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Fixed point theorems for cyclic Meir-Keeler type mappings in complete metric spaces

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Abstract

In this article, by using the Meir-Keeler type mappings, we obtain some new fixed point theorems for the cyclic orbital stronger (weaker) Meir-Keeler contractions and generalized cyclic stronger (weaker) Meir-Keeler contractions. Our results generalize or improve many recent fixed point theorems in the literature.

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1 Introduction and preliminaries

Throughout this article, by $\mathbb{R}^+$, we denote the set of all non-negative numbers, while $\mathbb{N}$ is the set of all natural numbers. It is well known and easy to prove that if (X, d) is a complete metric space, and if $f: X \to X$ is continuous and f satisfies

$$d(fx, f^2x) \le k \cdot d(x, fx), \quad \text{for all } x \in X \text{ and } k \in (0, 1),$$

then f has a fixed point in X. Using the above conclusion, Kirk et al. [1] proved the following fixed point theorem.

Theorem 1 [1]*Let A and B be two nonempty closed subsets of a complete metric space (X, d), and suppose $f: A \cup B \to A \cup B$ satisfies*

(i) $f(A) \subset B$ and $f(B) \subset A$,
(ii) $d(fx, fy) \le k\cdot d(x, y)$ for all $x \in A$, $y \in B$ and $k \in (0,1)$.

Then $A \cap B$ is nonempty and f has a unique fixed point in $A \cap B$.

The following definitions and results will be needed in the sequel. Let A and B be two nonempty subsets of a metric space (X, d). A mapping $f : A \cup B \to A \cup B$ is called a cyclic map if $f(A) \subseteq B$ and $f(B) \subseteq A$. In the recent, Karpagam and Agrawal [2] introduced the notion of cyclic orbital contraction, and obtained a unique fixed point theorem for such a map.

Definition 1 [2]*Let A and B be nonempty subsets of a metric space (X, d), $f : A \cup B \to A \cup B$ be a cyclic map such that for some $x \in A$, there exists a $\kappa_x \in (0,1)$ such that*

$$d(f^{2n}x, fy) \le k_x \cdot d(f^{2n-1}x, y), \quad n \in \mathbb{N}, \quad y \in A. \tag{1}$$

Then f is called a cyclic orbital contraction.

Theorem 2 [2]*Let A and B be two nonempty closed subsets of a complete metric space (X, d), and let* $f : A \cup B \to A \cup B$ *be a cyclic orbital contraction. Then f has a fixed point in* $A \cap B$.

Furthermore, Kirk et al. [1] introduced the notion of the generalized cyclic mapping and obtained some fixed point results. Let $\{A_i\}_{i=1}^k$ be nonempty subsets of a metric space (X, d), and let $f : \cup_{i=1}^k A_i \to \cup_{i=1}^k A_i$ Then f is called a generalized cyclic map if $f(A_i) \subseteq A_{i+1}$ for $i = 1, 2,..., k$ and $A_{k+1} = A_1$. Kirk et al. [1] first extended the question of wherther Edelstein's [3] classical result for contractive mappings, and they obtained the following theorem.

Theorem 3 [1]*Let* $\{A_i\}_{i=1}^k$ *be nonempty closed subsets of a complete metric space (X, d), at least one of which is compact, and suppose* $f : \cup_{i=1}^k A_i \to \cup_{i=1}^k A_i$ *satisfies the following conditions (where* $A_{k+1} = A_1$*):*

(*i*) $f(A_i) \subseteq A_{i+1}$ *for* $i = 1,2,...,k$,
(*ii*) $d(fx, fy) < d(x, y)$ *whenever* $x \in A_i$, $y \in A_{i+1}$ *and* $x \neq y$, $(i = 1, 2,..., k)$.

Then f has a unique fixed point.

On the other hand, Kirk et al. [1] took up the question of whether condition (*ii*) of Theorem 3 can be replaced by contractive conditions which typically arise in extensions of Banachs theorem. The authors began with a condition introduced by Geraghty [4]. Let S denote the class of those functions $\alpha : \mathbb{R}^+ \to [0,1)$ that satisfy the simple condition:

$$\alpha(t_n) \to 1 \quad \Rightarrow \quad t_n \to 0.$$

Theorem 4 [4]*Let (X, d) be a complete metric space, let* $f : X \to X$, *and suppose that there exists* $\alpha \in S$ *such that*

$$d(fx, fy) \le \alpha(d(x, y)) \cdot d(x, y), \quad \text{for all } x, y \in X.$$

Then f has a unique fixed point z in X and $\{f^n x\}$ *converges to z for each* $x \in X$.

Applying Theorem 4, Kirk et al. [1] proved the below theorem.

Theorem 5 [1]*Let* $\{A_i\}_{i=1}^k$ *be nonempty closed subsets of a complete metric space (X, d), let* $\alpha \in \mathcal{S}$, *and suppose* $f : \cup_{i=1}^k A_i \to \cup_{i=1}^k A_i$ *satisfies the following conditions (where* $A_{k+1} = A_1$*):*

(i) $f(A_i) \subseteq A_{i+1}$ *for* $i = 1,2,...,k$,
(*ii*) $d(fx, fy) \le \alpha(d(x, y)) \cdot d(x, y)$ *for all* $x \in A_i$, $y \in A_{i+1}$, $i=1,2,...,k$.

Then f has a unique fixed point.

