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On generalized weakly directional contractions and approximate fixed point property with applications

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Abstract

In this article, we first introduce the concept of directional hidden contractions in metric spaces. The existences of generalized approximate fixed point property for various types of nonlinear contractive maps are also given. From these results, we present some new fixed point theorems for directional hidden contractions which generalize Berinde-Berinde's fixed point theorem, Mizoguchi-Takahashi's fixed point theorem and some well-known results in the literature.

MSC: 47H10; 54H25.

Keywords: τ-function, τ^0-metric, Reich's condition, $\mathscr{R}$-function($\mathscr{M}$T-function), directional hidden contraction, approximate fixed point property, generalized Mizoguchi-Takahashi's fixed point theorem, generalized Berinde-Berinde's fixed point theorem

1 Introduction and preliminaries

Let (X, d) be a metric space. The open ball centered in $x \in X$ with radius $r > 0$ is denoted by $B(x, r)$. For each $x \in X$ and $A \subseteq X$, let $d(x, A) = \inf_{y\in A} d(x, y)$. Denote by $\mathcal{N}(X)$ the class of all nonempty subsets of X, $\mathcal{C}(X)$ the family of all nonempty closed subsets of X and $\mathcal{CB}(X)$ the family of all nonempty closed and bounded subsets of X. A function $\mathcal{H} : \mathcal{CB}(X) \times \mathcal{CB}(X) \to [0, \infty)$ defined by

$$\mathcal{H}(A, B) = \max\left\{\sup_{x\in B} d(x, A), \sup_{x\in A} d(x, B)\right\}$$

is said to be the Hausdorff metric on $\mathcal{CB}(X)$ induced by the metric d on X. A point v in X is a fixed point of a map T if $v = Tv$ (when $T : X \to X$ is a single-valued map) or $v \in Tv$ (when $T : X \to \mathcal{N}(X)$ is a multivalued map). The set of fixed points of T is denoted by $\mathcal{F}(T)$. Throughout this article, we denote by $\mathbb{N}$ and $\mathbb{R}$, the sets of positive integers and real numbers, respectively.

The celebrated Banach contraction principle (see, e.g., [1]) plays an important role in various fields of applied mathematical analysis. It is known that Banach contraction principle has been used to solve the existence of solutions for nonlinear integral equations and nonlinear differential equations in Banach spaces and been applied to study the convergence of algorithms in computational mathematics. Since then a number of generalizations in various different directions of the Banach contraction principle have been investigated by several authors; see [1-36] and references therein. A interesting direction of research is the extension of the Banach contraction principle to multivalued maps, known as Nadler's fixed point theorem [2], Mizoguchi-Takahashi's fixed point theorem [3], Berinde-Berinde's fixed point theorem [5] and references therein. Another interesting direction of research led to extend to the multivalued maps setting previous fixed point results valid for single-valued maps with so-called directional contraction properties (see [20-24]). In 1995, Song [22] established the following fixed point theorem for directional contractions which generalizes a fixed point result due to Clarke [20].

Theorem S [22]. Let L be a closed nonempty subset of X and $T : L \to \mathcal{CB}(X)$ be a multivalued map. Suppose that

(i) T is H-upper semicontinuous, that is, for every $\varepsilon > 0$ and every $x \in L$ there exists $r > 0$ such that $\sup_{y \in Tx'} d(y, Tx) < \varepsilon$ for every $x' \in B(x, r)$;

(ii) there exist $\alpha \in (0, 1]$ and $\gamma \in [0, \alpha)$ such that for every $x \in L$ with $x \notin Tx$, there exists $y \in L \setminus \{x\}$ satisfying

$$\alpha d(x, y) + d(y, Tx) \leq d(x, Tx)$$

and

$$\sup_{z \in Tx} d(z, Ty) \leq \gamma d(x, y).$$

Then $\mathcal{F}(T) \cap L \neq \emptyset$.

Definition 1.1 [23]. Let L be a nonempty subset of a metric space (X, d). A multivalued map $T : L \to \mathcal{CB}(X)$ is called a *directional multivalued $k(\cdot)$-contraction* if there exist $\lambda \in (0, 1]$, $a : (0, \infty) \to [\lambda, 1]$ and $k : (0, \infty) \to [0, 1)$ such that for every $x \in L$ with $x \notin Tx$, there is $y \in L \setminus \{x\}$ satisfying the inequalities

$$a(d(x, y))d(x, y) + d(y, Tx) \leq d(x, Tx)$$

and

$$\sup_{z \in Tx} d(z, Ty) \leq k(d(x, y))d(x, y).$$

Subsequently Uderzo [23] generalized Song's result and some main results in [21] for directional multivalued $k(\cdot)$-contractions.

