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Coupled fixed point theorems for generalized Mizoguchi-Takahashi contractions with applications

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Abstract

We derive some new coupled fixed point theorems for nonlinear contractive maps that satisfied a generalized Mizoguchi-Takahashi's condition in the setting of ordered metric spaces. Presented theorems extends and generalize many well-known results in the literature. As an application, we give an existence and uniqueness theorem for the solution to a two-point boundary value problem.

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1 Introduction

Let (X, d) be a metric space. Denote by $P(X)$ the set of all nonempty subsets of X and $CB(X)$ the family of all nonempty closed and bounded subsets of X. A point x in X is a fixed point of a multivalued map $T : X \to P(X)$, if $x \in Tx$. Nadler [1] extended the Banach contraction principle to multivalued mappings.

Theorem 1.1 (Nadler [1]) *Let (X, d) be a complete metric space and let $T : X \to CB(X)$ be a multivalued map. Assume that there exists $r \in [0,1)$ such that*

$$H(Tx, Ty) \leq rd(x, y)$$

for all $x, y \in X$, where H is the Hausdorff metric with respect to d. Then T has a fixed point.

Reich [2] proved the following generalization of Nadler's fixed point theorem.

Theorem 1.2 (Reich [2]) *Let (X, d) be a complete metric space and $T : X \to C(X)$ be a multi-valued map with non empty compact values. Assume that*

$$H(Tx, Ty) \leq \varphi(d(x, y))d(x, y)$$

for all $x, y \in X$, where ϕ is a function from $[0, \infty)$ into $[0,1)$ satisfying $\limsup_{s\to t^+}\varphi(s) < 1$ for all $t > 0$. Then T has a fixed point.

Mizoguchi and Takahashi [3] proved the following generalization of Nadler's fixed point theorem for a weak contraction which is a partial answer of Problem 9 in Reich [4].

Theorem 1.3 (Mizoguchi and Takahashi [3]) *Let* (X, d) *be a complete metric space and* $T : X \to CB(X)$ *be a multivalued map. Assume that*

$$H(Tx, Ty) \le \varphi(d(x, y))d(x, y)$$

for all $x, y \in X$, *where* ϕ *is a function from* $[0, \infty)$ *into* $[0,1)$ *satisfying* $\limsup_{s\to t^+}\varphi(s) < 1$ *for all* $t \ge 0$. *Then T has a fixed point.*

Suzuki [5] gave a very simple proof of Theorem 1.3.

Very recently, Amini-Harandi and O'Regan [6] obtained a nice generalization of Mizoguchi and Takahashi's fixed point theorem. Throughout the article, let Ψ be the family of all functions $\psi : [0, \infty) \to [0, \infty)$ satisfying the following conditions:

(a) $\psi(s) = 0 \Leftrightarrow s = 0$,

(b) ψ is nondecreasing,

(c) $\limsup_{s\to 0^+} \frac{s}{\psi(s)} < \infty$.

We denote by Φ the set of all functions $\phi : [0, \infty) \to [0,1)$ satisfying $\limsup_{r\to t^+}\varphi(r) < 1$ for all $t \ge 0$.

Theorem 1.4 (Amini-Harandi and O'Regan [6]) *Let* (X, d) *be a complete metric space and* $T : X \to CB(X)$ *be a multivalued map. Assume that*

$$\psi(H(Tx, Ty)) \le \varphi(\psi(d(x, y)))\psi(d(x, y))$$

for all $x, y \in X$, *where* $\psi \in \Psi$ *is lower semicontinuous and* $\phi \in \Phi$. *Then T has a fixed point.*

The existence of fixed point in partially ordered sets has been investigated recently in [7-27] and references therein.

Du [13] proved some coupled fixed point results for weakly contractive single-valued maps that satisfy Mizoguchi-Takahashi's condition in the setting of quasiordered metric spaces. Before recalling the main results in [13], we need some definitions.

Definition 1.1 (Bhaskar and Lakshmikantham [11]) *Let X be a nonempty set and A* $: X \times X \to X$ *be a given map. We call an element* $(x, y) \in X \times X$ *a coupled fixed point of A if* $x = A(x, y)$ *and* $y = A(y, x)$.

Definition 1.2 (Bhaskar and Lakshmikantham [11]) *Let* $(X, \preccurlyeq)$ *be a quasiordered set and* $A : XX \to X$ *a map. We say that A has the mixed monotone property on X if A* (x, y) *is monotone nondecreasing in* $x \in X$ *and is monotone nonincreasing in* $y \in X$, *that is, for any* $x, y \in X$,

$$x_1, x_2 \in X \text{ with } x_1 \preccurlyeq x_2 \Rightarrow A(x_1, y) \preccurlyeq A(x_2, y),$$
$$y_1, y_2 \in X \text{ with } y_1 \preccurlyeq y_2 \Rightarrow A(x, y_1) \succcurlyeq A(x, y_2).$$

Definition 1.3 (Du [13]) *Let* (X, d) *be a metric space with a quasi-order* $\preccurlyeq$. *A nonempty subset M of X is said to be*

(i) sequentially $\preccurlyeq\uparrow$*-complete if every* $\preccurlyeq$*-nondecreasing Cauchy sequence in M converges;*

