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Weak compactness and the Eisenfeld-Lakshmikantham measure of nonconvexity

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Abstract

In this article, weakly compact subsets of real Banach spaces are characterized in terms of the Cantor property for the Eisenfeld-Lakshmikantham measure of nonconvexity. This characterization is applied to prove the existence of fixed points for condensing maps, nonexpansive maps, and isometries without convexity requirements on their domain.

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1. Introduction

Throughout this article, $(X, \|\cdot\|)$ will denote a real Banach space.

Definition 1.1. *The Eisenfeld-Lakshmikantham measure of nonconvexity (E-L measure of nonconvexity, for short) of a bounded subset A of X is defined by*

$$\mu(A) = \sup_{x \in \overline{\text{co}}A} \inf_{a \in A} \|x - a\| = H(A, \overline{\text{co}}A),$$

where $\overline{\text{co}}A$ denotes the closed and convex hull of A and $H(C, D)$ is the Hausdorff-Pompeiu distance between the bounded subsets C and D of X .

The E-L measure of nonconvexity was introduced in [1]. The following properties of μ can be derived in a fairly straightforward manner from its definition. Here, $A, B \subset X$ are assumed to be bounded and \overline{A} denotes the closure of A .

- (i) $\mu(A) = 0$ if, and only if, \overline{A} is convex.
- (ii) $\mu(\lambda A) = |\lambda| \mu(A)$ ($\lambda \in \mathbb{R}$).
- (iii) $\mu(A + B) \leq \mu(A) + \mu(B)$.
- (iv) $|\mu(A) - \mu(B)| \leq \mu(A - B)$.
- (v) $\mu(\overline{A}) = \mu(A)$.
- (vi) $\mu(A) \leq \delta(A)$, where

$$\delta(A) = \sup_{x, y \in A} \|x - y\|$$

is the diameter of A .

$$(vii) \quad |\mu(A) - \mu(B)| \leq 2H(A, B).$$

The following result was obtained in [2].

Lemma 1.2 ([2, Lemma 2.4]). *Let $\{A_n\}_{n=1}^\infty$ be a decreasing sequence of nonempty, closed, and bounded subsets of a Banach space X with*

$$\lim_{n \rightarrow \infty} \mu(A_n) = 0,$$

where μ is the E-L measure of nonconvexity of X , and let $A_\infty = \bigcap_{n=1}^\infty A_n$. Then $A_\infty = \bigcap_{n=1}^\infty \overline{\text{co}}A_n$.

Definition 1.3. *Let Y be a nonempty and closed subset of the Banach space X . The E-L measure of nonconvexity μ of X is said to have the Cantor property in Y if for every decreasing sequence $\{A_n\}_{n=1}^\infty$ of nonempty, closed, and bounded subsets of Y such that $\lim_{n \rightarrow \infty} \mu(A_n) = 0$, the closed and bounded (and, by Lemma 1.2, convex) set $A_\infty = \bigcap_{n=1}^\infty A_n$ is nonempty.*

Theorem 1.4 ([2, Theorem 2.5]). *For a Banach space X , the following statements are equivalent:*

- (i) X is reflexive.
- (ii) The E-L measure of nonconvexity of X satisfies the Cantor property in X .

In Section 2 below we prove a result (Theorem 2.1), more general than Theorem 1.4, which characterizes weak compactness also in terms of the Cantor property for the E-L measure of nonconvexity. As an application of this characterization, we show that the convexity requirements can be dropped from the hypotheses of a number of fixed point theorems in [3-5] for condensing maps (see Section 3.1), nonexpansive maps (see Section 3.2) and isometries (see Section 4).