In 1969, Boyd and Wong [5] introduced the notion of Φ-contraction. A mapping $f : X \to X$ on a metric space is called Φ-contraction if there exists an upper semi-continuous function $\psi : [0,\infty) \to [0,\infty)$ such that

$$d(fx, fy) \le \Phi(d(x, y)) \quad \text{for all } x, y \in X.$$

Kirk et al. [1] also proved the below theorem.

Theorem 6 [1]*Let* $\{A_i\}_{i=1}^k$ *be nonempty closed subsets of a complete metric space* (X, d). *Suppose* $f : \cup_{i=1}^k A_i \to \cup_{i=1}^k A_i$ *satisfies the following conditions (where* $A_{k+1} = A_1$*):*

(*i*) $f(A_i) \subseteq A_{i+1}$ *for* $i = 1,2,\ldots,k$,
(*ii*) $d(fx, fy) \le \Phi(d(x, y))$ *for all* $x \in A_i$, $y \in A_{i+1}$, $i = 1,2,\ldots,k$,

where $\Phi : [0, \infty) \to [0, \infty)$ *is upper semi-contionuous from the right and satisfies* $0 \le \psi(t) < t$ *for* $t > 0$. *Then f has a unique fixed point.*

In this article, we also recall the notion of the Meir-Keeler type mapping. A function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is said to be a Meir-Keeler type mapping (see [6]), if for each $\eta \in \mathbb{R}^+$, there exists $\delta > 0$ such that for $t \in \mathbb{R}^+$ with $\eta \le t < \eta + \delta$, we have $\psi(t) < \eta$. Subsequently, some authors worked on this notion (for example, [7-10]). This article will deal with two new mappings of the stronger Meir-Keeler type and weaker Meir-Keeler type in a metric space (X,d). We first introduce the below notion of stronger Meir-Keeler type mapping in a metric space.

Definition 2 *Let* (X, d) *be a metric space. We call* $\psi : \mathbb{R}^+ \to [0,1)$ *a stronger Meir-Keeler type mapping in X if the mapping* ψ *satisfies the following condition:*

$$\forall \eta > 0 \ \exists \delta > 0 \ \exists \gamma_\eta \in [0,1) \ \forall x, y \in X \ (\eta \le d(x,y) < \delta + \eta \quad \Rightarrow \quad \psi(d(x,y)) < \gamma_\eta).$$

Example 1 *Let* $X = \mathbb{R}^2$ *and we define* $d : X \times X \to \mathbb{R}^+$ *by*

$$d(x,y) = |x_1 - y_1| + |x_2 - y_2| \quad \text{for all } x = (x_1, x_2), \quad y = (y_1, y_2) \in X.$$

If $\psi : \mathbb{R}^+ \to [0,1)$, $\psi(d(x,y)) = \dfrac{d(x,y)}{d(x,y)+1}$, *then* ψ *is a stronger Meir-Keeler type mapping in X.*

The following provides an example of a Meir-Keeler type mapping which is not a stronger Meir-Keeler type mapping in a metric space (X, d).

Example 2 *Let* $X = \mathbb{R}^2$ *and we define* $d : X \times X \to \mathbb{R}^+$ *by*

$$d(x,y) = |x_1 - y_1| + |x_2 - y_2| \quad \text{for all } x = (x_1, x_2), \quad y = (y_1, y_2) \in X.$$

If $\phi : \mathbb{R}^+ \to \mathbb{R}^+$,

$$\varphi(d(x,y)) = \begin{cases} d(x,y) - 1, & \text{if } d(x,y) > 1; \\ 0, & \text{if } d(x,y) \le 1, \end{cases}$$

then ϕ *is a Meir-Keeler type mapping which is not a stronger Meir-Keeler type mapping in X.*

We next introduce the below notion of weaker Meir-Keeler type mapping in a metric space.

Definition 3 *Let* (X, d) *be a metric space, and* $\phi : \mathbb{R}^+ \to \mathbb{R}^+$. *Then* ϕ *is called a weaker Meir-Keeler type mapping in X, if the mapping* ϕ *satisfies the following condition:*

$$\forall \eta > 0 \ \exists \delta > 0 \ \forall x, y \in X \ (\eta \le d(x,y) < \delta + \eta \ \Rightarrow \ \exists n_0 \in \mathbb{N} \ \varphi^{n_0}(d(x,y)) < \eta).$$

Example 3 *Let* $X = \mathbb{R}^2$ *and we define* $d : X \times X \to \mathbb{R}^+$ *by*

$$d(x,y) = |x_1 - y_1| + |x_2 - y_2| \quad \text{for all } x = (x_1, x_2), \quad y = (y_1, y_2) \in X.$$

If $\phi : \mathbb{R}^+ \to \mathbb{R}^+$, $\varphi(d(x, y)) = \frac{1}{2}d(x, y)$, *then* ϕ *is a weaker Meir-Keeler type mapping in* X.

The following provides an example of a weaker Meir-Keeler type mapping which is not a Meir-Keeler type mapping in a metric space (X, d).

Example 4 *Let* $X = \mathbb{R}^2$ *and we define* $d : X \times X \to \mathbb{R}^+$ *by*

$$d(x, y) = |x_1 - y_1| + |x_2 - y_2| \quad \text{for all } x = (x_1, x_2), \quad y = (y_1, y_2) \in X.$$

If $\phi : \mathbb{R}^+ \to \mathbb{R}^+$,

$$\varphi(d(x, y)) = \begin{cases} 0, & \text{if } d(x, y) \leq 1, \\ 2 \cdot d(x, y), & \text{if } 1 < d(x, y) < 2; \\ 1, & \text{if } d(x, y) \geq 2, \end{cases}$$

then ϕ *is a weaker Meir-Keeler type mapping which is not a Meir-Keeler type mapping in* X.