(ii) sequentially $\preccurlyeq\downarrow$*-complete if every* $\preccurlyeq$*-nonincreasing Cauchy sequence in M converges;*

(iii) sequentially $\preccurlyeq\updownarrow$*-complete if it is both* $\preccurlyeq\uparrow$*-complete and* $\preccurlyeq\downarrow$*-complete.*

Theorem 1.5 (Du [13]) *Let* $(X, d, \preccurlyeq)$ *be a sequentially* $\preccurlyeq\updownarrow$*-complete metric space and* $A : X \times X \to X$ *be a continuous map having the mixed monotone property on X. Assume that there exists a function* $\phi \in \Phi$ *such that*

$$d(A(x,y), A(u,v)) \leq \frac{1}{2}\varphi(d(x,u) + d(y,v))(d(x,u) + d(y,v))$$

for all $x \succcurlyeq u$ *and* $y \preccurlyeq v$*. If there exist* $x_0, y_0 \in X$ *such that* $x_0 \preccurlyeq A(x_0, y_0)$ *and* $y_0 \succcurlyeq A(y_0, x_0)$*, then A has a coupled fixed point.*

Theorem 1.6 (Du [13]) *Let* $(X, d, \preccurlyeq)$ *be a sequentially* $\preccurlyeq\updownarrow$*-complete metric space and* $A : X \times X \to X$ *be a map having the mixed monotone property on X. Assume that*

(i) any $\preccurlyeq$*-nondecreasing sequence* (x_n) *with* $x_n \to x$ *implies* $x_n \preccurlyeq x$ *for all n,*
(ii) any $\preccurlyeq$*-nonincreasing sequence* (y_n) *with* $y_n \to y$ *implies* $\succcurlyeq$ *for all n.*

Assume also that there exists a function $\phi \in \Phi$ *such that*

$$d(A(x,y), A(u,v)) \leq \frac{1}{2}\varphi(d(x,u) + d(y,v))(d(x,u) + d(y,v))$$

for all $x \succcurlyeq u$ *and* $y \preccurlyeq v$*. If there exist* $x_0, y_0 \in X$ *such that* $x_0 \preccurlyeq A(x_0, y_0)$ *and* $y_0 \succcurlyeq A(y_0, x_0)$*, then A has a coupled fixed point.*

Very recently, Gordji and Ramezani [14] established a new fixed point theorem for a self-map $T : X \to X$ satisfying a generalized Mizoguchi-Takahashi's condition in the setting of ordered metric spaces. The main result in [14] is the following.

Theorem 1.7 (Gordji and Ramezani [14]) *Let* $(X, d, \preccurlyeq)$ *be a complete ordered metric space and* $T : X \to X$ *an increasing mapping such that there exists an element* $x_0 \in X$ *with* $x_0 \preccurlyeq Tx_0$*. Suppose that there exists a lower semicontinuous function* $\psi \in \Psi$ *and* $\phi \in \Phi$ *such that*

$$\psi(d(Tx, Ty)) \leq \varphi(\psi(d(x,y))\psi(d(x,y))$$

for all $x, y \in X$ *such that x and y are comparable. Assume that either T is continuous or X is such that the following holds: any* $\preccurlyeq$*-nondecreasing sequence* (x_n) *with* $x_n \to x$ *implies* $x_n \preccurlyeq x$ *for all n. Then T has a fixed point.*

In this article, we present new coupled fixed point theorems for mixed monotone mappings satisfying a generalized Mizoguchi-Takahashi's condition in the setting of ordered metric spaces. Presented theorems extend and generalize Du [[13], Theorems 2.8 and 2.10], Bhaskar and Lakshmikantham [[11], Theorems 2.1 and 2.2], Harjani et al. [[15], Theorems 2 and 3], and other existing results in the literature. Moreover, some applications to ordinary differential equations are presented.

2 Main results

Through this article, we will use the following notation: if $(X, \preccurlyeq)$ is an ordered set, we endow the product set $X \times X$ with the order $\preccurlyeq$ given by

$$(x,y), (u,v) \in X \times X, \quad (x,y) \preccurlyeq (u,v) \Leftrightarrow x \preccurlyeq u,\ y \succcurlyeq v.$$

Our first result is the following.

Theorem 2.1 *Let* $(X, d, \preccurlyeq)$ *be a sequentially* $\preccurlyeq\updownarrow$*-complete metric space and* $A : X \times X \to X$ *be a map having the mixed monotone property on X. Suppose that there exist*

$\psi \in \Psi$ and $\phi \in \Phi$ such that for any $(x, y),(u, v) \in X\times X$ with $(u, v) \preccurlyeq (x, y)$,

$$\psi(d(A(x,y),A(u,v))) \leq \varphi(\psi(\max\{d(x,u),d(y,v)\}))\psi(\max\{d(x,u),d(y,v)\}). \qquad (1)$$

Suppose also that either A is continuous or $(X, d, \preccurlyeq)$ has the following properties:

(i) any $\preccurlyeq$-nondecreasing sequence (x_n) with $x_n \to x$ implies $x_n \preccurlyeq x$ for each n,
(ii) any $\preccurlyeq$-nonincreasing sequence (y_n) with $y_n \to y$ implies $y_n \succcurlyeq y$ for each n.