2. A characterization of weak compactness

Theorem 2.1. *Let X be a Banach space with E-L measure of nonconvexity μ , and let C be a nonempty, weakly closed, and bounded subset of X . The following statements are equivalent:*

- (i) C is weakly compact.
- (ii) The measure μ satisfies the Cantor property in $\overline{\text{co}}C$.
- (iii) For every decreasing sequence $\{A_n\}_{n=1}^\infty$ of nonempty and closed subsets of $\overline{\text{co}}C$ such that $\lim_{n \rightarrow \infty} \mu(A_n) = 0$, the set $A_\infty = \bigcap_{n=1}^\infty A_n$ is nonempty.

Proof. Part (iii) is just a rephrasing of part (ii).

Suppose (i) holds. By the Krein-Šmulian theorem [6, Theorem V.6.4], $\overline{\text{co}}C$ is weakly compact. Let $\{A_n\}_{n=1}^\infty$ be a decreasing sequence of nonempty and closed subsets of $\overline{\text{co}}C$ with $\lim_{n \rightarrow \infty} \mu(A_n) = 0$. By Lemma 1.2, $A_\infty = \bigcap_{n=1}^\infty \overline{\text{co}}A_n$, where $\{\overline{\text{co}}A_n\}_{n=1}^\infty$ is a

decreasing sequence of nonempty closed, and convex subsets of the weakly compact and convex set $\overline{\text{co}}C$. The Šmulian theorem [6, Theorem V.6.2] then allows us to conclude that A_∞ is nonempty.

Conversely, assume (iii). If we take any decreasing sequence $\{C_n\}_{n=1}^\infty$ of nonempty, closed, and convex subsets of the bounded and convex set $\overline{\text{co}}C$, then $\mu(C_n) = 0$ ($n \in \mathbb{N}$), and therefore $C_\infty \neq \emptyset$. Appealing again to the Šmulian theorem [6, Theorem V.6.2] we find that the convex set $\overline{\text{co}}C$ is weakly compact. Finally, being a weakly closed subset of $\overline{\text{co}}C$, the set C itself is weakly compact.

Note that Theorem 1.4 can be easily derived from Theorem 2.1. For the sake of completeness, we give a proof of this fact.

Corollary 2.2. *For a Banach space X with E-L measure of nonconvexity μ , the following statements are equivalent:*

- (i) X is reflexive.
- (ii) The closed unit ball B_X of X is weakly compact.
- (iii) For every decreasing sequence $\{A_n\}_{n=1}^\infty$ of nonempty and closed subsets of B_X such that $\lim_{n \rightarrow \infty} \mu(A_n) = 0$, the set $A_\infty = \bigcap_{n=1}^\infty A_n$ is nonempty and convex.
- (iv) The measure μ satisfies the Cantor property in X .

Proof. The equivalence of (i) and (ii) is well known [6, Theorem V.4.7]. To see that (ii) and (iii) are equivalent, take $C = B_X$ in Theorem 2.1, bearing in mind that $\overline{\text{co}}C = B_X$. For the proof that (iii) implies (iv), let $\{A_n\}_{n=1}^\infty$ be a decreasing sequence of nonempty, closed, and bounded subsets of X such that $\lim_{n \rightarrow \infty} \mu(A_n) = 0$. Since A_1 is bounded and $\{A_n\}_{n=1}^\infty$ is decreasing, there exists $\lambda > 0$ such that

$$B_n = \lambda A_n \subset B_X \quad (n \in \mathbb{N}).$$

Now $\{B_n\}_{n=1}^\infty$ is a decreasing sequence of nonempty, closed, and bounded subsets of B_X with

$$\lim_{n \rightarrow \infty} \mu(B_n) = \lambda \lim_{n \rightarrow \infty} \mu(A_n) = 0.$$

Therefore $A_\infty = \lambda^{-1}B_\infty \neq \emptyset$, as asserted. Finally, it is apparent that (iv) implies (iii).

