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Fixed point theorems of convex-power 1-set-contraction operators in Banach spaces

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Abstract

In this article, we give the definition of a class of new operators, namely, convex-power 1-set-contraction operators in Banach spaces, and study the existence of fixed points of this class of operators. By using methods of approximation by operators, we obtain fixed point theorems of convex-power 1-set-contraction operators, which generalize fixed point theorems of 1-set-contraction operators in Banach spaces. By using the fixed point theorem, the existence of solutions of nonlinear Sturm-Liouville problems in Banach spaces is investigated under more general conditions than those used in former literatures.

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Keywords: convex-power 1-set-contraction, fixed point theorem, Banach spaces, Sturm-Liouville problems

0 Introduction

For the need of studying differential equations and integral equations, Sun and Zhang [1] gave the definition of convex-power condensing operators and obtained the fixed point theorem of this class of operators. Li [2] gave the fixed point theorem of semi-closed 1-set-contraction operators.

In this article, by combinating the definitions of convex-power condensing operators and 1-set-contraction operators, we give the definition of convex-power 1-set-contraction operators in Banach spaces and study the existence of fixed points of this class of new operators. The results in this article generalize the ones in [1-3]. By using the fixed point theorem, the existence of solutions of nonlinear Sturm-Liouville problems in Banach spaces is investigated under more general conditions than those used in former literatures.

1 Preliminaries

Before providing the main results, we introduce some basic definitions and results (see [1-6]).

In this article, we always assume that E is a Banach space, $D \subset E$, and $\alpha(S)$ denotes the Kuratowski measure of noncompactness of a bounded set $S \subset E$.

Let A: $D \to E$ be continuous. If there exists a constant $k \geq 0$, such that for any bounded subset $S \subset D$,

$$\alpha(A(S)) \leq k\alpha(S).$$

Then A is said to be k-set-contraction in D.

Let A: $D \to E$ be continuous and bounded. If there exist $x_0 \in D$ and a positive integer n_0, such that for any bounded and nonprecompact subset $S \subset D$,

$$\alpha\left(A^{(n_0,x_0)}(S)\right) < \alpha(S),$$

where $A^{(1,x_0)}(S) = A(S)$, $A^{(n,x_0)}(S) = A\left(\overline{co}\left\{A^{(n-1,x_0)}(S), x_0\right\}\right)$, $n = 2, 3, \ldots$.

Then A is said to be convex-power condensing.

Lemma 1.1 [1]. *Let $D \subset E$ be bounded, convex, and closed. Suppose that A: $D \to D$ is convex-power condensing, then A has at least one fixed point in D.*

Definition 1.1 [2]. *A: $D \to E$ is said to be semi-closed if for any closed set $F \subset D$, $(I - A)F$ is closed.*

Definition 1.2 [3]. *Let A: $D \to E$, $\{x_n\} \subset D$ bounded, $\{x_n - Ax_n\}$ strongly convergent. A is said to be semi-compact if $\{x_n\}$ has a strongly convergent subsequence.*

2 Main results

Next, we will give the definition of convex-power 1-set-contraction operators in Banach spaces.

Definition 2.1. *Let $A : D \to E$ be continuous and bounded. If there exist $x_0 \in D$ and a positive integer n_0, such that for any bounded subset $S \subset D$,*

$$\alpha\left(A^{(n_0,x_0)}(S)\right) \le \alpha(S),$$

where $A^{(1,x_0)}(S) = A(S)$, $A^{(n,x_0)}(S) = A\left(\overline{co}\left\{A^{(n-1,x_0)}(S), x_0\right\}\right)$, $n = 2, 3, \ldots$.

Then A is said to be convex-power 1-set-contraction in Banach spaces.