Theorem U [23]. Let L be a closed nonempty subset of a metric space (X, d) and $T : L \to \mathcal{CB}(X)$ be an u.s.c. directional multivalued $k(\cdot)$-contraction. Assume that there exist $x_0 \in L$ and $\delta > 0$ such that $d(x_0, Tx_0) \leq \alpha\delta$ and

$$\sup_{t \in (0, \delta]} k(t) < \inf_{t \in (0, \delta]} a(t),$$

where $\lambda \in (0, 1]$, a and k are the constant and the functions occuring in the definition of directional multivalued $k(\cdot)$-contraction. Then $\mathcal{F}(T) \cap L \neq \emptyset$.

Recall that a function $p : X \times X \to [0, \infty)$ is called a *w-distance* [1,25-30], if the following are satisfied:

($w1$) $p(x, z) \leq p(x, y) + p(y, z)$ for any $x, y, z \in X$;

($w2$) for any $x \in X$, $p(x, \cdot): X \to [0, \infty)$ is l.s.c;

($w3$) for any $\varepsilon > 0$, there exists $\delta > 0$ such that $p(z, x) \leq \delta$ and $p(z, y) \leq \delta$ imply $d(x, y) \leq \varepsilon$.

A function $p : X \times X \to [0, \infty)$ is said to be a *τ-function* [14,26,28-30], first introduced and studied by Lin and Du, if the following conditions hold:

($\tau 1$) $p(x, z) \leq p(x, y) + p(y, z)$ for all $x, y, z \in X$;

($\tau 2$) if $x \in X$ and $\{y_n\}$ in X with $\lim_{n\to\infty} y_n = y$ such that $p(x, y_n) \leq M$ for some $M = M(x) > 0$, then $p(x, y) \leq M$;

($\tau 3$) for any sequence $\{x_n\}$ in X with $\lim_{n\to\infty} \sup\{p(x_n, x_m): m > n\} = 0$, if there exists a sequence $\{y_n\}$ in X such that $\lim_{n\to\infty} p(x_n, y_n) = 0$, then $\lim_{n\to\infty} d(x_n, y_n) = 0$;

($\tau 4$) for $x, y, z \in X$, $p(x, y) = 0$ and $p(x, z) = 0$ imply $y = z$.

Note that not either of the implications $p(x, y) = 0 \Leftrightarrow x = y$ necessarily holds and p is nonsymmetric in general. It is well known that the metric d is a w-distance and any w-distance is a τ-function, but the converse is not true; see [26] for more detail.

The following result is simple, but it is very useful in this article.

Lemma 1.1. Let A be a nonempty subset of a metric space (X, d) and $p : X \times X \to [0, \infty)$ be a function satisfying ($\tau 1$). Then for any $x \in X$, $p(x, A) \leq p(x, z) + p(z, A)$ for all $z \in X$.

The following results are crucial in this article.

Lemma 1.2 [14]. Let A be a closed subset of a metric space (X, d) and $p : X \times X \to [0, \infty)$ be any function. Suppose that p satisfies ($\tau 3$) and there exists $u \in X$ such that $p(u, u) = 0$. Then $p(u, A) = 0$ *if and only if* $u \in A$, where $p(u, A) = \inf_{a\in A} p(u, a)$.

Lemma 1.3 [29, Lemma 2.1]. Let (X, d) be a metric space and $p : X \times X \to [0, \infty)$ be a function. Assume that p satisfies the condition ($\tau 3$). If a sequence $\{x_n\}$ in X with $\lim_{n\to\infty} \sup\{p(x_n, x_m): m > n\} = 0$, then $\{x_n\}$ is a Cauchy sequence in X.

Recently, Du first introduced the concepts of τ^0-functions and τ^0-metrics as follows.

Definition 1.2 [14]. Let (X, d) be a metric space. A function $p : X \times X \to [0, \infty)$ is called a τ^0-function if it is a τ-function on X with $p(x, x) = 0$ for all $x \in X$.

Remark 1.1. If p is a τ^0-function, then, from ($\tau 4$), $p(x, y) = 0$ if and only if $x = y$.

Example 1.1 [14]. Let $X = \mathbb{R}$ with the metric $d(x, y) = |x - y|$ and $0 < a < b$. Define the function $p : X \times X \to [0, \infty)$ by

$$p(x, y) = \max\{a(y - x), b(x - y)\}.$$

Then p is nonsymmetric and hence p is not a metric. It is easy to see that p is a τ^0-function.