3. Fixed points for condensing and nonexpansive maps

Definition 3.1 ([2, Definition 4.3]). *Let Y be a nonempty, closed, and bounded subset of a Banach space X . A map $f: Y \rightarrow Y$ is said to have property (C) if $\lim_{n \rightarrow \infty} \mu(Y_n) = 0$, where μ is the E-L measure of nonconvexity in X and $\{Y_n\}_{n=1}^\infty$ is the decreasing sequence of nonempty, closed, and bounded subsets of X defined by*

$$Y_1 = \overline{f(Y)}, \quad Y_{n+1} = \overline{f(Y_n)} \quad (n \in \mathbb{N}).$$

Proposition 3.2. *Let Y be a nonempty and weakly compact subset of a Banach space X , and let $f: Y \rightarrow Y$ be a map with property (C). Then Y contains a nonempty, closed, and convex (hence, weakly compact) set K such that $f(K) \subset K$.*

Proof. Let $\{Y_n\}_{n=1}^\infty$ be as above. Since f has property (C), we have

$$\lim_{n \rightarrow \infty} \mu(Y_n) = 0.$$

Theorem 2.1 yields that $K = Y_\infty = \bigcap_{n=1}^\infty Y_n$ is nonempty, closed, and convex. Clearly, $f(K) \subset K$. Closed convex sets are weakly closed [6, Theorem V.3.18] and therefore K is weakly compact, as claimed.

As an application of Proposition 3.2, some fixed point theorems for condensing and nonexpansive maps will be proved.

3.1. Condensing maps

Definition 3.3. Let Y be a nonempty and bounded subset of a Banach space X , and let γ denote some measure of noncompactness in X , in the sense of [7, Definition 3.2]. A map $f: Y \rightarrow Y$ is called γ -condensing provided that

$$\gamma(f(B)) < \gamma(B)$$

for every $B \subset Y$ with $f(B) \subset B$ and $\gamma(B) > 0$.

The following result is an extension of [3, Theorem 4]. It can be also viewed as a version of Sadovskii's theorem [8].

Theorem 3.4. Let γ be a measure of noncompactness in a Banach space X and let Y be a nonempty and closed subset of X such that $\overline{\text{co}}Y$ is weakly compact. Assume that the map $f: Y \rightarrow Y$ is continuous, γ -condensing and has property (C). Then f has at least one fixed point in Y .

Proof. Arguing as in the proof of Proposition 3.2 we get a nonempty, closed, and convex set $K \subset Y$ such that $f(K) \subset K$. The required conclusion follows from [7, Corollary 3.5].

3.2. Nonexpansive maps

Definition 3.5. Let $A \subset X$ be bounded. A point $x \in A$ is a diametral point of A provided that $\sup_{y \in A} \|x - y\| = \delta(A)$. The set A is said to have normal structure if for each convex subset B of A containing more than one point, there exists some $x \in B$ which is not a diametral point of B .

The following is a version of Kirk's seminal theorem (cf. [4, Theorem 4.1]) which does not require the convexity of the domain.

Theorem 3.6. Let Y be a nonempty and weakly compact subset of a Banach space X . Suppose Y has normal structure. If $f: Y \rightarrow Y$ has property (C) and is nonexpansive, that is, satisfies

$$\|f(x) - f(y)\| \leq \|x - y\| \quad (x, y \in Y),$$

then f has a fixed point.

Proof. The asserted conclusion can be derived from Proposition 3.2 and [4, Theorem 4.1].

4. Fixed points for isometries

Definition 4.1. Let Y be a nonempty and weakly compact subset of a Banach space X . We say that Y has the fixed point property, FPP for short, if every isometry $f: Y \rightarrow Y$ has

a fixed point. The set Y is said to have the hereditary FPP if every nonempty, closed, and convex subset of Y has the FPP.