If there exist $x_0,y_0 \in X$ such that $x_0 \preccurlyeq A(x_0, y_0)$ and $y_0 \succcurlyeq A(y_0, x_0)$, then there exist a, $b \in X$ such that $a = A(a, b)$ and $b = A(b, a)$.

Proof. Define the sequences (x_n) and (y_n) in X by

$$x_{n+1} = A(x_n, y_n), \quad y_{n+1} = A(y_n, x_n) \quad \text{for all } n \geq 0.$$

In order to make the proof more comprehensive we will divide it into several steps.

• Step 1. $x_n \preccurlyeq x_{n+1}$ and $y_n \succcurlyeq y_{n+1}$ for all $n \geq 0$.

We use mathematical induction.

As $x_0 \preccurlyeq A(x_0, y_0) = x_1$ and $y_0 \succcurlyeq A(y_0, x_0) = y_1$, our claim is satisfied for $n = 0$.

Suppose that our claim holds for some fixed $n \geq 0$. Then, since $x_n \preccurlyeq x_{n+1}$ and $y_n \succcurlyeq y_{n+1}$, and as A has the mixed monotone property, we get

$$x_{n+1} = A(x_n, y_n) \preccurlyeq A(x_{n+1}, y_n) \preccurlyeq A(x_{n+1}, y_{n+1}) = x_{n+2}$$

and

$$y_{n+1} = A(y_n, x_n) \succcurlyeq A(y_{n+1}, x_n) \succcurlyeq A(y_{n+1}, x_{n+1}) = y_{n+2}.$$

This proves our claim.

• Step 2. $\lim_{n\to\infty} \psi(\max\{d(x_{n+1},x_n),d(y_{n+1},y_n)\}) = 0$.

Since $x_n \preccurlyeq x_{n+1}$ and $y_n \succcurlyeq y_{n+1}$ (Step 1), we have $(x_n, y_n) \preccurlyeq (x_{n+1},y_{n+1})$, and by (1), we have

$$\begin{aligned}&\psi(d(A(x_{n+1},y_{n+1}),A(x_n,y_n)))\\ &\leq \varphi(\psi(\max\{d(x_{n+1},x_n),d(y_{n+1},y_n)\}))\psi(\max\{d(x_{n+1},x_n),d(y_{n+1},y_n)\})\\ &\leq \psi(\max\{d(x_{n+1},x_n),d(y_{n+1},y_n)\}).\end{aligned} \qquad (2)$$

Similarly, since $(y_{n+1},x_{n+1}) \preccurlyeq (y_n, x_n)$, by (1), we have

$$\begin{aligned}&\psi(d(A(y_n,x_n),A(y_{n+1},x_{n+1})))\\ &\leq \varphi(\psi(\max\{d(y_n,y_{n+1}),d(x_n,x_{n+1})\}))\psi(\max\{d(y_n,y_{n+1}),d(x_n,x_{n+1})\})\\ &\leq \psi(\max\{d(y_n,y_{n+1}),d(x_n,x_{n+1})\}).\end{aligned} \qquad (3)$$

From (2) and (3), we get

$$\max\{\psi(d(x_{n+2},x_{n+1}),\psi(d(y_{n+2},y_{n+1}))\} \leq \psi(\max\{d(x_{n+1},x_n),d(y_{n+1},y_n)\}).$$

Since ψ is nondecreasing, this implies that

$$\psi\left(\max\left\{d(x_{n+2},x_{n+1})d(y_{n+2},y_{n+1})\right\}\right) \leq \psi\left(\max\left\{d(x_{n+1},x_n),d(y_{n+1},y_n)\right\}\right) \qquad (4)$$

for all $n \geq 0$. Now, (4) means that $(\psi(\max\{d(x_{n+1},x_n),d(y_{n+1},y_n)\}))$ is a non increasing sequence. On the other hand, this sequence is bounded below; thus there exists $\mu \geq 0$ such that

$$\lim_{n\to\infty} \psi(\max\{d(x_{n+1},x_n),d(y_{n+1},y_n)\}) = \mu. \tag{5}$$

Since $\phi \in \Phi$, we have $\limsup_{r\to\mu^+}\varphi(r) < 1$ and $\phi(\mu) <1$. Then, there exist $\alpha \in [0,1)$ and $\varepsilon >0$ such that $\phi(r) \le \alpha$ for all $r \in [\mu,\mu+\varepsilon)$. From (5), we can take $n_0 \ge 0$ such that $\mu \le \psi(\max\{d(x_{n+1},x_n),d(y_{n+1},y_n)\}) \le \mu + \varepsilon$ for all $n \ge n_0$. Then, from (2), for all $n \ge n_0$, we have

$$\begin{aligned} &\psi(d(x_{n+1},x_{n+2})) \\ &\quad\le \varphi(\psi(\max\{d(x_{n+1},x_n),d(y_{n+1},y_n)\}))\psi(\max\{d(x_{n+1},x_n),d(y_{n+1},y_n)\}) \\ &\quad\le \alpha\psi(\max\{d(x_{n+1},x_n),d(y_{n+1},y_n)\}). \end{aligned} \tag{6}$$