2 The fixed point theorems for cyclic orbital Meir-Keeler contractions

Using the notions of the cyclic orbital contraction (see, Definition 1) and stronger Meir-Keeler type mapping (see, Definition 2), we introduce the below notion of cyclic orbital stronger Meir-Keeler contraction.

Definition 4 *Let* A *and* B *be nonempty subsets of a metric space* (X, d). *Suppose* $f : A \cup B \to A \cup B$ *is a cyclic map such that for some* $x \in A$, *there exists a stronger Meir-Keeler type mapping* $\psi : \mathbb{R}^+ \to [0,1)$ *in* X *such that*

$$d(f^{2n}x, fy) \leq \psi(d(f^{2n-1}x, y)) \cdot d(f^{2n-1}x, y), \quad n \in \mathbb{N}, \quad y \in A. \tag{2}$$

Then f *is called a cyclic orbital stronger Meir-Keeler* ψ*-contraction.*

Now, we are in a position to state the following theorem.

Theorem 7 *Let* A *and* B *be two nonempty closed subsets of a complete metric space* (X, d), *and let* $\psi : \mathbb{R}^+ \to [0,1)$ *be a stronger Meir-Keeler type mapping in* X. *Suppose* $f : A \cup B \to A \cup B$ *is a cyclic orbital stronger Meir-Keeler* ψ*-contraction. Then* $A \cap B$ *is nonempty and* f *has a unique fixed point in* $A \cap B$.

Proof. Since $f : A \cup B \to A \cup B$ *is a* cyclic orbital stronger Meir-Keeler ψ-contraction, there exists $x \in A$ satisfying (2), and we also have that for each $n \in \mathbb{N}$,

$$\begin{aligned} d(f^{2n}x, f^{2n+1}x) &\leq \psi d(f^{2n-1}x, f^{2n}x)) \cdot d(f^{2n-1}x, f^{2n}x) \\ &\leq d(f^{2n-1}x, f^{2n}x), \end{aligned}$$

and

$$\begin{aligned} d(f^{2n+1}x, f^{2n+2}x) &= d(f^{2n+2}x, f^{2n+1}x) \\ &\leq \psi(d(f^{2n+1}x, f^{2n}x)) \cdot d(f^{2n+1}x, f^{2n}x) \\ &\leq d(f^{2n+1}x, f^{2n}x) = d(f^{2n}x, f^{2n+1}x). \end{aligned}$$

Generally, we have

$$d(f^n x, f^{n+1}x) \leq d(f^{n-1}x, f^n x), \quad n \in \mathbb{N}.$$

Thus the sequence $\{d(f^n x, f^{n+1}x)\}$ is non-increasing and hence it is convergent. Let $\lim_{n\to\infty} d(f^n x, f^{n+1}x) = \eta$. Then there exists $\kappa_0 \in \mathbb{N}$ and $\delta > 0$ such that for all $n \geq \kappa_0$,

$$\eta \le d(f^n x, f^{n+1}x) < \eta + \delta.$$

Taking into account the above inequality and the definition of stronger Meir-Keeler type mapping ψ *in* X, corresponding to η use, there exists $\gamma_\eta \in [0,1)$ such that

$$\psi(d(f^{k_0+n}x, f^{k_0+n+1}x)) < \gamma_\eta \quad \text{for all} \quad n \in \mathbb{N} \cup \{0\}.$$

Therefore, by (2), we also deduce that for each $n \in \mathbb{N}$,

$$\begin{aligned} d(f^{k_0+n}x, f^{k_0+n+1}x) &\le \psi(d(f^{k_0+n-1}x, f^{k_0+n}x)) \cdot d(f^{k_0+n-1}x, f^{k_0+n}x) \\ &< \gamma_\eta \cdot d(f^{k_0+n-1}x, f^{k_0+n}x), \end{aligned}$$

and it follows that for each $n \in \mathbb{N}$,

$$\begin{aligned} d(f^{k_0+n}x, f^{k_0+n+1}x) &< \gamma_\eta \cdot d(f^{k_0+n-1}x, f^{k_0+n}x) \\ &< \cdots\cdots \\ &< \gamma_\eta^n \cdot d(f^{k_0}x, f^{k_0+1}x). \end{aligned}$$

So

$$\lim_{n\to\infty} d(f^{k_0+n}x, f^{k_0+n+1}x) = 0, \quad \text{since} \quad \gamma_\eta \in [0, 1).$$

We now claim that $\lim_{n\to\infty} d(f^{k_0+n}x, f^{k_0+m}x) = 0$ for $m > n$. For $m, n \in \mathbb{N}$ with $m > n$, we have

$$d(f^{k_0+n}x, f^{k_0+m}x) \le \sum_{i=n}^{m-1} d(f^{k_0+i}x, f^{k_0+i+1}x) < \frac{\gamma_\eta^{m-1}}{1-\gamma_\eta} d(f^{k_0}x, f^{k_0+1}x),$$

and hence $d(f^n x, f^m x) \to 0$, since $0 < \gamma_\eta < 1$. So $\{f^n x\}$ is a Cauchy sequence. Since (X, d) is a complete metric space, there exists $\nu \in A \cup B$ such that $\lim_{n\to\infty} f^n x = \nu$. Now $\{f^{2n}x\}$ is a sequence in A and $\{f^{2n-1}x\}$ is a sequence in B, and also both converge to ν. Since A and B are closed, $\nu \in A \cap B$, and so $A \cap B$ is nonempty. Since

$$\begin{aligned} d(\nu, f\nu) &= \lim_{n\to\infty} d(f^{2n}x, f\nu) \\ &\le \lim_{n\to\infty} [\psi(d(f^{2n-1}x, \nu)) \cdot d(f^{2n-1}x, \nu)] \\ &\le \lim_{n\to\infty} [\gamma_\eta \cdot d(f^{2n-1}x, \nu)] = 0, \end{aligned}$$

hence ν is a fixed point of f.