Remark 2.1. *Obviously, 1-set-contraction operators are convex-power 1-set-contraction operators. Convex-power 1-set-contraction operators are more general.*

Now, we establish the main theorem as follows:

Theorem 2.1. *Let E be a Banach space, $D \subset E$ bounded, convex, and closed. Suppose that A: $D \to D$ is semi-closed and convex-power 1-set-contraction, then A has at least one fixed point in D.*

Proof. Since A is convex-power 1-set-contraction, there exist $x_0 \in D$ and a positive integer n_0, such that for any bounded subset $S \subset D$,

$$\alpha\left(A^{(n_0,x_0)}(S)\right) \le \alpha(S).$$

$\forall x \in D$, let $A_n x = \left(1 - \frac{1}{n}\right)Ax + \frac{1}{n}x_0 \; (n = 2, 3, \ldots)$, then A_n: $D \to D$. $\forall y \in D - x_0 = \{x - x_0 | x \in D\}$, let $By = A(y + x_0) - x_0$.

For any bounded subset $S \subset D$,

$$\begin{aligned}
B^{(1,0)}(S - x_0) &= B(S - x_0) = A(S) - x_0 = A^{(1,x_0)}(S) - x_0, \\
A_n^{(1,x_0)}(S) &= A_n(S) = \left(1 - \frac{1}{n}\right)A(S) + \frac{1}{n}x_0 = \left(1 - \frac{1}{n}\right)A^{(1,x_0)}(S) + \frac{1}{n}x_0; \\
B^{(2,0)}(S - x_0) &= B\left(\overline{co}\left\{B^{(1,0)}(S - x_0), 0\right\}\right) = B\left(\overline{co}\left\{A^{(1.x_0)}(S) - x_0, 0\right\}\right) \\
&= A\left(\overline{co}\left\{A^{(1,x_0)}(S) - x_0, 0\right\} + x_0\right) - x_0 \\
&= A\left(\overline{co}\left\{A^{(1,x_0)}(S), x_0\right\}\right) - x_0 = A^{(2,x_0)}(S) - x_0,
\end{aligned}$$

$$\begin{aligned}
A_n^{(2,x_0)}(S) &= A_n\left(\overline{co}\left\{A_n^{(1,x_0)}(S), x_0\right\}\right) \\
&= A_n\left(\overline{co}\left\{\left(1-\frac{1}{n}\right)B^{(1,0)}(S-x_0)+x_0, x_0\right\}\right) \\
&= A_n\left(\overline{co}\left\{\left(1-\frac{1}{n}\right)B^{(1,0)}(S-x_0), 0\right\}+x_0\right) \\
&\subset A_n\left(\overline{co}\left\{B^{(1,0)}(S-x_0), 0\right\}+x_0\right) \\
&= A_n\left(\overline{co}\left\{B^{(1,0)}(S-x_0)+x_0, x_0\right\}\right) \\
&= A_n\left(\overline{co}\left\{A^{(1,x_0)}(S), x_0\right\}\right) = \left(1-\frac{1}{n}\right)A^{(2,x_0)}(S)+\frac{1}{n}x_0;
\end{aligned}$$

and generally,

$$\begin{aligned}
B^{(n_0,0)}(S-x_0) &= B\left(\overline{co}\left\{B^{(n_0-1,0)}(S-x_0), 0\right\}\right) = B\left(\overline{co}\left\{A^{(n_0-1,x_0)}(S)-x_0, 0\right\}\right) \\
&= A\left(\overline{co}\left\{A^{(n_0-1,x_0)}(S)-x_0, 0\right\}+x_0\right)-x_0 = A^{(n_0,x_0)}(S)-x_0,
\end{aligned}$$

$$\begin{aligned}
A_n^{(n_0,x_0)}(S) &= A_n\left(\overline{co}\left\{A_n^{(n_0-1,x_0)}(S), x_0\right\}\right) \\
&\subset A_n\left(\overline{co}\left\{\left(1-\frac{1}{n}\right)A^{(n_0-1,x_0)}(S)+\frac{1}{n}x_0, x_0\right\}\right) \\
&= A_n\left(\overline{co}\left\{\left(1-\frac{1}{n}\right)B^{(n_0-1,0)}(S-x_0)+x_0, x_0\right\}\right) \\
&\subset A_n\left(\overline{co}\left\{B^{(n_0-1,0)}(S-x_0), 0\right\}+x_0\right) \\
&= A_n\left(\overline{co}\left\{A^{(n_0-1,x_0)}(S), x_0\right\}\right) \\
&= \left(1-\frac{1}{n}\right)A^{(n_0,x_0)}(S)+\frac{1}{n}x_0.
\end{aligned}$$