Similarly, from (3), for all $n > n_0$, we have

$$\begin{aligned} &\psi(d(y_{n+1},y_{n+2})) \\ &\quad\le \varphi(\psi(\max\{d(y_n,y_{n+1}),d(x_n,x_{n+1})\}))\psi(\max\{d(y_n,y_{n+1}),d(x_n,x_{n+1})\}) \\ &\quad\le \alpha\psi(\max\{d(y_n,y_{n+1}),d(x_n,x_{n+1})\}). \end{aligned} \tag{7}$$

Now, from (6) and (7), we get

$$\psi\left(\max\left\{d(x_{n+1},x_{n+2})d(y_{n+1},y_{n+2})\right\}\right) \le \alpha\psi\left(\max\left\{d(y_n,y_{n+1}),d(x_n,x_{n+1})\right\}\right) \tag{8}$$

for all $n \ge n_0$. Letting $n \to \infty$ in the above inequality and using (5), we obtain that

$$\mu \le \alpha\mu.$$

Since $\alpha \in [0,1)$, this implies that $\mu = 0$. Thus, we proved that

$$\lim_{n\to\infty} \psi(\max\{d(x_{n+1},x_n),d(y_{n+1},y_n)\}) = 0. \tag{9}$$

- Step 3. $\lim_{n\to\infty} \max\{d(x_{n+1},x_n),d(y_{n+1},y_n)\} = 0$.

Since $(\psi(\max\{d(x_{n+1},x_n),d(y_{n+1},y_n)\}))$ is a decreasing sequence and ψ is nondecreasing, then $(\max\{d(x_{n+1},x_n),d(y_{n+1},y_n)\})$ is a decreasing sequence of positive numbers. This implies that there exists $\theta \ge 0$ such that

$$\lim_{n\to\infty} \max\{d(x_{n+1},x_n),d(y_{n+1},y_n)\}) = \theta^+.$$

Since ψ is nondecreasing, we have

$$\psi(\max\left\{d(x_{n+1},x_n),d(y_{n+1},y_n)\right\}) \ge \psi(\theta).$$

Letting $n \to \infty$ in the above inequality, from (9), we obtain that $0 \ge \psi(\theta)$, which implies that $\theta = 0$. Thus, we proved that

$$\lim_{n\to\infty} \max\left\{d(x_{n+1},x_n),d(y_{n+1},y_n)\right\} = 0. \tag{10}$$

- Step 4. (x_n) and (y_n) are Cauchy sequences in (X, d).

Suppose that $\max\{d(x_{m+1},x_m), d(y_{m+1},y_m)\} = 0$ for some $m \ge 0$. Then, we have $d(x_{m+1}, x_m) = d(y_{m+1},y_m) = 0$, which implies that $(x_m, y_m) = (x_{m+1},y_{m+1})$, that is, $x_m = A(x_m, y_m)$ and $y_m = A(y_m, x_m)$. Then, (x_m, y_m) is a coupled fixed point of A.

Now, suppose that $\max\{d(x_{n+1},x_n), d(y_{n+1},y_n)\} \ne 0$ for all $n \ge 0$.

Denote

$$a_n = \psi(\max\{d(x_{n+1},x_n),d(y_{n+1},y_n)\}) \quad \text{for all } n \ge 0.$$

From (8), we have

$$a_{n+1} \le \alpha a_n \quad \text{for all } n \ge n_0.$$

Then, we have

$$\sum_{n=0}^{\infty} a_n \le \sum_{n=0}^{n_0} a_n + \sum_{n=n_0+1}^{\infty} \alpha^{n-n_0} a_{n0} < \infty.$$

On the other hand, we have

$$\limsup_{n\to\infty} \frac{\max\{d(x_{n+1}, x_n), d(y_{n+1}, y_n)\}}{\psi(\max\{d(x_{n+1}, x_n), d(y_{n+1}, y_n)\})} \le \limsup_{s\to 0^+} \frac{s}{\psi(s)} < \infty.$$

Then $\sum_n \max\{d(x_n, x_{n+1}), d(y_n, y_{n+1})\} < \infty$. Hence, (x_n) and (y_n) are Cauchy sequences in X.

• Step 5. Existence of a coupled fixed point.

Since $(X, d, \preccurlyeq)$ is sequentially $\preccurlyeq\updownarrow$-complete metric space and (x_n) is $\preccurlyeq$-nondecreasing Cauchy sequence, there exists $a \in X$ such that

$$\lim_{n\to\infty} x_n = a. \tag{11}$$

Similarly, since $(X, d, \preccurlyeq)$ is sequentially $\preccurlyeq\updownarrow$-complete metric space and (y_n) is $\preccurlyeq$-noincreasing Cauchy sequence, there exists $b \in X$ such that

$$\lim_{n\to\infty} y_n = b. \tag{12}$$

Case 1. A is continuous.