Definition 1.3 [14]. Let (X, d) be a metric space and p be a τ^0-function. For any $A, B \in \mathcal{CB}(X)$, define a function $\mathcal{D}_p : \mathcal{CB}(X) \times \mathcal{CB}(X) \to [0, \infty)$ by

$$\mathcal{D}_p(A, B) = \max\{\delta_p(A, B), \delta_p(B, A)\},$$

where $\delta_p(A, B) = sup_{x\in A}\, p(x, B)$, then $\mathcal{D}_p$ is said to be the τ^0-*metric* on $\mathcal{CB}(X)$ induced by p.

Clearly, any Hausdorff metric is a τ^0-metric, but the reverse is not true. It is known that every τ^0-metric $\mathcal{D}_p$ is a metric on $\mathcal{CB}(X)$; see [14] for more detail.

Let f be a real-valued function defined on $\mathbb{R}$. For $c \in \mathbb{R}$, we recall that

$$\limsup_{x\to c} f(x) = \inf_{\varepsilon>0} \sup_{0<|x-c|<\varepsilon} f(x)$$

and

$$\limsup_{x\to c^+} f(x) = \inf_{\varepsilon>0} \sup_{0<x-c<\varepsilon} f(x).$$

Definition 1.4. A function $\alpha : [0, \infty) \to [0, 1)$ is said to be a *Reich's function* ($\mathcal{R}$ *-function*, for short) if

$$\limsup_{s\to t^+} \alpha(s) < 1 \quad \text{for all } t \in [0, \infty). \tag{1.1}$$

Remark 1.2. In [14-19,30], a function $\alpha : [0, \infty) \to [0, 1)$ satisfying the property (1.1) was called to be an $\mathcal{MT}$-function. But it is more appropriate to use the terminology $\mathcal{R}$ -function instead of $\mathcal{MT}$-function since Professor S. Reich was the first to use the property (1.1).

It is obvious that if $\alpha : [0, \infty) \to [0, 1)$ is a nondecreasing function or a nonincreasing function, then α is a $\mathcal{R}$ -function. So the set of $\mathcal{R}$ -functions is a rich class. It is easy to see that $\alpha : [0, \infty) \to [0, 1)$ is a $\mathcal{R}$ -function *if and only if* for each $t \in [0, \infty)$, there exist $r_t \in [0, 1)$ and $\varepsilon_t > 0$ such that $\alpha(s) \le r_t$ for all $s \in [t, t + \varepsilon_t)$; for more details of characterizations of $\mathcal{R}$ -functions, one can see [19, Theorem 2.1].

In [14], the author established some new fixed point theorems for nonlinear multivalued contractive maps by using τ^0-function, τ^0-metrics and $\mathcal{R}$ -functions. Applying those results, the author gave the generalizations of Berinde-Berinde's fixed point theorem, Mizoguchi-Takahashi's fixed point theorem, Nadler's fixed point theorem, Banach contraction principle, Kannan's fixed point theorems and Chatterjea's fixed point theorems for nonlinear multivalued contractive maps in complete metric spaces; for more details, we refer the reader to [14].

This study is around the following Reich's open question in [35] (see also [36]): Let (X, d) be a complete metric space and $T : L \to \mathcal{CB}(X)$ be a multivalued map. Suppose that

$$\mathcal{H}(Tx, Ty) \le \varphi(d(x, y))d(x, y) \quad \text{for all } x, y \in X,$$

where $\varphi : [0, \infty) \to [0, 1)$ satisfies the property (*) except for $t = 0$. Does T have a fixed point? In this article, our some new results give partial answers of Reich's open question and generalize Berinde-Berinde's fixed point theorem, Mizoguchi-Takahashi's fixed point theorem and some well-known results in the literature.

The article is divided into four sections. In Section 2, in order to carry on the development of metric fixed point theory, we first introduce the concept of directional hidden contractions in metric spaces. In Section 3, we present some new existence results concerning p-approximate fixed point property for various types of nonlinear contractive maps. Finally, in Section 4, we establish several new fixed point theorems for directional hidden contractions. From these results, new generalizations of Berinde-Berinde's fixed point theorem and Mizoguchi-Takahashi's fixed point theorem are also given.