By the definition of the convex-power 1-set-contraction operator and the properties of the measure of noncompactness, we have

$$\alpha\left(A_n^{(n_0,x_0)}(S)\right) \le \left(1-\frac{1}{n}\right)\alpha\left(A^{(n_0,x_0)}(S)\right) \le \left(1-\frac{1}{n}\right)\alpha(S) < \alpha(S), n = 2, 3, \ldots.$$

Therefore, $A_n : D \to D$ is convex-power condensing. By Lemma 1.1, A_n has a fixed point x_n in D, i.e., $A_n x_n = x_n$ (n = 2, 3,...). Since $||Ax - A_n x|| = \frac{1}{n}||Ax - x_0||$, $\forall x \in D$, and A is bounded in D, then for any $x \in D$, $||Ax - A_n x|| \to 0$ ($n \to +\infty$). Obviously,

$$||Ax_n - x_n|| = ||Ax_n - A_n x_n|| \to 0\,(n \to +\infty).$$

i.e., $Ax_n - x_n \to 0$ ($n \to +\infty$). Since A is semi-closed and D is closed, $0 \in (I - A)D$. Therefore, there exists $x_0 \in D$, such that $x_0 = Ax_0$. The proof is completed.

Remark 2.2. *In Theorem 2.1, let* $n_0 = 1$, *the fixed point theorem of semiclosed 1-set-contraction operators in* [2]*is obtained. Therefore, Theorem2.1. generalizes the fixed point theorem of semi-closed 1-set-contraction operators.*

Theorem 2.2. *Let E be a Banach space,* $D \subset E$ *bounded, convex and closed. Suppose that* A: $D \to D$ *is semi-compact and convex-power 1-set-contraction, then A has at least one fixed point in D.*

Proof. $\forall x \in D$, let $A_n x = \left(1 - \frac{1}{n}\right) Ax + \frac{1}{n} x_0 \, (n = 2, 3, \ldots)$. By the proof of Theorem 2.1, A_n: $D \to D$ has a fixed point x_n, and $\{A_n\}$ is uniformly convergent to A in D. By $x_n \in D$, then $||A_n x_n - Ax_n|| \to 0$. i.e., $||x_n - Ax_n|| \to 0$. Therefore, $(I - A)(x_n) \to 0$ $(n \to +\infty)$.

Since A is semi-compact and $\{x_n\} \subset D$ is bounded, $\{x_n\}$ has a convergent subsequence $\{x_{n_i}\}$. Let $x_{n_i} \to x_0 \, (n_i \to +\infty)$. Since D is closed, $x_0 \in D$. Since A is continuous in D, $x_{n_i} - Ax_{n_i} \to x_0 - Ax_0 \, (n_i \to +\infty)$. By $x_{n_i} - Ax_{n_i} \to 0 \, (n_i \to +\infty)$, we have $x_0 - Ax_0 = 0$. The proof is completed.

3 Application

Let E be a Banach space. Consider the existence of solutions of nonlinear Sturm-Liouville problems in E as follows:

$$\begin{cases} -(Lx)(t) = f(t, x), & t \in (0, 1); \\ ax(0) - bx'(0) = 0, \; cx(1) + dx'(1) = 0 \end{cases} \tag{3.1}$$

where $(Lx)(t) = (p(t)x')' + q(t)x$, $f \in C[I \times E, E]$ $(I = [0, 1])$.