Finally, to prove the uniqueness of the fixed point, let μ be another fixed point of f. By the cyclic character of f, we have $\nu, \mu \in A \cap B$. Since f is a cyclic orbital stronger Meir-Keeler ψ-contraction, we have

$$\begin{aligned} d(\nu, \mu) = d(\nu, f\mu) &= \lim_{n\to\infty} d(f^{2n}x, f\mu) \\ &\le \lim_{n\to\infty} [\psi(d(f^{2n-1}x, \mu)) \cdot d(f^{2n-1}x, \mu)] \\ &\le \lim_{n\to\infty} [\gamma_\eta \cdot d(f^{2n-1}x, \mu)] \\ &\le \gamma_\eta \cdot d(\nu, \mu) < d(\nu, \mu), \end{aligned}$$

a contradiction. Therefore $\mu = \nu$, and so ν is a unique fixed point of f.

Example 5 *Let* $A = B = X = \mathbb{R}^+$ *and we define* $d: X \times X \to \mathbb{R}^+$ *by*

$$d(x, y) = |x - y|, \quad \text{for } x, y \in X.$$

Define $f : X \to X$ *by*

$$f(x) = \begin{cases} 0, & \text{if } 0 \le x < 1; \\ \frac{1}{4}, & \text{if } x \ge 1. \end{cases}$$

and define $\psi : \mathbb{R}^+ \to [0,1)$ *by*

$$\psi(t) = \begin{cases} \frac{1}{3}, & \text{if } 0 \le t \le 1; \\ \frac{t}{t+1}, & \text{if } t > 1. \end{cases}$$

Then f *is a cyclic orbital stronger Meir-Keeler* ψ*-contraction and* 0 *is the unique fixed point.*

Using the notions of the cyclic orbital contraction (see, Definition 1) and weaker Meir-Keeler type mapping (see, Definition 3), we next introduce the notion of cyclic orbital weaker Meir-Keeler contraction. We first define the below notion of ϕ-mapping.

Definition 5 *Let* (X, d) *be a metric space. We call* $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ *a* ϕ*-mapping in* X *if the function* ϕ *satisfies the following conditions:*

(ϕ_1) ϕ *>is a weaker Meir-Keeler type mapping in* X *with* $\phi(0) = 0$;

(ϕ_2) (a) *if* $\lim_{n\to\infty} t_n = \gamma > 0$, *then* $\lim_{n\to\infty} \phi(t_n) < \gamma$, *and*

(b) *if* $\lim_{n\to\infty} t_n = 0$, *then* $\lim_{n\to\infty} \phi(t_n) = 0$;

(ϕ_3) $\{\phi^n(t)\}_{n\in\mathbb{N}}$ *is decreasing.*

Definition 6 *Let* A *and* B *be nonempty subsets of a metric space* (X, d). *Suppose* $f : A \cup B \to A \cup B$ *is a cyclic map such that for some* $x \in A$, *there exists a* ϕ*-mapping* $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ *in* X *such that*

$$d(f^{2n}x, fy) \le \varphi(d(f^{2n-1}x, y)), \quad n \in \mathbb{N}, \quad y \in A. \tag{3}$$

Then f *is called a cyclic orbital weaker Meir-Keeler* ϕ*-contraction.*

Now, we are in a position to state the following theorem.

Theorem 8 *Let* A *and* B *be two nonempty closed subsets of a complete metric space* (X, d), *and let* $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ *be a* ϕ*-mapping in* X. *Suppose* $f : A \cup B \to A \cup B$ *is a cyclic orbital weaker Meir-Keeler* ϕ*-contraction. Then* $A \cap B$ *is nonempty and* f *has a unique fixed point in* $A \cap B$.

Proof. Since $f : A \cup B \to A \cup B$ *is a* cyclic orbital weaker Meir-Keeler ϕ-contraction, there exists $x \in A$ satisfying (3), and we also have that for each $n \in \mathbb{N}$,

$$d(f^{2n}x, f^{2n+1}x) \le \varphi(d(f^{2n-1}x, f^{2n}x)),$$

and

$$\begin{aligned} d(f^{2n+1}x, f^{2n+2}x) &= d(f^{2n+2}x, f^{2n+1}x) \\ &\le \varphi(d(f^{2n+1}x, f^{2n}x)). \end{aligned}$$

Generally, we have

$$d(f^n x, f^{n+1}x) \le \varphi(d(f^{n-1}x, f^n x)), \quad n \in \mathbb{N}.$$

So we conclude that for each $n \in \mathbb{N}$

$$\begin{aligned} d(f^n x, f^{n+1}x) &\le \varphi(d(f^{n-1}x, f^n x)) \\ &\le \varphi^2(d(f^{n-2}x, f^{n-1}x)) \\ &\le \cdots\cdots \\ &\le \varphi^n(d(x, fx)). \end{aligned}$$

