+ \neq \emptyset$, where*

$$\Psi_s{}^+ = \left\{x \in K | F(u, x, x + y) \in \text{int } C, \quad \forall y \in K_\infty \setminus \{0\}, \quad u \in T(x)\right\}.$$

Definition 2.3 [18]*A set-valued map F: E → 2^X is said to be KKM mapping if $co\Lambda \subseteq \cup_{i=1}^n F(x_i)(x_i)$ for each finite set $\Lambda = x_1,..., x_n \subseteq E$, where co(.) stands for the convex hull.*

The main tools for proving our results are the following well-known KKM theorems.

Lemma 2.3 [19]*For topological vector space X, let E ⊆ X be a nonempty convex and F: E → 2^X be a KKM mapping with closed values. If there is a subset X_0 contained in a compact convex subset of E such that $\cap_{x \in X_0} F(x)$ is compact, then $\cap_{x \in E} F(x) \neq \emptyset$.*

Definition 2.4 [20,21]*Let K be a nonempty, closed and convex subset of a real reflexive Banach space X with its dual X*. K is said to be well-positioned if there exist $x_0 \in X$ and g∈ X* such that*

$$\langle g, x - x_0 \rangle \geq \|x - x_0\|, \quad \forall x \in K.$$

Lemma 2.4 [20,21]*Let K be a nonempty, closed and convex subset of a real reflexive Banach space X with its dual X*. Then K is well-positioned if and only if the barrier cone barr(K) of K has a nonempty interior. Furthermore, if K is well-positioned then there is no sequence $\{x_n\} \subseteq K$ with $\|x_n\| \to +\infty$ such that origin is a weak limit of $\left\{\frac{x_n}{\|x_n\|}\right\}$.*

Lemma 2.5 [22]*Let X and Y be two metric spaces and T: X → 2^Y be a nonempty-compact-valued mapping and upper semicontinuous at x*. Then, for any sequences $x_n \to x^*$ and $u_n \in T(x_n)$, there exist a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ and some $u^* \to T(x^*)$ such that $u_{n_k} \to u*$.*

3 Feasibility-solvability theorems for the GVEP

In this section, we mainly establish a link between S_K and $S_{K,\text{loc}}^D$ of the GVEP. First, we make the following assumptions.

Assumption 3.1 *The mapping F: D × K × K→Y is such that*

(i) for all x∈ K, F(u, x,x) = 0, ∀u∈ T (x);
(ii) F is generalized hemicontinuous.

Lemma 3.1 *Suppose that Assumption 3.1 holds andF(u,x,.) is C-convex for all $x \in K$, $u \in T(x)$ and $S_{K,\text{loc}}^D \neq \emptyset$. Then, $S_{K,\text{loc}}^D \subseteq S_K$.*

Proof. Take $\bar{x} \in S_{K,\text{loc}}^D$ and $y \in K$. From the definition of $S_{K,\text{loc}}^D$, there exists $r > 0$ such that

$$F(\upsilon, w, \bar{x}) \notin \text{int } C, \quad \forall w \in K, \quad \upsilon \in T(w), \quad \|\bar{x} - w\| < r.$$

Take any $\bar{y} \in (\bar{x}, \bar{y}]$ with $\|\bar{x} - \bar{y}\| < r$, and set $y_t = (1-t)\bar{x} + t\bar{y}$ for $t \in (0,1)$. Obviously,

$$F(\upsilon, y_t, \bar{x}) \notin \text{int } C, \quad \forall \upsilon \in T(y_t).$$

Since $F(u,x,.)$ is C-convex for all $x \in K, u \in T(x)$, it holds that

$$(1-t)F(\upsilon, y_t, \bar{x}) + tF(\upsilon, y_t, \bar{y}) \in F(\upsilon, y_t, y_t) + C.$$

Furthermore, it follows from $F(u,x,x) = 0$ that

$$tF(\upsilon, y_t, \bar{y}) \in -(1-t)F(\upsilon, y_t, \bar{x}) + F(\upsilon, y_t, y_t) + C \subseteq -(1-t)F(\upsilon, y_t, \bar{x}) + C \subseteq Y \setminus -\text{int } C.$$

Hence,

$$F(\upsilon, y_t, \bar{y}) \notin -\text{int } C, \quad \forall \upsilon \in T(y_t).$$