Assume that

$$\begin{aligned} (\mathbf{H_1}) p(t) \in C^1[I, R], p(t) > 0, q(t) \in C[I, R], q(t) \le 0, \\ a \ge 0, b \ge 0, c \ge 0, d \ge 0, a^2 + b^2 \ne 0, c^2 + d^2 \ne 0, \end{aligned}$$

and the homogeneous equations of (3.1)

$$\begin{cases} -(Lx)(t) = 0, & t \in (0, 1); \\ ax(0) - bx'(0) = 0, \; cx(1) + dx'(1) = 0 \end{cases} \tag{3.2}$$

has only zero solution in $C^2[I, R]$.

Let $G(t, s)$ be Green function of (3.2), i.e.,

$$G(t, s) = \begin{cases} \frac{1}{\rho} u(t) v(s), \; 0 \le t \le s \le 1; \\ \frac{1}{\rho} u(s) v(t), \; 0 \le s \le t \le 1. \end{cases} \tag{3.3}$$

Lemma 3.1 [6]. *Assume that* (H_1) *holds, then Green function* $G(t, s)$ *of (3.3) has the following properties:*

(i) $G(t, s)$ *is continuous and symmetric in* $[0, 1] \times [0, 1]$;
(ii) $u(t) \in C^2[0, 1]$ *is monotonically increasing, and* $u(t) > 0$, $t \in (0, 1]$;
(iii) $v(t) \in C^2[0, 1]$ *is monotonically decreasing, and* $v(t) > 0$, $t \in [0, 1)$;
(iv) $(Lu)(t) \equiv 0$, $u(0) = b$, $u'(0) = a$;
(v) $(Lv)(t) \equiv 0$, $v(0) = d$, $v'(0) = -c$;
(vi) ρ *is a positive constant.*

Let

$$\begin{aligned} (Tx)(t) = \int_0^1 G(t, s) f(s, x(s)) \, ds, \; t \in [0, 1], \; x \in C[I, E], \\ (K\varphi)(t) = \int_0^1 G(t, s) \varphi(s) \, ds, \; t \in [0, 1], \; \varphi \in C[I, R]. \end{aligned}$$

We can prove that the solution in $C^2[I, E]$ of (3.1) is equivalent to the fixed point of T (see [7]).

Since $G(t, s)$ is continuous, it can be easily proved that K: $C\,[I, R] \to C\,[I, R]$ is linear and completely continuous. By Lemma 3.1, $\forall t, s \in [0, 1]$,

$$\frac{u(s)\,v(s)}{u(1)\,v(0)} G(t,t) \le G(t,s) \le G(t,t).$$

Therefore, by Krein-Rutman Theorem [6], the first characteristic value of K is $\lambda_1 > 0$, and $\lambda_1 = (r(K))^{-1}$.

Now we give some conditions:

($\mathbf{H_2}$) $f \in C[I \times E, E]$, for any bounded subset B in E, f is uniformly continuous in $I \times B$, and there exists $k \in [0, \lambda_1)$, such that

$$\alpha\left(f(t, B(t))\right) \le k\alpha(B(t)),\ \forall t \in [0, 1],$$

where λ_1 is the first characteristic value of K.

($\mathbf{H_3}$) there exist $M \in (0, \lambda_1)$ and $h(t) \in C[I, R^+]$, such that for any $(t, x) \in I \times E$,

$$\|f(t, x)\| \le M\|x\| + h(t).$$

Theorem 3.1. *Suppose that* (H_1), (H_2), (H_3) *hold, then Sturm-Liouville problems (3.1) has at least one solution in* $C^2[I, E]$.

To prove Theorem 3.1, here we introduce some lemmas.

Lemma 3.2 [7]. *For* $M < \lambda_1$ *as above, let* $K_1 = M\,K$, *then there exists a norm* $\|\cdot\|^*_{C[I,R]}$ *which is equivalent to* $\|\cdot\|_{C[I,\ R]}$ *and satisfies:*

(1) $\|K_1\varphi\|^*_{C[I,R]} \le \sigma\|\varphi\|^*_{C[I,R]}$, *where* $\sigma = \dfrac{M+\lambda_1}{2\lambda_1}$,

(2) if $0 \le \phi(t) \le \psi(t)$, $\forall\ t \in I$, *then* $\|\varphi\|^*_{C[I,R]} \le \|\psi\|^*_{C[I,R]}$, where $\|\phi\|_{C[I,\ R]} = \max_{t\in I}|\phi(t)|$.