Definition 4.2. Given a nonempty, closed, and bounded subset Y of a Banach space X , let

$$r(x) = r(x, Y) = \sup_{y \in Y} \|x - y\| \quad (x \in X),$$

$$r(Y) = \inf_{x \in Y} r(x),$$

and

$$\tilde{Y} = \{x \in Y : r(x) = r(Y)\}.$$

The number $r(Y)$ and the members of \tilde{Y} are respectively called Chebyshev radius and Chebyshev centers of Y . Further, define

$$\begin{aligned} \tilde{Y}_n &= \left\{ x \in Y : r(Y) \leq r(x) \leq r(Y) + \frac{1}{n} \right\} \\ &= \bigcap_{y \in Y} \left[y + \left(r(Y) + \frac{1}{n} \right) B_X \right] \cap Y \quad (n \in \mathbb{N}). \end{aligned}$$

We say that Y has property (S) provided that $\lim_{n \rightarrow \infty} \mu(\tilde{Y}_n) = 0$, where μ is the E-L measure of nonconvexity in X .

Lemma 4.3. Let Y be a nonempty and weakly compact subset of a Banach space X . If Y has property (S), then \tilde{Y} is nonempty, closed, and convex.

Proof. Note that $\{\tilde{Y}_n\}_{n=1}^{\infty}$ is a decreasing sequence of nonempty and closed subsets of Y , with $\lim_{n \rightarrow \infty} \mu(\tilde{Y}_n) = 0$. From Theorem 2.1, the set of Chebyshev centers

$$\tilde{Y} = \tilde{Y}_{\infty} = \bigcap_{n=1}^{\infty} \tilde{Y}_n$$

is nonempty closed, and convex.

Theorem 4.4. Let Y be a nonempty and weakly compact subset of a Banach space X . Assume further that Y has both property (S) and the hereditary FPP. Then every isometry $f: Y \rightarrow Y$ such that $f(\tilde{Y}) \subset \tilde{Y}$ has a fixed point in \tilde{Y} .

Proof. From Lemma 4.3, \tilde{Y} is nonempty, closed, and convex. It suffices to invoke the hereditary FPP of Y .

Definition 4.5. Let Y be a nonempty, closed, and bounded subset of a Banach space X . Given an isometry $f: Y \rightarrow Y$, let us consider

$$R_{f,0}(x) = r(x, Y) = \sup_{z \in Y} \|x - z\| \quad (x \in X),$$

$$R_{f,m}(x) = r(x, Y_m) = \sup_{z \in Y_m} \|x - z\|$$

$$= r(x, f^m(Y)) = \sup_{y \in Y} \|x - f^m(y)\| \quad (x \in X, m \in \mathbb{N}),$$

$$R_f(x) = \lim_{m \rightarrow \infty} R_{f,m}(x) = \inf_{m \in \mathbb{Z}_+} R_{f,m}(x) \quad (x \in X),$$

$$R_f(Y) = \inf_{x \in Y} R_f(x),$$

and

$$\hat{Y}_f = \{x \in Y : R_f(x) = R_f(Y)\}.$$

The number $R_f(Y)$ and the set \hat{Y}_f are respectively called asymptotic Chebyshev radius and asymptotic Chebyshev center of $\{Y_m\}_{m=0}^\infty = \{f^m(Y)\}_{m=0}^\infty$ with respect to Y . Further, define

$$\hat{Y}_{f,n} = \left\{ x \in Y : R_f(Y) \leq R_f(x) \leq R_f(Y) + \frac{1}{n} \right\}$$

$$= \bigcap_{m \in \mathbb{Z}_+} \bigcap_{z \in Y_m} \left[z + \left(R_f(Y) + \frac{1}{n} \right) B_X \right] \cap Y \quad (n \in \mathbb{N}).$$

We say that f has property (A) provided that $\lim_{n \rightarrow \infty} \mu(\hat{Y}_{f,n}) = 0$, where μ is the E-L measure of nonconvexity in X .