Letting $t \to 0+$, we obtain by generalized hemicontinuity of $F(.,.,y)$ and Lemma 2.5 that there exists $\bar{u} \in T(\bar{x})$ such that

$$F(\bar{u}, \bar{x}, \bar{y}) \notin -\text{int } C.$$

Now, we show that $F(\bar{u}, \bar{x}, y) \notin -\text{int } C$ by contradiction. On the contrary, suppose that $F(\bar{u}, \bar{x}, y) \in -\text{int } C$, then by the C-convexity of $F(u,x,.)$,

$$tF(\bar{u}, \bar{x}, \bar{x}) + (1-t)F(\bar{u}, \bar{x}, y) \in F(\bar{u}, \bar{x}, t\bar{x} + (1-t)y) + C, \quad \forall t \in (0,1).$$

It follows from $F(\bar{u}, \bar{x}, \bar{x}) = 0$ and int $C + C \subseteq$ int C that

$$F(\bar{u}, \bar{x}, t\bar{x} + (1-t)y) \in tF(\bar{u}, \bar{x}, \bar{x}) + (1-t)F(\bar{u}, \bar{x}, y) - C \subseteq (1-t)F(\bar{u}, \bar{x}, y) - C \subseteq -\text{int } C,$$

that is,

$$F(\bar{u}, \bar{x}, t\bar{x} + (1-t)y) \in -\text{int } C, \quad \forall t \in (0,1).$$

Since $\bar{y} \in (\bar{x}, y]$, there exists t_0 such that $\bar{y} = (1-t_0)\bar{x} + t_0 y$ with $F(\bar{u}, \bar{x}, \bar{y}) \in -\text{int } C$ which contradicts $F(\bar{u}, \bar{x}, \bar{y}) \notin -\text{int } C$. So, $F(\bar{u}, \bar{x}, y) \notin -\text{int } C$. By the arbitrariness of $y \in K$, we have $\bar{x} \in S_K$ and $S_{K,\text{loc}}^D \subseteq S_K$.

Lemma 3.2 *Suppose that Assumption* 3.1 *holds and $F(u,x,.)$ is C-explicitly quasiconvex for all$x \in K, u \in T(x)$ and $S_{K,\text{loc}}^D \neq \emptyset$. Then, $S_{K,\text{loc}}^D \subseteq S_K$.*

Proof. Take $\bar{x} \in S_{K,\text{loc}}^D$ and $y \in K$. From the definition of $S_{K,\text{loc}}^D$, there exists $r > 0$ such that

$$F(\upsilon, w, \bar{x}) \notin \text{int } C, \quad \forall w \in K, \quad \upsilon \in T(w), \quad \|\bar{x} - w\| < r.$$

Take any $\bar{y} \in (\bar{x}, y]$ with $\|\bar{x} - \bar{y}\| < r\, r$, and set $y_t = (1-t)\bar{x} + t\bar{y}$ for $t \in (0,1)$. Obviously,

$$F(\upsilon, y_t, \bar{x}) \notin \text{int } C, \quad \forall \upsilon \in T(y_t).$$

Now we show that $F(\upsilon, y_t, \bar{y}) \notin -\text{int } C, \quad \forall \upsilon \in T(y_t)$. by contradiction. On the contrary, suppose the conclusion does not hold, then there exists $\bar{t} \in (0,1), \bar{u} \in T(y_{\bar{t}})$ such that

$$F(\bar{u}, y_{\bar{t}}, \bar{y}) \in -\text{int } C.$$

We will break the arguments into two cases.

Case 1. If $F(\bar{u}, y_{\bar{t}}, \bar{x}) \in \partial C \subseteq C$, then

$$F(\bar{u}, y_{\bar{t}}, \bar{x}) - F(\bar{u}, y_{\bar{t}}, \bar{y}) \in C + \text{int } C \subseteq \text{int } C,$$

where ∂C is the boundary of C. Since $F(u,x,.)$ is C-explicitly quasiconvex, we have

$$F(\bar{u}, y_{\bar{t}}, \bar{x}) - F(\bar{u}, y_{\bar{t}}, y_{\bar{t}}) \in \text{int } C.$$

It follow from $F(\bar{u}, y_{\bar{t}}, y_{\bar{t}}) = 0$ that

$$F(\bar{u}, y_{\bar{t}}, \bar{x}) \in \text{int } C,$$

which contradicts the assumption that $F(\bar{u}, y_{\bar{t}}, \bar{x}) \in \partial C$.

Case 2. If $F(\bar{u}, y_{\bar{t}}, \bar{x}) \notin C$, then by Lemma 2.2, there exists $p \notin C$ such that

$$F(\bar{u}, y_{\bar{t}}, \bar{x}) \leq cp, \quad F(\bar{u}, y_{\bar{t}}, \bar{y}) \leq cp.$$

By the quasiconvexity of $F(u,x,.)$, one has

$$F(\bar{u}, y_{\bar{t}}, y_{\bar{t}}) \leq cp.$$

Noticing $F(\bar{u}, y_{\bar{t}}, y_{\bar{t}}) = 0$, we obtain that $p \in C$, which contradicts $p \notin C$.