Lemma 3.3 [7]. *If* $B \subset C\,[I, E]$ *is equicontinuous,* $u_0 \in C[I, E]$, *then* $\overline{co}\{B, u_0\}$ *is also equicontinuous in* $C[I, E]$.

Lemma 3.4 [7]. *If* $B \subset C[I, E]$ *is equicontinuous and bounded, then* $\alpha(B) = \max_{t\in I} \alpha(B(t))$.

Lemma 3.5 [7]. *If* $B \subset C[I, E]$ *is equicontinuous and bounded, then* $\alpha(B(t)) \in C[I, R^+]$, *and*

$$\alpha\left(\int_{t_0}^{t} B(s)\,ds\right) \le \int_{t_0}^{t} \alpha(B(s))\,ds,\ \forall t \in I.$$

Proof of Theorem 3.1. Set $R_1 > \dfrac{2\lambda_1}{\lambda_1 - M}\|Kh\|^*_{C[I,R]}$, where $(Kh)(t) = \int_0^1 G(t,s)\,h(s)\,ds$.

Let $D = \left\{x \in C[I,E]\,|\,\varphi(t) = \|x(t)\| \text{and} \|\varphi\|^*_{C[I,R]} \le R_1\right\}$. Since $\|\cdot\|_{C[I,\ R]}$ is equivalent to $\|\cdot\|^*_{C[I,R]}$, D is bounded, convex, and closed in $C[I, E]$.

First $\forall x \in D$, $\|x\|_{C[I,\ E]} = \max_{t\in I} \|x(t)\| = \max_{t\in I} \phi(t) = \max_{t\in I} |\phi(t)| = \|\phi\|_{C[I,\ R]}$, then D is bounded.

Second, $\forall x_n \in D$, $x_n \to x_0$, $n \to +\infty$. Therefore, $\varphi_n(t) = \|x_n(t)\|, \|\varphi_n\|^*_{C[I,R]} \le R_1, \|x_n - x_0\|_{C[I,E]} \to 0$, i.e., $\max_{t\in I} \|x_n(t) - x_0(t)\| \to 0$.

Let $\tilde{\varphi}_n(t) = ||x_n(t) - x_0(t)||$, $\phi_0(t) = ||x_0(t)||$, then $\varphi_0(t) \leq \tilde{\varphi}_n(t) + \varphi_n(t)$, $||\tilde{\varphi}_n||_{C[I,R]} \to 0$. By Lemma 3.2,

$$\begin{aligned}||\varphi_0||^*_{C[I,R]} &\leq ||\tilde{\varphi}_n + \varphi_n||^*_{C[I,R]}\\ &\leq ||\tilde{\varphi}_n||^*_{C[I,R]} + ||\varphi_n||^*_{C[I,R]}\\ &\leq ||\tilde{\varphi}_n||^*_{C[I,R]} + R_1\end{aligned}$$

Let $n \to +\infty$, then $||\varphi_0||^*_{C[I,R]} \leq R_1$, i.e., $x_0 \in D$, D is closed.

Finally, $\forall x_1, x_2 \in D$, $0 \leq \alpha \leq 1$. Let $\phi_i(t) = ||x_i(t)||$, $i = 1,2$; $\phi_3(t) = ||\alpha x_1(t)+(1-\alpha)x_2(t)||$. Obviously, $\phi_3 \leq \alpha\phi_1(t)+(1-\alpha)\phi_2(2)$. By Lemma 3.2,

$$||\varphi_3||^*_{C[I,R]}|| \leq \alpha||\varphi_1||^*_{C[I,R]}|| + (1-\alpha)\,||\varphi_2||^*_{C[I,R]} \leq R_1.$$

Then D is convex. Therefore, D is bounded, convex, and closed.