Lemma 4.6. *Let Y be a nonempty and weakly compact subset of a Banach space X , and let $f: Y \rightarrow Y$ be an isometry with property (A). Then \hat{Y}_f is nonempty, closed, and convex.*

Proof. Note that $\{\hat{Y}_{f,n}\}_{n=1}^\infty$ is a decreasing sequence of nonempty and closed subsets of Y , with $\lim_{n \rightarrow \infty} \mu(\hat{Y}_{f,n}) = 0$. From Theorem 2.1, the asymptotic Chebyshev center

$$\hat{Y}_f = \hat{Y}_{f,\infty} = \bigcap_{n=1}^\infty \hat{Y}_{f,n}$$

is nonempty closed, and convex.

Lemma 4.7. *Let Y be a nonempty and weakly compact subset of a Banach space X , and let $f: Y \rightarrow Y$ be an isometry. Assume $c \in \hat{Y}_f$ is such that $f(c) = c$. then $c \in \check{Y}$.*

Proof. We argue as in the proof of [5, Theorem 2]. Since f is an isometry and $f(c) = c$, we have

$$R_{f,m}(c) = R_{f,m}(f(c)) = R_{f,m-1}(c) \quad (m \in \mathbb{N}),$$

whence

$$R_{f,m}(c) = R_{f,0}(c) \quad (m \in \mathbb{N}).$$

From Definition 4.5 and the hypothesis that $c \in \hat{Y}_f$ it follows that

$$r(c, Y) = R_{f,0}(c) = \lim_{m \rightarrow \infty} R_{f,m}(c) = R_f(c) = R_f(Y).$$

Now, for any $x \in Y$ we get

$$r(c, Y) = R_f(Y) \leq \inf_{m \in \mathbb{Z}_+} R_{f,m}(x) \leq R_{f,0}(x) = r(x, Y),$$

which proves that $c \in \check{Y}$.

Theorem 4.8. *Let Y be a nonempty and weakly compact subset of a Banach space X . Suppose Y has the hereditary FPP. Then every isometry $f:Y \rightarrow Y$ with property (A) has a fixed point in \check{Y} .*

Proof. Let $f:Y \rightarrow Y$ be an isometry with property (A). From Lemma 4.6, \hat{Y}_f is nonempty, closed, and convex. Moreover, $f(\hat{Y}_f) \subset \hat{Y}_f$ (cf. [5, Proposition 3]). The hereditary FPP of Y then yields $c \in \hat{Y}_f$ such that $f(c) = c$, and Lemma 4.7 ensures that $c \in \check{Y}$.

Corollary 4.9 ([5, Theorem 2]). *Let Y be a nonempty, weakly compact, and convex subset of a Banach space X . Suppose Y has the hereditary FPP. Then every isometry $f:Y \rightarrow Y$ has a fixed point in \check{Y} .*

Proof. Since Y is convex, every isometry $f:Y \rightarrow Y$ has property (A). Theorem 4.8 completes the proof.

The following is an extension of Kirk's theorem [4, Theorem 4.1] for isometries.

Theorem 4.10. *Let Y be a nonempty and weakly compact subset of a Banach space X . Assume further that Y has normal structure. Then every isometry $f:Y \rightarrow Y$ with property (A) has a fixed point in \check{Y} .*

Proof. Let $f:Y \rightarrow Y$ be an isometry with property (A). From Lemma 4.6, \hat{Y}_f is nonempty, closed, and convex. Moreover, $f(\hat{Y}_f) \subset \hat{Y}_f$ (cf. [5, Proposition 3]). Kirk's theorem [4, Theorem 4.1] along with Lemma 4.7 yield $c \in \check{Y}$ such that $f(c) = c$.

Corollary 4.11 ([5, Corollary 1]). *Let Y be a nonempty, weakly compact, and convex subset of a Banach space X . Assume further that Y has normal structure. Then every isometry $f:Y \rightarrow Y$ has a fixed point in \check{Y} .*

Proof. The convexity of Y guarantees that every isometry $f:Y \rightarrow Y$ satisfies property (A). The desired conclusion follows from Theorem 4.10.

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Competing interests

The author declares that she has no competing interests.

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