From Cases (1) and (2), it holds that $F(\upsilon, y_t, \bar{y}) \notin -\text{int } C, \quad \forall \upsilon \in T(y_t)$. Letting $t \in 0^+$, we obtain by generalized hemicontinuity of $F(.,.,y)$ and Lemma 2.5 that there exists $\bar{u} \in T(\bar{x})$ such that

$$F(\bar{u}, \bar{x}, \bar{y}) \notin -\text{int } C.$$

Now we show that $F(\bar{u}, \bar{x}, y) \notin -\text{int } C$. Suppose on the contrary, $F(\bar{u}, \bar{x}, y) \in -\text{int } C$, then from the facts that $F(\bar{u}, \bar{x}, \bar{x}) = 0, F(\bar{u}, \bar{x}, y) \in -\text{int } C.$, it follows that

$$F(\bar{u}, \bar{x}, \bar{x}) - F(\bar{u}, \bar{x}, y) \in \text{int } C.$$

By the C-explicitly quasiconvexity of $F(u,x,.)$, one has

$$0 - F(\bar{u}, \bar{x}, t\bar{x} + (1-t)y) \in \text{int } C, \quad \forall t \in (0,1).$$

That is,

$$F(\bar{u}, \bar{x}, t\bar{x} + (1-t)y) \in -\text{int } C, \quad \forall t \in (0,1).$$

Since $\bar{y} \in (\bar{x}, y]]$, there exists t_0 such that $\bar{y} = (1-t_0)\bar{x} + t_0 y$ with $F(\bar{u}, \bar{x}, \bar{y}) \in -\text{int } C$ which contradicts $F(\bar{u}, \bar{x}, \bar{y}) \notin -\text{int } C$. So, $F(\bar{u}, \bar{x}, \bar{y}) \notin -\text{int } C$. By the arbitrariness of $y \in K$, we have $\bar{x} \in S_K$ and $S_{K,\text{loc}}^D \subseteq S_K$.

By virtue of C-pseudomonotonity of F, the equivalence between solution set of the GVEP and that of the DGVEP can be established.

Theorem 3.1 *Let $K \subset X$ be a nonempty and convex closed bounded set and suppose Assumption* 3.1 *holds. If F:$D \times K \times K \to Y$ satisfies the followings*

(i) F is C-pseudomonotone;

(ii) F(u,x,.) is C-convex and weakly lower semicontinuous for x ⌊ *K,u* ⌊ *T(x), then* $S_K = S_K^D \neq \emptyset$.

Proof. For any y ⌊ K, set

$$\Gamma(y) = \left\{x \in K | F(v, y, x) \notin \text{int } C, \quad \forall \upsilon \in T(y)\right\}.$$

We claim that Γ is a KKM mapping with closed values. Suppose on the contrary, it does not hold, then there exists a finite set $\{x_1,..., x_n\} \subseteq K$ and z ⌊ $co\{x_1,..., x_n\}$ such that $z \notin \cup_{i=1}^n \Gamma(x_i)$, where $co\{x_1,..., x_n\}$ denotes the convex hull generated by $x_1,..., x_n$. Thus, there exists v_i ⌊ $T(x_i)$, such that $F(v_i, x_i, z)$ ⌊ int C. Since F is C-pseudomonotone, it follows that

$$F(w, z, x_i) \in -\text{int } C, \quad \forall w \in T(z), \quad i = 1, 2, \ldots, n.$$

So, $\sum_1^n t_i F(w, z, x_i) \in -\text{int } C$, where $\sum_1^n t_i = 1, t_i \geq 0, i = 1, 2, \ldots, n$ For $z = \sum_1^n t_i x_i$, due to that the function *F(u,x,.)* is C-convex, we have

$$\sum_1^n t_i F(w, z, x_i) \in F(w, z, z) + C \subset C \subset Y \backslash - \text{int } C,$$

which contradicts $\sum_1^n t_i F(w, z, x_i) \in -\text{int } C$. So, $\{\Gamma(y) \ |y$ ⌊ $K\}$ satisfies the finite-intersection property. In combination with the assumption in (ii), we know that Γ is a KKM mapping with closed values. Since $K \subset X$ is a nonempty and convex closed bounded set, we deduce that K is weakly compact. From Lemma 2.3, there exists x^* ⌊ K such that $x^* \in \cap_{y \in K} \Gamma(y) = S_K^D$. It follows from Lemma 3.1 that $S_K^D \subseteq S_{K,\text{loc}}^D \subseteq S_K$. Furthermore, $S_K \subseteq S_K^D$ due to the C-pseudomonotonity of the F. Thus, $S_K = S_K^D = \emptyset$ and the proof is completed.