By (H_2), f is uniformly continuous in $I \times D$, then T: $D \to C[I, E]$ is continuous.

First, we prove that T: $D \to D$. For any given x in D, let $\phi(t) = ||Tx(t)||\psi(t) = ||x(t)||$. By (H_3),

$$\begin{aligned}\varphi(t) = ||Tx(t)|| &= ||\int_0^1 G(t,s)f(s,x(s))ds||\\ &\leq \int_0^1 G(t,s)\,||f(s,x(s))||ds\\ &\leq \int_0^1 G(t,s)M\psi(s)\,ds + \int_0^1 G(t,s)\,h(s)\,ds\\ &= (K_1\psi)(t) + (Kh)(t).\end{aligned}$$

By Lemma 3.2,

$$\begin{aligned}||\varphi||^*_{C[I,R]} &\leq ||K_1\psi + Kh||^*_{C[I,R]}\\ &\leq ||K_1\psi||^*_{C[I,R]} + ||Kh||^*_{C[I,R]}\\ &\leq \sigma||\psi||^*_{C[I,R]} + ||Kh||^*_{C[I,R]}\\ &\leq \sigma R_1 + \frac{\lambda_1 - M}{2\lambda_1}R_1 = R_1.\end{aligned}$$

Therefore, $T : D \to D$ is continuous

Next, we prove that $T(D)$ is equicontinuous in $C[I, E]$. By (H_2), $\exists M_1 > 0$, $||f(t, x)|| \leq M_1$, $\forall(t, x) \in I \times D$. Then,

$$||Tx(t_1) - Tx(t_2)|| \leq M_1\int_0^1 |G(t_1,s) - G(t_2,s)|ds, \forall t_1, t_2 \in I, x \in D.$$

Therefore, $T(D)$ is equicontinuous.

Let $F = \overline{co}T(D) \subset D$. Obviously, F is bounded, convex, and closed, and $T(\overline{co}T(D)) \subset T(D) \subset \overline{co}T(D)$, i.e., T: $F \to F$. By Lemma 3.3, F is equicontinuous in $C[I, E]$.

Next, we prove that T: $F \to F$ is convex-power 1-set-contraction. Obviously, T is bounded and continuous. Set $x_0 \in F$, we'll prove that there exists n_0, such that for any bounded $B \subset F$,

$$\alpha\left(T^{(n_0,x_0)}(B)\right) \le \alpha(B).$$

By $B \subset F \subset D$, $T(B)$ is equicontinuous. Then $T^{(2,x_0)}(B)$ is equicontinuous from $T^{(2,x_0)}(B) = T(\overline{co}\{T(B), x_0\}) \subset T(D)$ Generally, $\forall\ n \in N$, $T^{(n,x_0)}(B)$ is equicontinuous. Since $T^{(n,x_0)}(B)$ is bounded, By Lemma 3.4,

$$\alpha\left(T^{(n,x_0)}(B)\right) = \max_{t\in I}\alpha\left(\left(T^{(n,x_0)}(B)\right)(t)\right) n = 2, 3, \dots. \tag{3.4}$$

Since $G(t, s)$ is continuous in $I \times I$, f is uniformly continuous in $I \times D$, then

$$\begin{gathered} ||G(t,s_1) f(s_1, x(s_1)) - G(t,s_2) f(s_2, x(s_2))|| \\ \le ||G(t,s_1) - G(t,s_2)||\ ||f(s_1, x(s_1))|| + ||G(t,s_2)||\ ||f(s_1, x(s_1)) - f(s_2, x(s_2))|| \\ (\forall s_1, s_2 \in I, x \in B) \end{gathered}$$