2 Directional hidden contractions

Let (X, d) be a metric space and $p : X \times X \to [0, \infty)$ be any function. For each $x \in X$ and $A \subseteq X$, let

$$p(x, A) = \inf_{y \in A} p(x, y).$$

Recall that a multivalued map $T : X \to \mathcal{N}(X)$ is called

(1) a *Nadler's type contraction* (or a *multivalued k-contraction* [3]), if there exists a number $0 < k < 1$ such that

$$\mathcal{H}(Tx, Ty) \leq kd(x, y) \quad \text{for all } x, y \in X.$$

(2) a *Mizoguchi-Takahashi's type contraction*, if there exists a $\mathcal{R}$ -function $\alpha : [0, \infty) \to [0, 1)$ such that

$$\mathcal{H}(Tx, Ty) \leq \alpha(d(x, y))d(x, y) \quad \text{for all } x, y \in X;$$

(3) a *multivalued* (θ, L)*-almost contraction* [5-7], if there exist two constants $\theta \in (0, 1)$ and $L \geq 0$ such that

$$\mathcal{H}(Tx, Ty) \leq \theta d(x, y) + Ld(y, Tx) \quad \text{for all } x, y \in X.$$

(4) a *Berinde-Berinde's type contraction* (or a *generalized multivalued almost contraction* [5-7]), if there exist a $\mathcal{R}$ -function $\alpha : [0, \infty) \to [0, 1)$ and $L \geq 0$ such that

$$\mathcal{H}(Tx, Ty) \leq \alpha(d(x, y))d(x, y) + Ld(y, Tx) \quad \text{for all } x, y \in X.$$

Mizoguchi-Takahashi's type contractions and Berinde-Berinde's type contractions are relevant topics in the recent investigations on metric fixed point theory for contractive maps. It is quite clear that any Mizoguchi-Takahashi's type contraction is a Berinde-Berinde's type contraction. The following example tell us that a Berinde-Berinde's type contraction may be not a Mizoguchi-Takahashi's type contraction in general.

Example 2.1. Let ℓ^∞ be the Banach space consisting of all bounded real sequences with supremum norm d_∞ and let $\{e_n\}$ be the canonical basis of ℓ^∞. Let $\{\tau_n\}$ be a sequence of positive real numbers satisfying $\tau_1 = \tau_2$ and $\tau_{n+1} < \tau_n$ for $n \geq 2$ (for example, let $\tau_1 = \frac{1}{2}$ and $\tau_n = \frac{1}{n}$ for $n \in \mathbb{N}$ with $n \geq 2$). Thus $\{\tau_n\}$ is convergent. Put $v_n = \tau_n e_n$ for $n \in \mathbb{N}$ and let $X = \{v_n\}_{n \in \mathbb{N}}$ be a bounded and complete subset of ℓ^∞. Then (X, d_∞) be a complete metric space and $d_\infty(v_n, v_m) = \tau_n$ if $m > n$.

Let $T : L \to \mathcal{CB}(X)$ be defined by

$$Tv_n := \begin{cases} \{v_1, v_2\}, & \text{if } n \in \{1, 2\}, \\ X \backslash \{v_1, v_2, \ldots, v_n, v_{n+1}\}, & \text{if } n \geq 3. \end{cases}$$

and define $\phi : [0, \infty) \to [0, 1)$ by

$$\varphi(t) := \begin{cases} \frac{\tau_{n+2}}{\tau_n}, & \text{if } t = \tau_n \text{ for some } n \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$$

Then the following statements hold.

(a) T is a Berinde-Berinde's type contraction;
(b) T is not a Mizoguchi-Takahashi's type contraction.

Proof. Observe that $\limsup_{s \to t^+} \varphi(s) = 0 < 1$ for all $t \in [0, \infty)$, so ϕ is a $\mathcal{R}$ -function. It is not hard to verify that

$$\mathcal{H}_\infty(Tv_1, Tv_m) = \tau_1 > \tau_3 = \varphi(d_\infty(v_1, v_m))d_\infty(v_1, v_m) \quad \text{for all } m \geq 3.$$

Hence T is not a Mizoguchi-Takahashi's type contraction. We claim that T is a Berinde-Berinde's type contraction with $L \geq 1$; that is,

$$\mathcal{H}_\infty(Tx, Ty) \leq \varphi(d_\infty(x, y))d_\infty(x, y) + Ld_\infty(y, Tx) \text{ for all } x, y \in X,$$

where $\mathcal{H}_\infty$ is the Hausdorff metric induced by d_∞. Indeed, we consider the following four possible cases:

(i) $\varphi(d(v_1, v_2))d_\infty(v_1, v_2) + Ld_\infty(v_2, Tv_1) = \tau_3 > 0 = \mathcal{H}_\infty(Tv_1, Tv_2)$.