Theorem 3.2 *Let* $K \subset X$ *be a nonempty and convex closed bounded set and assume Assumption* 3.1 *holds. If* $F{:}D \times K \times K \rightarrow Y$ *satisfies that*

(i) F is C-pseudomonotone;
(ii) F(u, x,.) is C-explicitly quasiconvex and weakly lower semicontinuous for $x \in K$, $u \in T(x)$,

then $S_K = S_K^D = \emptyset$.

Proof. For any $y \in K$, set

$$\Gamma(y) = \left\{x \in K | F(v, y, x) \notin \text{int } C, \quad \forall v \in T(y)\right\}.$$

We claim that Γ is a KKM mapping. Suppose on the contrary, it does not hold. Then there exists a finite set $\{x_1,..., x_n\} \subseteq K$ and $z \in co\{x_1,..., x_n\}$ such that $z \notin \cup_{i=1}^n \Gamma(x_i)$. Thus, there exists $v_i \in T(x_i)$, such that $F(v_i, x_i, z) \in$ int C. Since F is C-pseudomonotone, it follows that

$$F(w, z, x_i) \in -\text{int } C, \quad \forall w \in T(z), \quad i = 1, 2, \ldots, n.$$

By the quasiconvexity of $F(w,z,.)$, we deduce the set $\{y \in K \mid F(w,z,y) \in -\text{int } C\}$ is convex. For $z = \sum_1^n t_i x_i, \sum_1^n t_i = 1, t_i \geq 0, i = 1, 2, \ldots, n$ $i = 1, 2,\ldots,n$, one has

$$F(w, z, z) \in -\text{int } C,$$

which contradicts $F(w, z, z) = 0$. So, $\{\Gamma(y) \mid y \in K\}$ satisfies the finite-intersection property. By the assumption (ii), we conclude that the Γ is closed value. Hence Γ is a KKM mapping with closed values. Following the similar arguments in the proof of Theorem 3.1, we can obtain the desired result.

In the sequel, we shall present some sufficient conditions for the nonemptiness and bound-edness of the solution set of the GVEP provided that it is strictly feasible in the strong sense.

Theorem 3.3 *Let K be a nonempty, closed, convex and well-positioned subset of a real reflexive Banach space X and Assumption* 3.1 *hold. If F: D × K × K→Y satisfies the followings*

(i) F is C-pseudomonotone;
(ii) $F(u,x,.)$ is C-convex and weakly lower semicontinuous for $x \in K, u \in T(x)$, then, the GVEP has a nonempty bounded solution set whenever it is strictly feasible in the strong sense.

Proof. Suppose that the GVEP is strictly feasible in the strong sense. Then there exists $x_0 \in K$ such that $x_0 \in \Psi_s^+$, i.e.,

$$F(u, x_0, x_0 + z) \in \text{int } C, \quad \forall u \in T(x_0), \quad z \in K_\infty.$$

Set

$$M = \left\{x \in K | F(u, x_0, x) \notin \text{int } C\right\}, \quad \forall u \in T(x_0).$$

By Assumption 3.1 and (ii), $x_0 \in M$ and M is weakly closed. We assert that M is bounded. Suppose on the contrary it does not holds, then there exists a sequence $\{x_n\} \subseteq M$ with $\|x_n\| \to +\infty$ as $n \to +\infty$. Since X is a real reflexive Banach space, without loss of generality, we may take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\frac{1}{\|x_{n_k} - x_0\|} \in (0, 1), \quad \lim_{k\to+\infty} \frac{x_{n_k}}{\|x_{n_k}\|} \rightharpoonup z \in K_\infty$$

and

$$\lim_{k\to+\infty} \frac{x_{n_k} - x_0}{\|x_{n_k} - x_0\|} = \lim_{k\to+\infty} \frac{x_{n_k}}{\|x_{n_k}\|} \rightharpoonup z \in K_\infty.$$

Indeed, since $\lim_{k\to+\infty} \frac{x_0}{\|x_{n_k} - x_0\|} = 0$, there holds

$$\begin{aligned}\lim_{k\to+\infty} \frac{x_{n_k} - x_0}{\|x_{n_k} - x_0\|} &= \lim_{k\to+\infty} \frac{x_{n_k}}{\|x_{n_k} - x_0\|} - \lim_{k\to+\infty} \frac{x_0}{\|x_{n_k} - x_0\|} \\ &= \lim_{k\to+\infty} \frac{\|x_{n_k}\|}{\|x_{n_k} - x_0\|} \cdot \frac{x_{n_k}}{\|x_{n_k}\|}.\end{aligned}$$