Since $\{\phi^n(d(x, fx))\}_{n\in\mathbb{N}}$ is decreasing, it must converge to some $\eta \ge 0$. We claim that $\eta = 0$. On the contrary, assume that $\eta > 0$. Then by the definition of weaker Meir-Keeler type mapping ϕ in X, there exists $\delta > 0$ such that for $x, y \in X$ with $\eta \le d(x, y) < \delta + \eta$, there exists $n_0 \in \mathbb{N}$ such that $\varphi^{n_0}(d(x, y)) < \eta$. Since $\lim_{n\to\infty} \phi^n(d(x, fx)) = \eta$, there exists $m_0 \in \mathbb{N}$ such that $\eta \le \phi^m(d(x, fx)) < \delta + \eta$, for all $m > m_0$. Thus, we conclude that $\varphi^{m_0+n_0}(d(x_0, x_1)) < \eta$, and we get a contradiction. So $\lim_{n\to\infty} \phi^n(d(x, fx)) = 0$, that is, $\lim_{n\to\infty} d(f^n x, f^{n+1}x) = 0$.

Next, we let $c_m = d(f^n x, f^{n+1}x)$, and we claim that the following result holds:

for each $\varepsilon > 0$, there is $n_0(\varepsilon) \in \mathbb{N}$ such that for all $m, n \ge n_0(\varepsilon)$,

$$d(f^m x, f^{m+1}x) < \varepsilon. \qquad (*)$$

We shall prove (*) by contradiction. Suppose that (*) is false. Then there exists some $\varepsilon > 0$ such that for all $p \in \mathbb{N}$, there are $m_p, n_p \in \mathbb{N}$ with $m_p > n_p \ge p$ satisfying:

(i) m_p is even and n_p is odd,
(ii) $d(f^{m_p}x, f^{n_p}x) \ge \varepsilon$, and
(iii) m_p is the smallest even number such that the conditions (*i*), (*ii*) hold.

Since $c_m \searrow 0$, by (*ii*), we have $\lim_{k\to\infty} d(f^{m_p}x, f^{n_p}x) = \varepsilon$, and

$$\begin{aligned} \varepsilon &\le d(f^{m_p}x, f^{n_p}x) \\ &\le d(f^{m_p}x, f^{m_p+1}x) + d(f^{m_p+1}x, f^{n_p+1}x) + d(f^{n_p+1}x, f^{n_p}x) \\ &\le d(f^{m_p}x, f^{m_p+1}x) + \varphi(d(f^{m_p}x, f^{n_p}x)) + d(f^{n_p+1}x, f^{n_p}x). \end{aligned}$$

Letting $p \to \infty$. Then by the condition (ϕ_2)-(a) of ϕ-mapping, we have

$$\varepsilon \le 0 + \lim_{p\to\infty} \varphi(d(f^{m_p}x, f^{n_p}x)) + 0 < \varepsilon,$$

a contradiction. So $\{f^n x\}$ is a Cauchy sequence. Since (X, d) is a complete metric space, there exists $\nu \in A \cup B$ such that $\lim_{n\to\infty} f_n x = \nu$. Now $\{f^{2n}x\}$ is a sequence in A and $\{f^{2n-1}x\}$ is a sequence in B, and also both converge to ν. Since A and B are closed, $\nu \in A \cap B$, and so $A \cap B$ is nonempty. By the condition (ϕ_2)-(b) of ϕ-mapping, we have

$$\begin{aligned} d(\nu, f\nu) &= \lim_{n\to\infty} d(f^{2n}x, f\nu) \\ &\le \lim_{n\to\infty} \varphi(d(f^{2n-1}x, \nu)) = 0, \end{aligned}$$

hence ν is a fixed point of f. Let μ be another fixed point of f. Since f is a cyclic orbital weaker Meir-Keeler ϕ-contraction, we have

$$\begin{aligned} d(\nu,\mu) &= d(\nu, f\mu) = \lim_{n\to\infty} d(f^{2n}x, f\mu) \\ &\leq \lim_{n\to\infty} \varphi(d(f^{2n-1}x,\mu)) \\ &< d(\nu,\mu), \end{aligned}$$

a contradiction. Therefore, $\mu = \nu$. Thus ν is a unique fixed point of f.

Example 6 *Let* $A = B = X = \mathbb{R}^+$ *and we define* $d : X \times X \to \mathbb{R}^+$ *by*

$$d(x,y) = |x - y|, \quad \text{for } x, y \in X.$$

Define $f : X \to X$ *by*

$$f(x) = \begin{cases} 0, \ \text{if } 0 \leq x < 1; \\ \frac{1}{4}, \ \text{if } x \geq 1. \end{cases}$$

and define $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ *by*

$$\varphi(t) = \frac{1}{3}t \quad \text{for } t \in \mathbb{R}^+.$$

Then f is a cyclic orbital weaker Meir-Keeler ϕ*-contraction and* 0 *is the unique fixed point.*

3 The fixed point theorems for generalized cyclic Meir-Keeler contractions

Using the notions of the generalized cyclic contraction [1] and stronger Meir-Keeler type mapping, we introduce the below notion of generalized cyclic stronger Meir-Keeler contraction.