(ii) For any $m \geq 3$, we have

$$\varphi(d_\infty(v_1, v_m))d_\infty(v_1, v_m) + Ld_\infty(v_m, Tv_1) = \tau_3 + L\tau_2 > \tau_1 = \mathcal{H}_\infty(Tv_1, Tv_m).$$

(iii) For any $m \geq 3$, we obtain

$$\varphi(d_\infty(v_2, v_m))d_\infty(v_2, v_m) + Ld_\infty(v_m, Tv_2) = \tau_4 + L\tau_2 > \tau_1 = \mathcal{H}_\infty(Tv_2, Tv_m).$$

(iv) For any $n \geq 3$ and $m > n$, we get

$$\varphi(d_\infty(v_n, v_m))d_\infty(v_n, v_m) + Ld_\infty(v_m, Tv_n) = \tau_{n+2} = \mathcal{H}_\infty(Tv_n, Tv_m).$$

Hence, by (i)-(iv), we prove that T is a Berinde-Berinde's type contraction with $L \geq 1$.

In order to carry on such development of classic metric fixed point theory, we first introduce the concept of directional hidden contractions as follows. Using directional hidden contractions, we will present some new fixed point results and show that several already existent results could be improved.

Definition 2.1. Let L be a nonempty subset of a metric space (X, d), $p : X \times X \to [0, \infty)$ be any function, $c \in (0, 1)$, $\eta : [0, \infty) \to (c, 1]$ and $\varphi : [0, \infty) \to [0, 1)$ be functions. A multivalued map $T : X \to \mathcal{N}(X)$ is called a *directional hidden contraction* with respect to p, c, η and φ ((p, c, η, φ)-**DHC**, for short) if for any $x \in L$ with $x \notin Tx$, there exist $y \in L \setminus \{x\}$ and $z \in Tx$ such that

$$p(z, Ty) \leq \phi(p(x, y))p(x, y)$$

and

$$\eta(p(x, y))p(x, y) + p(y, z) \leq p(x, Tx).$$

In particular, if $p \equiv d$, then we use the notation (c, η, φ)-**DHC** instead of (d, c, η, φ)-**DHC**.

Remark 2.1. We point out the fact that the concept of directional hidden contractions really generalizes the concept of directional multivalued $k(\cdot)$-contractions. Indeed, let T be a directional multivalued $k(\cdot)$-contraction. Then there exist $\lambda \in (0, 1]$, $a : (0, \infty) \to [\lambda, 1]$ and $k : (0, \infty) \to [0, 1)$ such that for every $x \in L$ with $x \notin Tx$, there is $y \in L \setminus \{x\}$ satisfying the inequalities

$$a(d(x,y))d(x,y) + d(y,Tx) \leq d(x,Tx) \tag{2.1}$$

and

$$\sup_{z\in Tx} d(z,Ty) \leq k(d(x,y))d(x,y). \tag{2.2}$$

Note that $x \neq y$ and hence $d(x, y) > 0$. We consider the following two possible cases:

(i) If $\lambda = 1$, then $a(t) = 1$ for all $t \in (0, \infty)$. Choose $c_1, r \in (0, 1)$ with $c_1 < r$. By (2.1), we have

$$rd(x,y) + d(y,Tx) < d(x,Tx),$$

which it is thereby possible to find $z_r \in Tx$ such that

$$rd(x,y) + d(y,z_r) < d(x,Tx).$$

Define $\eta_1 : [0, \infty) \to (c_1, 1]$ by

$$\eta_1(t) = r$$

and let $\varphi_1 : [0, \infty) \to [0, 1)$ be defined by

$$\phi_1(t) = \begin{cases} 0, & \text{if } t = 0, \\ k(t), & \text{if } t \in (0,\infty). \end{cases}$$

Hence T is a (c_1, η_1, φ_1)-**DHC**.

(ii) If $\lambda \in (0, 1)$, we choose c_2 satisfying $0 < c_2 < \lambda$. Then

$$c_2 < \frac{\lambda + c_2}{2} \leq \frac{a(t) + c_2}{2} < a(t) \leq 1 \quad \text{for all } t \in (0,\infty).$$

So we can define $\eta_2 : [0, \infty) \to (c_2, 1]$ by

$$\eta_2(t) = \begin{cases} 0, & \text{if } t = 0, \\ \dfrac{a(t) + c_2}{2}, & \text{if } t \in (0,\infty). \end{cases}$$

Since $\eta_2(t) < a(t)$ for all $t \in (0, \infty)$, the inequality (2.1) admits that there exists $z \in Tx$ such that

$$\eta(d(x,y))d(x,y) + d(y,z) < d(x,Tx).$$

Let $\varphi_2 = \varphi_1$. Therefore T is a (c_2, η_2, φ_2)-**DHC**.