Noting that

$$\frac{\|x_{n_k}\| - \|x_0\|}{\|x_{n_k}\|} \leq \frac{\|x_{n_k} - x_0\|}{\|x_{n_k}\|} \leq \frac{\|x_{n_k}\| + \|x_0\|}{\|x_{n_k}\|},$$

one has $\lim_{k\to+\infty} \frac{\|x_{n_k}\|}{\|x_{n_k} - x_0\|} = 1,$, which yields

$$\lim_{k\to+\infty} \frac{x_{n_k} - x_0}{\|x_{n_k} - x_0\|} = \lim_{k\to+\infty} \frac{x_{n_k}}{\|x_{n_k}\|} \rightharpoonup z \in K_\infty.$$

Since K is well-positioned, by Lemma 2.4, we have $z \neq 0$. It follows from $x_0 \in \Psi_s+$ that

$$F(u, x_0, x_0 + z) \in \text{int } C.$$

Noting that F is C-convex, we have

$$F\left(u, x_0, x_0 + \frac{x_{n_k} - x_0}{\|x_{n_k} - x_0\|}\right) = F\left(u, x_0, \left(1 - \frac{1}{\|x_{n_k} - x_0\|}\right) x_0 + \frac{x_{n_k}}{\|x_{n_k} - x_0\|}\right)$$
$$\in \left(1 - \frac{1}{\|x_{n_k} - x_0\|}\right) F(u, x_0, x_0) + \frac{1}{\|x_{n_k} - x_0\|} F(u, x_0, x_{nk}) - C.$$

It follows from $F(u,x_0,x_0) = 0$ that

$$F\left(u, x_0, x_0 + \frac{x_{n_k} - x_0}{\|x_{n_k} - x_0\|}\right) \in \frac{1}{\|x_{n_k} - x_0\|} F(u, x_0, x_{n_k}) - C.$$

By virtue of $F(u, x_0, x_{n_k}) \notin \text{int } C$, one has $F(u, x_0, x_{n_k}) \in Y \backslash \text{int } C$. Consequently,

$$F\left(u, x_0, x_0 + \frac{x_{n_k} - x_0}{\|x_{n_k} - x_0\|}\right) \varepsilon \frac{1}{\|x_{n_k} - x_0\|} F(u, x_0, x_{nk}) - C \subseteq Y \backslash \text{int } C - C \subseteq Y \backslash \text{int } C,$$

that is,

$$F(u, x_0, x_0 + \frac{x_n - x_0}{\|x_n - x_0\|}) \notin \text{int } C.$$

Taking into account that $F(u,x,.)$ is weakly lower semicontinuous, we obtain

$$F(u, x_0, x_0 + z) \notin \text{int } C,$$

which contradicts $F(u, x_0,x_0 + z) \in$ int C. Thus, M is bounded and so it is weakly compact. For each $p \in K$, set

$$M_p = \left\{x \in M | F(\upsilon, p, x) \notin \text{int } C\right\}, \quad \upsilon \in T(p).$$

We assert $M_p \neq \emptyset$, $\forall\ p \in K, \nu \in T(p)$. Indeed, given $p \in K, \nu \in T(p)$, set $K_0 = \text{conv}(M \cup p) \subseteq K$, where *conv* means the convex hull of a set. Then K_0 is nonempty, convex and weakly compact. By Theorem 3.1, there exists $\bar{x} \in K_0$ such that

$$F(\upsilon, y, \bar{x}) \notin \text{int } C, \quad \forall y \in K_0, \quad \upsilon \in T(p).$$

Since $F(u, x_0, \bar{x}) \notin \text{int}\, C$ implies $\bar{x} \in M$ and $F(v, p, \bar{x}) \notin \text{int}\, C$ implies $\bar{x} \in M_p$, we obtain $M_p = \emptyset$. Obviously, M_p is nonempty and weakly compact.

Next we prove that $\{M_p \mid p \in K\}$ has the finite intersection property. For any finite set $\{p_i | i = 1,2,...,n\} \subseteq K$, let $K_1 = conv\{M \cup \{p_1, p_2,..., p_n\}\}$. Then K_1 is nonempty, convex and weakly compact. By Theorem 3.1, there exists $\hat{x} \in K_1$ such that

$$F(\upsilon, y, \hat{x}) \notin \text{int}\, C, \quad \forall y \in K_1, \quad \upsilon \in T(y).$$

In particular, we have

$$F(u, x_0 \hat{x}) \notin \text{int}\, C, \quad F(\upsilon, p_i, \hat{x}) \notin \text{int}\, C, \quad i = 1, 2, \ldots, n.$$

This means that $\hat{x} \in \cap_{i=1}^{n} M_{pi}$. Thus $\{M_p \mid p \in K\}$ has the finite intersection property. Since M is weakly compact and $M_p \subseteq M$ is weakly closed for all $p \in K$, $v \in T(p)$, it follows that

$$\bigcap_{u \in K} M_p \neq \emptyset.$$

Let $x^* \in \cap_{u \in K} \mathrm{M}_p$, then

$$F(\upsilon, y, x^*) \notin \text{int}\, C, \quad \forall y \in K, \quad \upsilon \in T(y).$$

By Theorem 3.1, x^* is a solution of the GVEP. As for the boundedness of the solution set of the GVEP, it follows from Theorem 3.1 that the solution set of the GVEP is a subset of M.