Definition 7 *Let* $\{A_i\}_{i=1}^k$ *be nonempty subsets of a metric space* (X, d), *let* $\psi : \mathbb{R}^+ \to [0,1)$ *be a stronger Meir-Keeler type mapping in X, and suppose* $f : \cup_{i=1}^k A_i \to \cup_{i=1}^k A_i$ *satisfies the following conditions* (*where* $A_{k+1} = A_1$):

(*i*) $f(A_i) \subseteq A_{i+1}$ *for* $i = 1,2,\ldots,k$;
(*ii*) $d(fx, fy) \leq \psi(d(x, y)) \cdot d(x, y)$ *for all* $x \in A_i$, $y \in A_{i+1}$, $i=1,2,\ldots,k$.

Then we call f a generalized cyclic stronger Meir-Keeler ψ*-contraction.*

We state the main fixed point theorem for the generalized cyclic stronger Meir-Keeler ψ-contraction, as follows:

Theorem 9 *Let* $\{A_i\}_{i=1}^k$ *be nonempty closed subsets of a complete metric space* (X, d), *let* $\psi : \mathbb{R}^+ \to [0,1)$ *be a stronger Meir-Keeler type mapping in X, and let* $f : \cup_{i=1}^k A_i \to \cup_{i=1}^k A_i$ *be a generalized cyclic stronger Meir-Keeler* ψ*-contraction. Then f has a unique fixed point in* $\cap_{i=1}^k A_i$.

Proof. Given $x_0 \in X$ and let $x_n = f^n x_0$, $n \in \mathbb{N}$. Since f is a generalized cyclic stronger Meir-Keeler ψ-contraction, we have that for each $n \in \mathbb{N}$

$$\begin{aligned} d(x_n, x_{n+1}) &= d(f^n x_0, f^{n+1} x_0) \\ &\leq \psi(d(f^{n-1}x_0, f^n x_0)) \cdot d(f^{n-1}x_0, f^n x_0) \\ &\leq d(f^{n-1}x_0, f^n x_0) = d(x_{n-1}, x_n). \end{aligned}$$

Thus the sequence $\{d(x_n, x_{n+1})\}$ is non-increasing and hence it is convergent. Let $\lim_{n\to\infty} d(x_n, x_{n+1}) = \eta \geq 0$. Then there exists $\kappa_0 \in \mathbb{N}$ and $\delta > 0$ such that for all $n \geq \kappa_0$

$$\eta \le d(x_n, x_{n+1}) < \eta + \delta.$$

Taking into account the above inequality and the definition of stronger Meir-Keeler type mapping ψ in X, corresponding to η use, there exists $\gamma_n \in [0,1)$ such that

$$\psi(d(x_{k_0+n}, x_{k_0+n+1})) < \gamma_\eta,$$

for all $n \in \mathbb{N} \cup \{0\}$. Thus, we can deduce that for each $n \in \mathbb{N}$

$$\begin{aligned} d(x_{k_0+n}, x_{k_0+n+1}) &= d(f^{k_0+n}x_0, f^{k_0+n+1}x_0) \\ &\le \psi(d(f^{k_0+n-1}x_0, f^{k_0+n}x_0)) \cdot d(f^{k_0+n-1}x_0, f^{k_0+n}x_0) \\ &< \gamma_\eta d(f^{k_0+n-1}x_0, f^{k_0+n}x_0), \end{aligned}$$

and it follows that for each $n \in \mathbb{N}$

$$\begin{aligned} d(x_{k_0+n}, x_{k_0+n+1}) &< \gamma_\eta d(f^{k_0+n-1}x_0, f^{k_0+n}x_0) \\ &< \cdots \\ &< \gamma_\eta^n d(f^{k_0+1}x_0, f^{k_0+2}x_0). \end{aligned}$$

So

$$\lim_{n\to\infty} d(x_{k_0+n}, x_{k_0+n+1}) = 0, \quad \text{since} \quad \gamma_\eta < 1.$$

We now claim that $\lim_{n\to\infty} d(x_{k_0+n}, x_{k_0+m}) = 0$ for $m > n$. For $m, n \in \mathbb{N}$ with $m > n$, we have

$$\begin{aligned} d(x_{k_0+n}, x_{k_0+m}) &= d(f^{k_0+n}x_0, f^{k_0+m}x_0) \\ &\le \sum_{i=n}^{m-1} d(f^{k_0+i}x_0, f^{k_0+i+1}x_0) \\ &< \frac{\gamma_\eta^{m-1}}{1-\gamma_\eta} d(f^{k_0}x_0, f^{k_0+1}x_0)), \end{aligned}$$

and hence $d(f^n x_0, f^m x_0) \to 0$, since $0 < \gamma_\eta < 1$. So $\{f^n x_0\}$ is a Cauchy sequence. Since X is complete, there exists $\nu \in \cup_{i=1}^k A_i$ such that $\lim_{n\to\infty} f^n x_0 = \nu$. Now for all $i = 0,1, 2,..., k - 1$, $\{f^{kn-i}x\}$ is a sequence in A_i and also all converge to ν. Since A_i is clsoed for all $i = 1, 2,..., k$, we conclude $\nu \in \cap_{i=1}^k A_i$, and also we conclude that $\cap_{i=1}^k A_i \neq \phi$ Since

$$\begin{aligned} d(\nu, f\nu) &= \lim_{n\to\infty} d(f^{kn}x, f\nu) \\ &\le \lim_{n\to\infty} [\psi(d(f^{kn-1}x, \nu)) \cdot d(f^{kn-1}x, \nu)] \\ &\le \lim_{n\to\infty} [\gamma_\eta \cdot d(f^{kn-1}x, \nu)] = 0, \end{aligned}$$

hence ν is a fixed point of f.