From the continuity of A and using (11) and (12), we get

$$a = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} A\left(x_n, y_n\right) = A(\lim_{n\to\infty} x_n, \lim_{n\to\infty} y_n) = A(a, b)$$

and

$$b = \lim_{n\to\infty} y_{n+1} = \lim_{n\to\infty} A\left(y_n, x_n\right) = A(\lim_{n\to\infty} y_n, \lim_{n\to\infty} x_n) = A(b, a).$$

Case 2. $(X, d, \preccurlyeq)$ satisfies (i) and (ii).

Since (x_n) is $\preccurlyeq$-nondecreasing and $\lim_{n\to\infty} x_n = a$, then from (i), we have $x_n \preccurlyeq a$ for all n. Similarly, from (ii), since (y_n) is $\preccurlyeq$-nonincreasing and $\lim_{n\to\infty} y_n = b$, we have $y_n \succcurlyeq b$ for all n. Then, we have $(x_n, y_n) \preccurlyeq (a, b)$ for all n. Now, applying our contractive condition (1), we get

$$\begin{aligned} \psi(d(x_{n+1}, A(a, b))) &= \psi(d(A(x_n, y_n), A(a, b))) \\ &\le \varphi(\psi(\max\{d(x_n, a), d(y_n, b)\}))\psi(\max\{d(x_n, a), d(y_n, b)\}) \\ &\le \psi(\max\{d(x_n, a), d(y_n, b)\}). \end{aligned}$$

Since ψ is nondecreasing, this implies that

$$d(x_{n+1}, A(a, b)) \le \max\{d(x_n, a), d(y_n, b)\}.$$

letting $n \to \infty$ in the above inequality, we obtain that $d(a, A(a, b)) \le 0$, that is, $a = A(a, b)$. Similarly, we can show that $b = A(b, a)$. □

Remark 2.1 *In our presented theorems we don't need the hypothesis: ψ is lower semi-continuous. Such hypothesis is considered in Theorem 1.7 of Gordji and Ramezani* [14].

In what follows, we give a sufficient condition for the uniqueness of the coupled fixed point in Theorem 2.1. We consider the following hypothesis:

(H): For all $(x, y), (u, v) \in X \times X$, there exists $(w, z) \in X \times X$ such that $(x, y) \preccurlyeq (w, z)$ and $(u, v) \preccurlyeq (w, z)$.

Theorem 2.2 *Adding condition (H) to the hypotheses of Theorem 2.1, we obtain uniqueness of the coupled fixed point of A.*

Proof. Suppose that (a, b) and (c, e) are coupled fixed points of A, that is, $a = A(a, b)$, $b = A(b, a)$, $c = A(c, e)$ and $e = A(e, c)$. From (H), there exists $(f_0, g_0) \in X \times X$ such that $(a, b) \preccurlyeq (f_0, g_0)$ and $(c, e) \preccurlyeq (f_0, g_0)$.

We construct the sequences (f_n) and (g_n) in X defined by

$$f_{n+1} = A(f_n, g_n), \quad g_{n+1} = A(g_n, f_n) \quad \text{for all } n \geq 0.$$

We claim that $(a, b) \preccurlyeq (f_n, g_n)$ for all $n \geq 0$.

In fact, we will use mathematical induction.

Since $(a, b) \preccurlyeq (f_0, g_0)$, then our claim is satisfied for $n = 0$.

Suppose that our claim holds for some fixed $n \geq 0$. Then, we have $(a, b) \preccurlyeq (f_n, g_n)$, that is, $a \preccurlyeq f_n$ and $b \succcurlyeq g_n$. Using the mixed monotone property of A, we get

$$f_{n+1} = A(f_n, g_n) \succcurlyeq A(a, g_n) \succcurlyeq A(a, b) = a$$

and

$$g_{n+1} = A(g_n, f_n) \preccurlyeq A(b, f_n) \preccurlyeq A(b, a) = b.$$

This proves that $(a, b) \preccurlyeq (f_{n+1}, g_{n+1})$. Then, our claim holds.

Now, we can apply (1) with $(u, v) = (a, b)$ and $(x, y) = (f_n, g_n)$. We get

$$\begin{aligned} \psi(d(f_{n+1}, a)) &= \psi(d(A(f_n, g_n), A(a, b))) \\ &\leq \varphi(\psi(\max\{d(f_n, a), d(g_n, b)\}))\psi(\max\{d(f_n, a)d(g_n, b)\}). \end{aligned} \tag{13}$$

Similarly, we have

$$\begin{aligned} \psi(d(g_{n+1}, b)) &= \psi(d(A(g_n, f_n), A(b, a))) \\ &\leq \varphi(\psi(\max\{d(f_n, a), d(g_n, b)\}))\psi(\max\{d(f_n, a)d(g_n, b)\}). \end{aligned} \tag{14}$$

Combining (13) with (14), we obtain

$$\begin{aligned} \psi(\max\{d(f_{n+1}, a), d(g_{n+1}, b)\}) &\leq \varphi(\psi(\max\{d(f_n, a), d(g_n, b)\}))\psi(\max\{d(f_n, a), d(g_n, b)\}) \\ &\leq \psi(\max\{d(f_n, a), d(g_n, b)\}). \end{aligned} \tag{15}$$

Consequently, $(\psi(\max\{d(f_n, a), d(g_n, b)\}))$ is a nonnegative decreasing sequence and hence possesses a limit $\gamma \geq 0$. Following the same strategy used in the proof of Theorem 2.1, one can show that $\gamma = 0$ and $\lim_{n\to\infty} \max\{d(f_n, a), d(g_n, b)\} = 0$.