Remark 3.1 *The authors of* [16]*discuss a special case of the GVEP when T(x) is singleton. In general the GVEP, the condition that F is positively homogeneous with degree* $\alpha > 0$ *in* [16], *is not easily satisfied. Compared with* [16], *we remove the condition that F is positively homogeneous with degree* $\alpha > 0$, *when F(u,x,.) is C-convex rather than C-explicitly quasiconvex. As an application of Theorem* 3.3, *we can obtain the solvability of generalized vector variational inequality under strict feasibility in the strong sense.*

In the sequel, we present the new solvability condition for the GVEP, when *F(u,x,.)* is C-explicitly quasiconvex. First, we present a technical lemma.

Lemma 3.3 *Suppose that* $a,b \in Y$, *with* $a = 0$ *and* $b \notin \text{int}\, C$. *Then, there exists* $c \notin \text{int}\, C$ *such that* $a \leq_c c$ *and* $b \leq_c c$.

Proof. Since $\text{int}\, C \neq \emptyset$, there exists $d \in \text{int}\, C$ such that $d - b \in C$, see [23]. For $t \in [0,1]$, set $dt = td+(1- t)b$. Since C is closed and convex, there exists $t_0 \in (0,1)$ such that

$$d_t \in C, \quad \forall t \in [t_0, 1].$$

Furthermore, there exists $t^* \in [t_0,1]$ such that $d_{t^*} \in \partial C$. That is, $d_t^* \in C$ and $d_t^* \notin \text{int}\, C$. Set $c = d_t^*$. We can verify $c - b = t^*(d - b) \in C$ and $c - 0 = d_t^* \in C$.

Theorem 3.4 *Let K be a nonempty, closed, convex and well-positioned subset of a real reflexive Banach space X and Assumption* 3.1 *hold. If F:* $D \times K \times K \to Y$ *satisfies the followings*

(i) F is C-pseudomonotone;

(ii) F(u, x,.) is C-explicitly quasiconvex and weakly lower semicontinuous for $x \in K, u \in T(x)$;

(iii) there exists $b \notin \operatorname{int} C$ *such that* $F(u,x_0,m) \leq_C b$ *for* $m \in M$*, where M is defined in Theorem* 3.3*, then, the GVEP has a nonempty bounded solution set whenever it is strictly feasible in the strong sense.*

Proof. Suppose that the GVEP is strictly feasible in the strong sense. Then there exists $x_0 \in K$ such that $x_0 \in \Psi_s^+$,]i.e.,

$$F(u, x_0, x_0 + z) \in \operatorname{int} C, \quad \forall u \in T(x_0), \quad z \in K_\infty.$$

Set

$$D = \{x \in K | F(u, x_0, x) \notin \operatorname{int} C\}, \quad \forall u \in T(x_0).$$

By Assumption 3.1 and assumption (ii), $x_0 \in D$ and D is weakly closed. We assert that D is bounded. Indeed, if it is not the case, there exists a sequence $\{x_n\} \subseteq D$ with $\|x_n\| \to +\infty$ as $n \to +\infty$. Without loss of generality, we may take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\frac{1}{\|x_{n_k} - x_0\|} \in (0, 1), \quad \lim_{k \to +\infty} \frac{x_{n_k} - x_0}{\|x_{n_k} - x_0\|} = \lim_{k \to +\infty} \frac{x_{n_k}}{\|x_{n_k}\|} \rightharpoonup z \in K_\infty.$$

By Lemma 2.3, $z \neq 0$ since K is well-positioned. It follows from $x_0 \in \Psi_s+$ that

$$F(u, x_0, x_0 + z) \in \operatorname{int} C.$$

Since $F(u,x_0,x_0) = 0$, $\forall u \in T(x)$ and condition (iii) holds, i.e., $F(u, x_0, x_{n_k}) \leq_C b, b \notin \operatorname{int} C$, by Lemma 3.3, there exists $c \notin \operatorname{int} C$ such that

$$0 = F(u, x_0, x_0) \leq_C c, F(u, x_0, x_{n_k}) \leq_C c.$$

Taking into account that $F(u,x,.)$ is C-explicitly quasiconvex, we obtain

$$F\left(u, x_0, x_0 + \frac{x_{n_k} - x_0}{\|x_{n_k} - x_0\|}\right) = F\left(u, x_0\left(1 - \frac{1}{\|x_{n_k} - x_0\|}\right)x_0 + \frac{x_{n_k}}{\|x_{n_k} - x_0\|}\right) \leq_C c.$$