Finally, to prove the uniqueness of the fixed point, let μ be another fixed point of f. By the cyclic character of f, we have $\mu \in \cap_{i=1}^k A_i$. Since f is a generalized cyclic stronger Meir-Keeler ψ-contraction, we have

$$\begin{aligned} d(\nu,\mu) &= d(\nu, f\mu) = \lim_{n\to\infty} d(f^{kn}x, f\mu) \\ &\le \lim_{n\to\infty} [\psi(d(f^{kn-1}x,\mu)) \cdot d(f^{kn-1}x,\mu)] \\ &\le \lim_{n\to\infty} [\gamma_\eta \cdot d(f^{kn-1}x,\mu)] \\ &\le \gamma_\eta \cdot d(\nu,\mu) < d(\nu,\mu), \end{aligned}$$

a contradiction. Therefore, $\mu = \nu$. Thus ν is a unique fixed point of *f*.

Example 7 *Let* $X = \mathbb{R}^3$ *and we define* $d : X \times X \to \mathbb{R}^+$ *by*

$$d(x,y) = |x_1 - y_1| + |x_2 - y_2| + |x_3 - y_3|, \quad \text{for } x = (x_1,x_2,x_3), y = (y_1,y_2,y_3) \in X,$$

and let $A = \{(x, 0,0): x \in \mathbb{R}\}$, $B = \{(0,y,0): y \in \mathbb{R}\}$, $C = \{(0,0, z): z \in \mathbb{R}\}$ *be three subsets of X. Define* $f: A \cup B \cup C \to A \cup B \cup C$ *by*

$$\begin{aligned} f((x,0,0)) &= (0,x,0); \quad \text{for all } x \in \mathbb{R}; \\ f((0,y,0)) &= (0,0,y); \quad \text{for all } y \in \mathbb{R}; \\ f((0,0,z)) &= (z,0,0); \quad \text{for all } z \in \mathbb{R}. \end{aligned}$$

and define $\psi : \mathbb{R}^+ \to [0,1)$ *by*

$$\psi(t) = \frac{t}{t+1} \quad ; \text{for } t \in \mathbb{R}^+.$$

Then f is a generalized cyclic stronger Meir-Keeler ψ-contraction and (0,0,0) *is the unique fixed point.*

Using the notions of the generalized cyclic contraction and weaker Meir-Keeler type mapping, we introduce the below notion of generalized cyclic weaker Meir-Keeler contraction.

Definition 8 *Let* $\{A_i\}_{i=1}^k$ *be nonempty subsets of a metric space* (X, d), *let* $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ *be a ϕ-mapping in X, and suppose* $f : \cup_{i=1}^k A_i \to \cup_{i=1}^k A_i$ *satisfies the following conditions (where* $A_{k+1} = A_1$*):*

(*i*) $f(A_i) \subseteq A_{i+1}$ *for* $i = 1,2,\ldots,k$;
(*ii*) $d(fx, fy) \le \phi(d(x, y))$ *for all* $x \in A_i$, $y \in A_{i+1}$, $i = 1,2,\ldots,k$.

Then we call f a generalized cyclic weaker Meir-Keeler ϕ-contraction.

Now, we are in a position to state the following theorem.

Theorem 10 *Let* $\{A_i\}_{i=1}^k$ *be nonempty closed subsets of a complete metric space* (X, d), *let* $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ *be a ϕ-mapping in X, and let* $f : cup_{i=1}^k A_i \to \cup_{i=1}^k A_i$ *be a generalized cyclic weaker Meir-Keeler ϕ-contraction. Then f has a unique fixed point in* $\cap_{i=1}^k A_i$.

Proof. Given $x_0 \in X$ and let $x_n = f^n x_0$, $n \in \mathbb{N}$. Since *f* is a generalized cyclic weaker Meir-Keeler ϕ-contraction, we have that for each $n \in \mathbb{N}$

$$\begin{aligned} d(x_n, x_{n+1}) &= d(f^n x_0, f^{n+1} x_0) \\ &\le \varphi(d(f^{n-1}x_0, f^n x_0)) = \varphi(d(x_{n-1}, x_n)) \\ &\le \cdots\cdots \\ &\le \varphi^n(d(x_0, x_1)). \end{aligned}$$

Since $\{\phi^n(d(x_0,x_1))\}_{n\in\mathbb{N}}$ is decreasing, it must converge to some $\eta \geq 0$. We claim that $\eta = 0$. On the contrary, assume that $\eta > 0$. Then by the definition of weaker Meir-Keeler type mapping ϕ in X, there exists $\delta > 0$ such that for $x, y \in X$ with $\eta \leq d(x, y) < \delta + \eta$, there exists $n_0 \in \mathbb{N}$ such that $\varphi^{n_0}(d(x,y)) < \eta < \eta$. Since $\lim_{n\to\infty} \phi^n(d(x_0,x_1)) = \eta$, there exists $m_0 \in \mathbb{N}$ such that $\eta < \phi^m(d(x_0,x_1)) < \delta + \eta$, for all $m > m_0$. Thus, we conclude that $\varphi^{m_0+n_0}(d(x_0,x_1)) < \eta$, a contradiction. So $\lim_{n\to\infty} \phi^n(d(x_0,x_1)) = 0$, that is, $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$.