Therefore $G(t, s)f(s, B(s))(\forall\ s,\ t \in\ I)$ is equicontinuous in $C[I, E]$. By (H_2), Lemmas 3.4 and 3.5,

$$\begin{aligned} \alpha\left(\left(T^{(1,x_0)}(B)\right)(t)\right) &= \alpha\left((T(B))(t)\right) \\ &= \alpha\left(\int_0^1 G(t,s) f(s, B(s))\, ds\right) \\ &\le \int_0^1 G(t,s)\, \alpha\left(f(s, B(s))\right) ds \\ &\le k\int_0^1 G(t,s)\, \alpha(B(s))\, ds \\ &\le k\alpha(B)\int_0^1 G(t,s)\, ds \\ &= k\alpha(B)\cdot K\varphi_0(t) \end{aligned}$$

where $\phi_0(t) \equiv 1$, $\forall\ t \in\ I$.

By the equicontinuity of $T^{(1,x_0)}(B) = T(B)$ and the uniform continuity of f, $G(t,s) f\left(s, \overline{co}\left\{\left(T^{(1,x_0)}(B)\right)(s), x_0\right\}\right) (\forall s, t \in I)$ is equicontinuous. Therefore,

$$\begin{aligned} \alpha\left(\left(T^{(2,x_0)}(B)\right)(t)\right) &= \alpha\left(T\overline{co}\left\{\left(T^{(1,x_0)}(B)\right)(t), x_0\right\}\right) \\ &= \alpha\left(\int_0^1 G(t,s) f\left(s, \overline{co}\left\{\left(T^{(1,x_0)}(B)\right)(s), x_0\right\}\right) ds\right) \\ &\le \int_0^1 G(t,s)\, \alpha\left(f\left(s, \overline{co}\left\{\left(T^{(1,x_0)}(B)\right)(s), x_0\right\}\right)\right) ds \\ &\le k\int_0^1 G(t,s)\, \alpha\left(\overline{co}\left\{\left(T^{(1,x_0)}(B)\right)(s), x_0\right\}\right) ds \\ &= k\int_0^1 G(t,s)\, \alpha\left(\left(T^{(1,x_0)}(B)\right)(s)\right) ds \\ &\le k^2\alpha(B)\int_0^1 G(t,s)\, K\varphi_0(s)\, ds \\ &= k^2\alpha(B)\cdot\ K^2\varphi_0(t). \end{aligned}$$

Generally,

$$\alpha\left(\left(T^{(n,x_0)}(B)\right)(t)\right) \le k^n\alpha(B)\cdot K^n\varphi_0(t)$$

We have $r(kK) = kr(K) = k \cdot \lambda_1^{-1} < \lambda_1 \cdot \lambda_1^{-1} = 1$. By the definition of spectral radius, let $\varepsilon = \dfrac{1 - r(kK)}{2}$, then $\exists\ m_0 > 0$, when $n > m_0$,

$$\begin{aligned}\max_{t\in I}|k^n K^n \varphi_0(t)| &= ||k^n K^n \varphi_0|| \\ &\leq ||k^n K^n||\,||\varphi_0|| = ||k^n K^n|| \\ &\leq (r(kK) + \varepsilon)^n = \left(\frac{1 + r(kK)}{2}\right)^n < 1.\end{aligned}$$

Set $n_0 > m_0$, then $\forall t \in I$,

$$\begin{aligned}\alpha\left(\left(T^{(n_0,x_0)}(B)\right)(t)\right) &\leq k^{n_0}\alpha(B) \cdot K^{n_0}\varphi_0(t) \\ &\leq ||k^{n_0} \cdot K^{n_0}\varphi_0||\alpha(B) \\ &\leq \left(\frac{1 + r(kK)}{2}\right)^{n_0}\alpha(B) \leq \alpha(B).\end{aligned}$$

By (3.4), $\alpha\left(T^{(n_0,x_0)}(B)\right) \leq \alpha(B)$. Therefore, T: $F \to F$ is convex-power 1-set-contraction. Since f is uniformly continuous, T is semi-closed. By Theorem 2.1, T has one fixed point in $C[I, E]$, i.e., Sturm-Liouville problems (3.1) has at least one solution in $C^2[I, E]$.

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Competing interests
The authors declare that they have no competing interests.

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