Thus, $F\left(u, x_0, x_0 + \frac{x_n - x_0}{\|x_n - x_0\|}\right) \notin \operatorname{int} C$. Since $F(u,x,.)$ is weakly lower semicontinuous, one has

$$F(u, x_0, x_0 + z) \notin \operatorname{int} C$$

This is a contradiction to $F(u,x_0,x_0 + z) \in \operatorname{int} C$. Thus, D is bounded and it is weakly compact. Following the similar arguments in the proof of Theorem 3.3, we can obtain the desired result.

Remark 3.2 *Compared with* [16], *we substitute the condition that F is positively homogeneous by the condition* (iii), *when* $F(u,x,.)$ *is C-explicitly quasiconvex.*

Similar to [16], we can establish the existence of the solution of the GVEP, when F is positively homogeneous with $\alpha > 0$.

Theorem 3.5 *Let K be a nonempty, closed, convex and well-positioned subset of a real reflexive Banach space X and Assumption* 3.1 *hold. If* $F: D \times K \times K \to Y$ *satisfies the followings* *(i) F is C-pseudomonotone;*

(ii)F(u,x,.) is C-explicitly quasiconvex and weakly lower semicontinuous for $x \in K, u \in T(x)$;

(iii) F is positively homogeneous with degree $\alpha > 0$*, i.e., there exists* $\alpha > 0$ *such that*

$$F(u, x, x + t(y - x)) = t^{\alpha} F(u, x, y), \quad \forall x, y \in K, \quad u \in T(x), \quad t \in (0, 1),$$

then, the GVEP has a nonempty bounded solution set whenever it is strictly feasible in the strong sense.

The following example shows that the converse of Theorem 3.3 is not true in general.

Example 3.1 *Let* $X = R$, $K = R$, $D = [0,1]$, $Y = R$, $C = R_+^2$ *and*

$$T(x) = \begin{cases} 1, & \text{if } x > 0 \\ \{0, 1\} & \text{if } x = 0. \end{cases}$$

Let $F: D \times K \times K \to 2^Y$ be defined by

$$F(u, x, y) = \begin{cases} \langle u, y^2 - x^2 \rangle, & \forall x, y \in K, \quad u \in T(x), \\ \langle u, y - x \rangle, & \forall x, y \in K, \quad u \in T(x). \end{cases}$$

It is easy to see that K is well-positioned and F satisfies assumptions of Theorem 3.3. It can be verified that the GVEP has a nonempty bounded solution set $S_K = S_K^D = \{0\}$. On the other hand, it can also be verified that $\Psi_s^+ = \emptyset$.

The following example illustrates the conclusion of Theorem 3.4.

Example 3.2 *Let* $X = R$, $K = (-\infty, -1]$, $D = [0,1]$, $Y = R$, $C = R_+^2$ *and*

$$T(x) = \begin{cases} 1, & \text{if } x < -1 \\ \{0, 1\}, & \text{if } x = -1. \end{cases}$$

Let $F: D \times K \times K \to 2^Y$ be defined by

$$F(u, x, y) = \begin{cases} \langle u, y^2 - x^2 \rangle, & \forall x, y \in K, \quad u \in T(x), \\ \left\langle u, \frac{1}{y} - \frac{1}{x} \right\rangle, & \forall x, y \in K, \quad u \in T(x). \end{cases}$$

For this problem, it can be verified that K is well-positioned and F satisfies assumptions (i) and (ii) of Theorem 3.4 and $F(u,x,.)$ is not C-convex. However, Theorem 3.3 is not applicable. Furthermore, we can verify that $-1 \in \Psi_s^+$ and $M = \{-1\}$. So, there exists $0 \notin \text{int } C$ such that $F(u,x_0,x_0) \leq_C 0$. This means that assumptions (iii) in Theorem 3.4 holds. In summary, all the assumptions of Theorem 3.4 are satisfied for this example. Thus, the GVEP is solvable. In fact, $x^* = -1$ is its a solution. However, F is not positively homogeneous with α 0. Thus, Theorem 3.5 is not applicable.