Next, we claim that $\{x_n\}$ is a Cauchy sequence. We claim that the following result holds:

for each $\varepsilon > 0$, there is $n_0(\varepsilon) \in \mathbb{N}$ such that for all $m, n \geq n_0(\varepsilon)$,

$$d(x_m, x_n) < \varepsilon, \qquad (**)$$

We shall prove (**) by contradiction. Suppose that (**) is false. Then there exists some $\varepsilon > 0$ such that for all $p \in \mathbb{N}$, there are $m_p, n_p \in \mathbb{N}$ with $m_p > n_p \geq p$ satisfying:

(i) $d(x_{m_p}, x_{n_p}) \geq \varepsilon$, and
(ii) m_p is the smallest number greater than n_p such that the condition (i) holds.

Since

$$\begin{aligned}\varepsilon &\leq d(x_{m_p}, x_{n_p})\\ &\leq d(x_{m_p}, x_{m_{p-1}}) + d(x_{m_{p-1}}, x_{n_p})\\ &\leq d(x_{m_p}, x_{m_{p-1}}) + \varepsilon,\end{aligned}$$

hence we conclude $\lim_{p\to\infty} d(x_{m_p}, x_{n_p}) = \varepsilon$. Since

$$d(x_{m_p}, x_{n_p}) - d(x_{m_p}, x_{m_{p+1}}) \leq d(x_{m_p+1}, x_{n_p}) \leq d(x_{m_p}, x_{m_p+1}) + d(x_{m_p}, x_{n_p}),$$

we also conclude $\lim_{p\to\infty} d(x_{m_p+1}, x_{n_p}) = \varepsilon$. Thus, there exists i, $0 \leq i \leq k - 1$ such that $m_p - n_p + i = 1 \mod k$ for infinitely many p. If $i = 0$, then we have that for such p,

$$\begin{aligned}\varepsilon &\leq d(x_{m_p}, x_{n_p})\\ &\leq d(x_{m_p}, x_{m_p+1}) + d(x_{m_p+1}, x_{n_p+1}) + d(x_{n_p+1}, x_{n_p})\\ &\leq d(x_{m_p}, x_{m_p+1}) + \varphi(d(x_{m_p}, x_{n_p})) + d(x_{n_p+1}, x_{n_p}).\end{aligned}$$

Letting $p \to \infty$. Then by the condition (ϕ_2)-(a) of ϕ-mapping, we have

$$\varepsilon \leq 0 + \lim_{p\to\infty} \varphi(d(x_{m_p}, x_{n_p})) + 0 < \varepsilon,$$

a contradiction. The case $i \neq 0$ similar. Thus, $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists $\nu \in \cup_{i=1}^k A_i$ such that $\lim_{n\to\infty} x_n = \nu$. Now for all $i = 0, 1, 2, ..., k - 1$, $\{f^{kn-i}x\}$ is a sequence in A_i and also all converge to ν. Since A_i is closed for all $i = 1, 2, ..., k$, we conclude $\nu \in \cup_{i=1}^k A_i$, and also we conclude that $\cap_{i=1}^k A_i \neq \phi$. By the condition (ϕ_2)-(b) of ϕ-mapping, we have

$$\begin{aligned}d(\nu, f\nu) &= \lim_{n\to\infty} d(f^{kn}x, f\nu)\\ &\leq \lim_{n\to\infty} \varphi(d(f^{kn-1}x, \nu)) = 0,\end{aligned}$$

hence ν is a fixed point of f. Let μ be another fixed point of f. Since f is a generalized cyclic weaker Meir-Keeler ϕ-contraction, we have

$$\begin{aligned} d(\nu,\mu) &= d(\nu, f\mu) = \lim_{n\to\infty} d(f^{kn}x, f\mu) \\ &\leq \lim_{n\to\infty} \varphi(d(f^{kn-1}x,\mu)) \\ &< d(\nu,\mu), \end{aligned}$$

a contradiction. Therefore, $\mu = \nu$. Thus ν is a unique fixed point of f.

Example 8 *Let* $X = \mathbb{R}^3$ *and we define* $d : X \times X \to \mathbb{R}^+$ *by*

$$d(x,y) = |x_1 - y_1| + |x_2 - y_2| + |x_3 - y_3|, \text{ for } x = (x_1, x_2, x_3), y = (y_1, y_2, y_3) \in X,$$

and let $A = \{(x,0,0): x \in \mathbb{R}\}$, $B = \{(0,y,0): y \in \mathbb{R}\}$, $C = \{(0,0,z): z \in \mathbb{R}\}$ *be three subsets of X. Define* $f : A \cup B \cup C \to A \cup B \cup C$ *by*

$$f((x,0,0)) = \left(0, \frac{1}{4}x, 0\right); \quad \text{for all } x \in \mathbb{R};$$

$$f((0,y,0)) = (0,0,\frac{1}{4}y); \quad \text{for all } y \in \mathbb{R};$$

$$f((0,0,z)) = (\frac{1}{4}z, 0, 0); \quad \text{for all } z \in \mathbb{R}.$$

and define $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ *by*

$$\varphi(t) = \frac{1}{3}t \quad ; \text{for } t \in \mathbb{R}^+.$$

Then f is a generalized cyclic weaker Meir-Keeler ϕ*-contraction and* (0, 0, 0) *is the unique fixed point.*

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The author declares that they have no competing interests.

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