Analogously, it can be proved that $\lim_{n\to\infty} \max\{d(f_n, c), d(g_n, e)\} = 0$.

Now, we have

$$d(a, c) \leq d(a, f_n) + d(f_n, c) \leq \max\{d(f_n, a), d(g_n, b)\} + \max\{d(f_n, c), d(g_n, e)\}$$

and

$$d(b, e) \leq d(b, g_n) + d(g_n, e) \leq \max\{d(f_n, a), d(g_n, b)\} + \max\{d(f_n, c), d(g_n, e)\}.$$

Then, we have

$$\max\{d(a,c),d(b,e)\} \le \max\{d(f_n,a),d(g_n,b)\} + \max\{d(f_n,c),d(g_n,e)\}.$$

Letting $n \to \infty$ in the above inequality, we get

$$\max\{d(a,c),d(b,e)\} = 0,$$

which implies that $d(a, c) = d(b, e) = 0$. Then, $(a, b) = (c, e)$. □

Theorem 2.3 *Under the assumptions of Theorem 2.1, suppose that x_0 and y_0 are comparable, then the coupled fixed point $(a, b) \in X \times X$ satisfies $a = b$.*

Proof. Assume that $x_0 \preccurlyeq y_0$ (the same strategy can be used if $y_0 \preccurlyeq x_0$). Using the mixed monotone property of A, it is easy to show that $x_n \preccurlyeq y_n$ for all $n \geq 0$.

Now, using the contractive condition, as $x_n \preccurlyeq y_n$, we have

$$\begin{aligned} \psi(d(y_{n+1},x_{n+1})) &= \psi(d(A(y_n,x_n),A(x_n,y_n))) \\ &\le \varphi(\psi(d(x_n,y_n)))\psi(d(x_n,y_n)) \\ &\le \psi(d(x_n,y_n)). \end{aligned} \tag{16}$$

Thus $\lim_{n\to\infty} \psi(d(x_n, y_n)) = \theta$ for certain $\theta \geq 0$. Since $\phi \in \Phi$, we have $\limsup_{r\to\theta^+}\varphi(r) < 1$ and $\phi(\theta) <1$. Then, there exist $\alpha \in [0,1)$ and $\varepsilon >0$ such that $\phi(r) <\alpha$ for all $r \in [\theta,\theta + \varepsilon)$.

Now, we take $n_0 \geq 0$ such that $\theta \leq \psi(d(x_n, y_n)) \leq \theta + \varepsilon$ for all $n \geq n_0$. Then, from (16), for all $n \geq n_0$, we have

$$\psi(d(y_{n+1},x_{n+1})) \le \alpha\psi(d(x_n,y_n)).$$

Letting $n \to \infty$ in the above inequality, we obtain that

$$\theta \le \alpha\theta.$$

Since $\alpha \in [0,1)$, this implies that $\theta = 0$. Thus, we proved that

$$\lim_{n\to\infty} \psi(d(x_n,y_n)) = 0,$$

which implies that $\lim_{n\to\infty} d(x_n, y_n) = 0$. Now, we have

$$0 = \lim_{n\to\infty} d(x_n,y_n) = d(\lim_{n\to\infty} x_n, \lim_{n\to\infty} y_n) = d(a,b)$$

and thus $a = b$. This finishes the proof. □

Now, we present some consequences of our theorems.

Corollary 2.1 *Let $(X, d, \preccurlyeq)$ be a sequentially $\preccurlyeq\updownarrow$-complete metric space and $A : X \times X \to X$ be a map having the mixed monotone property on X. Suppose that there exist $\psi \in \Psi$ and $\tilde{\varphi} : [0,\infty) \to [0,\infty)$ with $\liminf_{s\to t^+}(\tilde{\varphi}(s)/\psi(s)) > 0$ for all $t \geq 0$ such that for any $(x, y),(u, v) \in X \times X$ with $(u, v)\preccurlyeq (x, y)$,*

$$\psi(d(A(x,y),A(u,v))) \le \psi(\max\{d(x,u),d(y,v)\}) - \tilde{\varphi}(\psi(\max\{d(x,u),d(y,v)\})).$$

Suppose also that either A is continuous or $(X, d, \preccurlyeq)$ has the following properties:

(i) any $\preccurlyeq$-nondecreasing sequence (x_n) with $x_n \to x$ implies $x_n \preccurlyeq x$ for each n,
(ii) any $\preccurlyeq$-nonincreasing sequence (y_n) with $y_n \to y$ implies $y_n \succcurlyeq y$ for each n.

If there exist $x_0, y_0 \in X$ *such that* $x_0 \preccurlyeq A(x_0, y_0)$ *and* $y_0 \succcurlyeq A(y_0, x_0)$, *then there exist a*, $b \in X$ *such that* $a = A(a, b)$ *and* $b = A(b, a)$.