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Author details
[1]School of Management Science, Qufu Normal University, Shandong Rizhao 276826, China [2]School of Mathematics and Information Science, Weifang University, Shandong Weifang 261000, China

Authors' contributions
All authors carried out the proof. All authors conceived of the study, and participated in its design and coordination. All authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

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References

1. Chiang, Y, Chadli, O, Yao, JC: Generalized vector equilibrium problems with trifunctions. J Glob Optim. **30**, 135–154 (2004). doi:10.1007/s10898-004-8273-0
2. Bianchi, M, Pini, R: Coercivity conditions for equilibrium problems. J Optim Theory Appl. **124**, 79–92 (2005). doi:10.1007/s10957-004-6466-9
3. Bianchi, M, Hadjisavvas, N, Schaible, S: Vector equilibrium problems with generalized monotone bifunctions. J Optim Theory Appl. **92**, 527–542 (1997). doi:10.1023/A:1022603406244
4. Hadjisavvas, N, Schaible, S: From scalar to vector equilibrium problems in the quasimonotone case. J Optim Theory Appl. **96**, 297–309 (1998). doi:10.1023/A:1022666014055
5. Ansari, QH, Konnov, IV, Yao, JC: Characterizations of solutions for vector equilibrium problems. J Optim Theory Appl. **113**, 435–447 (2002). doi:10.1023/A:1015366419163
6. Gong, XH: Strong vector equilibrium problems. J Glob Optim. **36**, 339–349 (2006). doi:10.1007/s10898-006-9012-5
7. Huang, LG: Existence of solutions for vector equilibrium problems. Acta Math Sin. **52**, 905–910 (2009)
8. Liu, QY, Long, XJ, Huang, NJ: Connectedness of the sets of weak efficient solutions for generalized vector equilibrium problems. Mathematica Slovaca. **62**, 123–136 (2012). doi:10.2478/s12175-011-0077-3
9. Long, XJ, Peng, JW: Connectedness and compactness of weak efficient solutions for vector equilibrium Problems. Bull Korean Math Soc. **48**, 1225–1233 (2011)
10. Long, XJ, Huang, NJ, Teo, KL: Existence and stability of solutions for generalized strong vector quasi equilibrium problems. Math Comput Model. **47**, 445–451 (2008). doi:10.1016/j.mcm.2007.04.013
11. He, YR, Ng, KF: Strict feasibility of generalized complementarity problems. J Austral Math Soc Ser A. **81**(1):15–20 (2006). doi:10.1017/S1446788700014609
12. He, YR, Mao, XZ, Zhou, M: Strict feasibility of variational inequalities in reflexive Banach spaces. Acta Math Sin. **23**, 563–570 (2007)
13. Fang, YP, Huang, NJ: Feasibility and solvability for vector complementarity problems. J Optim Theory Appl. **129**, 373–390 (2006). doi:10.1007/s10957-006-9073-0
14. Fang, YP, Huang, NJ: Feasibility and solvability of vector variational inequalities with moving cones in Banach spaces. Nonlinear Anal. **47**, 2024–2034 (2009)
15. Hu, R, Fang, YP: Feasibility-solvability theorem for a generalized system. J Optim Theory Appl. **142**(3):493–499 (2009). doi:10.1007/s10957-009-9510-y
16. Hu, R, Fang, YP: Strict feasibility and solvability for vector equilibrium problems in in reflexive Banach spaces. Optim Lett. **5**, 505–514 (2011). doi:10.1007/s11590-010-0215-9
17. Auslender, A, Teboulle, M: Asymptotic Cones and Functions in Optimization and-Variational Inequalities. Springer New York (2003)
18. Fan, K: Some properties of sets related to fixed point theorems. Math Ann. **266**, 519–537 (1984). doi:10.1007/BF01458545
19. Tarafdar, E: A fixed point theorem quivalent to the Fan-Knaster-Kuratowski-Mazurkiewcz theorem. J Math Anal Appl. **128**, 475–479 (1987). doi:10.1016/0022-247X(87)90198-3
20. Adly, S, Thera, M, Ernst, E: On the closedness of the algebraic difference of closed convex sets. J Math Pures Appl. **82**(9):1219–1249 (2003). doi:10.1016/S0021-7824(03)00024-2
21. Adly, S, Thera, M, Ernst, E: Well-positioned closed convex sets and well-positioned closed convexfunctions. J Glob Optim. **29**, 337–351 (2004)
22. Wang, G, Huang, XX, Zhang, J, Chen, GY: Levitin-Polyak weii-posedness of generalized vector equilibrium problems with functional constraints. Acta Math Sci. **30**, 1400–1412 (2010)
23. Jameson, G: Ordered Linear Spaces: Lecture Notes in Mathematics. SpringerVerlag Berlin Germany (1970)

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