The following example show that the concept of directional hidden contractions is indeed a proper extension of classic contractive maps.

Example 2.2. Let $X = [0, 1]$ with the metric $d(x, y) = |x - y|$ for $x, y \in X$. Let $T : X \to \mathcal{C}(X)$ be defined by

$$Tx = \begin{cases} \{0,1\}, & \text{if } x = 0, \\ \{\frac{1}{2}x^4, 1\}, & \text{if } x \in (0, \frac{1}{4}], \\ \{0, \frac{1}{2}x^4\}, & \text{if } x \in (\frac{1}{4}, 1), \\ \{1\}, & \text{if } x = 1. \end{cases}$$

Define $\eta : [0, \infty) \to (\frac{1}{2}, 1]$ and $\varphi : [0, \infty) \to [0, 1)$ by

$$\eta(s) = \frac{3}{4} \quad \text{for all } s \in [0, \infty)$$

and

$$\phi(t) = \begin{cases} 2t, & \text{if } t \in [0, \frac{1}{2}), \\ 0, & \text{if } t \in [\frac{1}{2}, \infty), \end{cases}$$

respectively. It is not hard to verify that T is a $(\frac{1}{2}, \eta, \phi)$-**DHC**. Notice that

$$\mathcal{H}(T(0), T(1)) = 1 = d(0, 1),$$

so T is not a Mizoguchi-Takahashi's type contraction (hence it is also not a Nadler's type contraction).

We now present some existence theorems for directional hidden contractions.

Theorem 2.1. Let (X, d) be a metric space, p be a τ^0-function, $T : X \to \mathcal{C}(X)$ be a multivalued map and $\gamma \in [0, \infty)$. Suppose that

$(\mathcal{P})$ there exists a function $\phi : (0, \infty) \to [0, 1)$ such that

$$\limsup_{s \to \gamma^+} \varphi(s) < 1$$

and for each $x \in X$ with $x \notin Tx$, it holds

$$p(y, Ty) \leq \varphi(p(x, y))p(x, y) \quad \text{for all } y \in Tx. \tag{2.3}$$

Then there exist $c \in (0, 1)$ and functions $\eta : [0, \infty) \to (c, 1]$ and $\varphi : [0, \infty) \to [0, 1)$ such that

(a) $\limsup_{s \to \gamma^+} \phi(s) < \liminf_{s \to \gamma^+} \eta(s)$;

(b) T is a (p, c, η, φ)-**DHC**.

Proof. Set $L \equiv X$. Let $\varphi : [0, \infty) \to [0, 1)$ be defined by

$$\phi(s) := \begin{cases} 0, & \text{if } s = 0, \\ \varphi(s), & \text{if } s \in (0, \infty). \end{cases}$$

By $(\mathcal{P})$, there exists $c \in (0, 1)$ such that

$$\limsup_{s \to \gamma^+} \varphi(s) < c < 1.$$

Put $\alpha = \frac{c+1}{2}$. Then $0 < c < \alpha < 1$. Define $\eta : [0, \infty) \to (c, 1]$ by $\eta(s) = \alpha$ for all $s \in [0, \infty)$. So we obtain

$$\limsup_{s \to \gamma^+} \phi(s) < \alpha = \liminf_{s \to \gamma^+} \eta(s).$$

Given $x \in X$ with $x \notin Tx$. Since p is a τ^0-function and Tx is a closed set in X, by Lemma 1.2, $p(x, Tx) > 0$. Since $p(x, Tx) < \frac{p(x, Tx)}{\alpha}$, there exists $y \in Tx$, such that

$$p(x,y) < \frac{p(x,Tx)}{\alpha}. \tag{2.4}$$

Clearly, $y \neq x$. Let $z = y \in Tx$. Since p is a τ^0-function, we have $p(y, z) = 0$. From (2.3) and (2.4), we obtain

$$p(z,Ty) \le \phi(p(x,y))p(x,y)$$

and

$$\eta(p(x,y))p(x,y) + p(y,z) \le p(x,Tx),$$

which show that T is a (p, c, η, φ)-**DHC**. □

If we put $p \equiv d$ in Theorem 2.1, then we have the following result.