Proof. It follows immediately from Theorem 2.1 by considering $\varphi(s) = 1 - \tilde{\varphi}(s)/\psi(s)$. □

Remark 2.2 *Corollary 2.1 is an extension of Harjani et al.* [[15], *Theorems 2 and 3].*

Corollary 2.2 *Let* $(X, d, \preccurlyeq)$ *be a sequentially* $\preccurlyeq\updownarrow$*-complete metric space and* $A : X \times X \to X$ *be a map having the mixed monotone property on X. Suppose that there exists a nondecreasing function* $\phi : [0, \infty) \to [0, 1)$ *such that for any* $(x, y), (u, v) \in X \times X$ *with* $(u, v) \preccurlyeq (x, y)$,

$$d(A(x,y), A(u,v)) \le \varphi(2\max\{d(x,u), d(y,v)\}) \max\{d(x,u), d(y,v)\}.$$

Suppose also that either A is continuous or $(X, d, \preccurlyeq)$ *has the following properties:*

(i) any $\preccurlyeq$*-nondecreasing sequence* (x_n) *with* $x_n \to x$ *implies* $x_n \preccurlyeq x$ *for each n,*
(ii) any $\preccurlyeq$*-nonincreasing sequence* (y_n) *with* $y_n \to y$ *implies* $y_n \succcurlyeq y$ *for each n.*

If there exist $x_0, y_0 \in X$ *such that* $x_0 \preccurlyeq A(x_0, y_0)$ *and* $y_0 \succcurlyeq A(y_0, x_0)$, *then there exist a*, $b \in X$ *such that* $a = A(a, b)$ *and* $b = A(b, a)$.

Proof. It follows from Theorem 2.1 by considering $\psi(s) = 2s$ □

Remark 2.3 *If* ϕ *is nondecreasing, Corollary 2.2 generalizes Du* [[13], *Theorems 1.5 and 1.6].*

Corollary 2.3 *Let* $(X, d, \preccurlyeq)$ *be a sequentially* $\preccurlyeq\updownarrow$*-complete metric space and* $A : X \times X \to X$ *be a map having the mixed monotone property on X. Suppose that there exists* $k \in [0, 1)$ *such that for any* $(x, y),(u, v) \in X \times X$ *with* $(u, v) \preccurlyeq (x, y)$,

$$d(A(x,y), A(u,v)) \le k \max\{d(x,u), d(y,v)\}.$$

Suppose also that either A is continuous or $(X, d, \preccurlyeq)$ *has the following properties:*

(i) any $\preccurlyeq$*-nondecreasing sequence* (x_n) *with* $x_n \to x$ *implies* $x_n \preccurlyeq x$ *for each n,*
(ii) any $\preccurlyeq$*-nonincreasing sequence* (y_n) *with* $y_n \to y$ *implies* $y_n \succcurlyeq y$ *for each n.*

If there exist $x_0, y_0 \in X$ *such that* $x_0 \preccurlyeq A(x_0, y_0)$ *and* $y_0 \succcurlyeq A(y_0, x_0)$, *then there exist a*, $b \in X$ *such that* $a = A(a, b)$ *and* $b = A(b, a)$.

Proof. It follows immediately from Corollary 2.2 by considering $\phi(s) = k$ □

Remark 2.4 *Corollary 2.3 is a generalization of Bhaskar and Lakshmikantham* [[11], *Theorems 2.1 and 2.2].*

3 An application

In this section, we apply our main results to study the existence and uniqueness of solution to the two-point boundary value problem

$$\begin{cases} -\dfrac{d^2x}{dt^2}(t) = f(t, x(t), x(t)), & t \in [0, 1] \\ x(0) = x(1) = 0, \end{cases} \tag{17}$$

where $f : [0, 1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

Previously we considered the space $X = C(I, \mathbb{R})(I = [0, 1])$ of continuous functions defined on I. Obviously, this space with the metric given by

$$d(x,y) = \max\{|x(t) - y(t)| : t \in I\} \quad \text{for } x, y \in I$$

is a complete metric space. The space X can also be equipped with a partial order given by

$$x, y \in I, \quad x \preccurlyeq y \Leftrightarrow x(t) \leq y(t) \text{ for all } t \in I.$$

Obviously, $(X, \preccurlyeq)$ satisfies condition (H) since for $x, y \in X$ the functions $\max\{x, y\}$ and $\min\{x, y\}$ are least upper and greatest lower bounds of x and y, respectively. Moreover, in [21] it is proved that $(X, d, \preccurlyeq)$ satisfies conditions (i) and (ii) of Theorem 2.1.

Now, we consider the following assumptions:

(a) $f : [0, 1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous.

(b) For *all* $t \in I$, $z \geq h$, $w \leq r$,

$$0 \leq f(t, z, w) - f(t, h, r) \leq 4[\ln(z - h + 1) + \ln(r - w + 1)].$$

(c) There exists $(\alpha, \beta) \in C^2(I, \mathbb{R}) \times C^2(I, \mathbb{R})$ solution to

$$\begin{cases} -\dfrac{d^2\alpha}{dt^2}(t) \leq f(t, \alpha(t), \beta(t)), & t \in [0, 1] \\ -\dfrac{d^2\beta}{dt^2}(t) \geq f(t, \beta(t), \alpha(t)), & t \in [0, 1] \\ \alpha(0) = \alpha(1) = \beta(0) = \beta(1) = 0. \end{cases} \tag{18}$$

(d) $\alpha \preccurlyeq \beta$ or $\beta \preccurlyeq \alpha$.