Theorem 2.2. Let (X, d) be a metric space, $T : X \to \mathcal{C}(X)$ be a multivalued map and $\gamma \in [0, \infty)$. Suppose that

$(\mathcal{P}_d)$ there exists a function $\phi : (0, \infty) \to [0, 1)$ such that

$$\limsup_{s\to\gamma^+} \varphi(s) < 1$$

and for each $x \in X$ with $x \notin Tx$, it holds

$$d(y,Ty) \le \varphi(d(x,y))d(x,y) \quad \text{for all } y \in Tx.$$

Then there exist $c \in (0, 1)$ and functions $\eta : [0, \infty) \to (c, 1]$ and $\varphi : [0, \infty) \to [0, 1)$ such that

(a) $\limsup_{s\to\gamma^+} \phi(s) < \liminf_{s\to\gamma^+} \eta(s)$;

(b) T is a (c, η, φ)-**DHC**.

Theorem 2.3. Let (X, d) be a metric space, p be a τ^0-function, $\mathcal{D}_p$ be a τ^0-metric on $\mathcal{CB}(X)$ induced by p, $T : X \to \mathcal{CB}(X)$ be a multivalued map, $h : X \times X \to [0, \infty)$ be a function and $\gamma \in [0, \infty)$. Suppose that

$(\mathcal{A})$ there exists a function $\phi : (0, \infty) \to [0, 1)$ such that

$$\limsup_{s\to\gamma^+} \varphi(s) < 1$$

and

$$\mathcal{D}_p(Tx,Ty) \le \varphi(p(x,y))p(x,y) + h(x,y)p(y,Tx) \quad \text{for all } x,y \in X \text{ with } x \neq y. \tag{2.5}$$

Then there exist $c \in (0, 1)$ and functions $\eta : [0, \infty) \to (c, 1]$ and $\varphi : [0, \infty) \to [0, 1)$ such that

(a) $\limsup_{s\to\gamma^+} \phi(s) < \liminf_{s\to\gamma^+} \eta(s)$;

(b) T is a (p, c, η, φ)-**DHC**.

Proof. Let $x \in X$ with $x \notin Tx$ and let $y \in Tx$ be given. So $x \neq y$. By Lemma 1.2, $p(y, Tx) = 0$. It is easy to see that (2.5) implies (2.3). Therefore the conclusion follows from Theorem 2.1. □

Theorem 2.4. Let (X, d) be a metric space, $T : X \to \mathcal{CB}(X)$ be a multivalued map, $h : X \times X \to [0, \infty)$ be a function and $\gamma \in [0, \infty)$. Suppose that

$(\mathcal{A}_d)$ there exists a function $\phi : (0, \infty) \to [0, 1)$ such that

$$\limsup_{s \to \gamma^+} \varphi(s) < 1$$

and

$$\mathcal{H}(Tx, Ty) \leq \varphi(d(x, y))d(x, y) + h(x, y)d(y, Tx) \quad \text{for all } x, y \in X \text{ with } x \neq y.$$

Then there exist $c \in (0, 1)$ and functions $\eta : [0, \infty) \to (c, 1]$ and $\varphi : [0, \infty) \to [0, 1)$ such that

(a) $\limsup_{s \to \gamma^+} \phi(s) < \liminf_{s \to \gamma^+} \eta(s)$;

(b) T is a (c, η, φ)-**DHC**.

The following result is immediate from Theorem 2.4.

Theorem 2.5. Let (X, d) be a metric space and $T : X \to \mathcal{CB}(X)$ be a multivalued map. Assume that one of the following conditions holds.

(1) T is a Berinde-Berinde's type contraction;

(2) T is a multivalued (θ, L)-almost contraction;

(3) T is a Mizoguchi-Takahashi's type contraction;

(4) T is a Nadler's type contraction.

Then there exist $c \in (0, 1)$ and functions $\eta : [0, \infty) \to (c, 1]$ and $\varphi : [0, \infty) \to [0, 1)$ such that T is a (c, η, φ)-**DHC**.

3 Nonlinear conditions for *p*-approximate fixed point property

Let K be a nonempty subset of a metric space (X, d). Recall that a multivalued map $T : K \to \mathcal{N}(X)$ is said to have the *approximate fixed point property* [7] in K provided $\inf_{x \in K} d(x, Tx) = 0$. Clearly, $\mathcal{F}(T) \neq \emptyset$ implies that T has the approximate fixed point property. A natural generalization of the approximate fixed point property is defined as follows.

Definition 3.1. Let K be a nonempty subset of a metric space (X, d) and p be a τ-function. A multivalued map $T : K \to \mathcal{N}(X)$ is said to have the *p-approximate fixed point property* in K provided $\inf_{x \in K} p(x, Tx) = 0$.