Theorem 3.1 *Under the assumptions (a)-(d), problem (17) has one and only one solution* $x^* \in C^2(I, \mathbb{R})$.

Proof. It is well known that the solution (in $C^2(I, \mathbb{R})$) of problem (17) is equivalent to the solution (in $C(I, \mathbb{R})$) of the following Hammerstein integral equation:

$$x(t) = \int_0^1 G(t, s) f(x(s))\, ds, \quad t \in I,$$

where $G(t, s)$ is the Green function of differential operator $-d^2/dt^2$ with Dirichlet boundary condition $x(0) = x(1) = 0$, i.e.,

$$G(t, s) = \begin{cases} s(1 - t), & 0 \leq s \leq t \leq 1, \\ t(1 - s), & 0 \leq t \leq s \leq 1. \end{cases}$$

Define $A : X \times X \to X$ by

$$A(x, y)(t) = \int_0^1 G(t, s) f(s, x(s), y(s))\, ds, \quad t \in I,$$

for all $x, y \in X$.

From (b), it is clear that A has the mixed monotone property with respect to the partial order $\preccurlyeq$ in X.

Let $x, y, u, v \in X$ such that $x \succcurlyeq u$ and $y \preccurlyeq v$. From (b), we have

$$\begin{aligned} d(A(x,y),A(u,v)) &= \sup_{t\in I} |A(x,y)(t) - A(u,v)(t)| \\ &= \sup_{t\in I} \int_0^1 G(t,s)\left[f(s,x(s),y(s)) - f(s,u(s),v(s))\right] ds \\ &\le \sup_{t\in I} \int_0^1 4G(t,s)[\ln(x(s)-u(s)+1) + \ln(v(s)-y(s)+1)]\, ds \\ &\le (\ln(d(x,u)+1) + \ln(d(y,v)+1)) \sup_{t\in I} \int_0^1 4G(t,s)\, ds \\ &\le \left(\sup_{t\in I} \int_0^1 8G(t,s)\, ds\right) \ln\left(\max\{d(x,u), d(y,v)\} + 1\right) \end{aligned}$$

On the other hand, for all $t \in I$, we have

$$\int_0^1 G(t,s)\, ds = \frac{1}{2} t(1-t),$$

which implies that

$$\sup_{t\in I} \int_0^1 G(t,s)\, ds = \frac{1}{8}.$$

Then, we get

$$d(A(x,y),A(u,v)) \le \ln\left(\max\{d(x,u), d(y,v)\} + 1\right).$$

This implies that

$$\begin{aligned} \ln\left(d(A(x,y),A(u,v)) + 1\right) &\le \ln\left(\ln\left(\max\{d(x,u), d(y,v)\} + 1\right) + 1\right) \\ &= \frac{\ln\left(\ln\left(\max\{d(x,u), d(y,v)\} + 1\right) + 1\right)}{\ln\left(\max\{d(x,u), d(y,v)\} + 1\right)} \ln\left(\max\{d(x,u), d(y,v)\} + 1\right). \end{aligned}$$

Thus, the contractive condition (1) of Theorem (2.1) is satisfied with $\psi(t) = \ln(t+1)$ and $\phi(t) = \psi(t)/t$.

Now, let $(\alpha, \beta) \in C^2(I, \mathbb{R}) \times C^2(I, \mathbb{R})$ be a solution to (18). We will show that $\alpha \preccurlyeq A(\alpha, \beta)$ and $\beta \succcurlyeq A(\beta, \alpha)$. Indeed,

$$-\alpha''(s) \le f(s, \alpha(s), \beta(s)), \quad s \in [0,1].$$

Multiplying by $G(t, s)$, we get

$$\int_0^1 -\alpha''(s) G(t,s)\, ds \le A(\alpha,\beta)(t), \quad t \in [0,1].$$

Then, for all $t \in [0, 1]$, we have

$$-(1-t)\int_0^1 s\alpha''(s)\,ds \; - t\int_t^1 (1-s)\alpha''(s)\,ds \; \le A(\alpha,\beta)(t).$$

Using an integration by parts, and since $\alpha(0) = \alpha(1) = 0$, for all $t \in [0, 1]$, we get

$$-(1-t)(t\alpha'(t)-\alpha(t)) - t(-(1-t)\alpha'(t)-\alpha(t)) \le A(\alpha,\beta)(t).$$

Thus, we have

$$\alpha(t) \le A(\alpha,\beta)(t), \quad t \in [0,1].$$

This implies that $\alpha \preccurlyeq A(\alpha, \beta)$. Similarly, one can show that $\beta \succcurlyeq A(\beta, \alpha)$.

Now, applying our Theorems 2.1 and 2.2, we deduce the existence of a unique $(x, y) \in X^2$ solution to $x = A(x, y)$ and $y = A(y, x)$. Moreover, from (d), and using Theorem 2.3, we get $x = y$. Thus, we proved that $x^* = x = y \in C^2([0, 1], \mathbb{R})$ is the unique solution to (17). □

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Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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