Lemma 3.1. Let $\phi : (0, \infty) \to [0, 1)$ be a function and $\gamma \in (0, \infty)$. If $\limsup_{s \to \gamma^+} \varphi(s) < 1$, then for any strictly decreasing sequence $\{\xi_n\}_{n \in \mathbb{N}}$ in $(0, \infty)$ with $\lim_{n \to \infty} \xi_n = \gamma$, we have $0 \leq \sup_{n \in \mathbb{N}} \varphi(\xi_n) < 1$.

Proof. Since $\limsup_{s \to \gamma^+} \varphi(s) < 1$, there exists $\varepsilon > 0$ such that

$$\sup_{\gamma < s < \gamma + \varepsilon} \varphi(s) < 1.$$

By the denseness of $\mathbb{R}$, there exists $\alpha \in [0, 1)$ such that

$$\sup_{\gamma < s < \gamma + \varepsilon} \varphi(s) \leq \alpha < 1.$$

Hence $\phi(s) \leq \alpha$ for all $s \in (\gamma, \gamma + \varepsilon)$. Let $\{\xi_n\}_{n \in \mathbb{N}}$ be a strictly decreasing sequence in $(0, \infty)$ with $\lim_{n\to\infty} \xi_n = \gamma$. Then

$$\gamma = \lim_{n\to\infty} \xi_n = \inf_{n\in\mathbb{N}} \xi_n \geq 0. \tag{3.1}$$

Since $\{\xi_n\}_{n \in \mathbb{N}}$ is strictly decreasing, it is obvious that $\xi_n > \gamma$ for all $n \in \mathbb{N}$. By (3.1), there exists $\ell \in \mathbb{N}$, such that

$$\gamma < \xi_n < \gamma + \varepsilon \ \text{ for all } n \in \mathbb{N} \text{ with } n \geq \ell.$$

Hence $\varphi(\xi_n) \leq \alpha$ for all $n \geq \ell$. Let

$$\varsigma := \max\{\varphi(\xi_1), \varphi(\xi_2), \ldots, \varphi(\xi_{\ell-1}), \alpha\} < 1.$$

Then $\varphi(\xi_n) \leq \zeta$ for all $n \in \mathbb{N}$ and hence $0 \leq \sup_{n\in\mathbb{N}} \varphi(\xi_n) \leq \varsigma < 1$. □

Theorem 3.1. Let (X, d) be a metric space, p be a τ^0-function and $T : X \to \mathcal{N}(X)$ be a multivalued map. Suppose that

($\mathcal{R}$) there exists a function $\phi : (0, \infty) \to [0, 1)$ satisfying Reich's condition; that is

$$\limsup_{s\to t^+} \varphi(s) < 1 \ \text{ for all } t \in (0, \infty)$$

and for each $x \in X$ with $x \notin Tx$, it holds

$$p(y, Ty) \leq \varphi(p(x, y))p(x, y) \ \text{ for all } y \in Tx. \tag{3.2}$$

Then the following statements hold.

(a) There exists a Cauchy sequence $\{x_n\}_{n \in \mathbb{N}}$ in X such that
  (i) $x_{n+1} \in Tx_n$ for each $n \in \mathbb{N}$;
  (ii) $\inf_{n\in\mathbb{N}} p(x_n, x_{n+1}) = \lim_{n\to\infty} p(x_n, x_{n+1}) = \lim_{n\to\infty} d(x_n, x_{n+1}) = \inf_{n\in\mathbb{N}} d(x_n, x_{n+1}) = 0.$

(b) $\inf_{x\in X} p(x, Tx) = \inf_{x\in X} d(x, Tx) = 0$; that is T have the p-approximate fixed point property and approximate fixed point property in X.

Proof. Let $x_1 \in X$ with $x_1 \notin Tx_1$ and $x_2 \in Tx_1$. Then $x_1 \neq x_2$. Since p is a τ^0-function, $p(x_1, x_2) > 0$. By (3.2), we have

$$p(x_2, Tx_2) \leq \varphi(p(x_1, x_2))p(x_1, x_2). \tag{3.3}$$

If $x_2 \in Tx_2$, then $x_2 \in \mathcal{F}(T)$. Since

$$\inf_{x\in X} p(x, Tx) \leq p(x_2, Tx_2) \leq p(x_2, x_2) = 0$$

and

$$\inf_{x\in X} d(x, Tx) \leq d(x_2, x_2) = 0,$$

we have $\inf_{x\in X} p(x, Tx) = \inf_{x\in X} d(x, Tx) = 0$. Let $\{z_n\}$ be a sequence defined by $z_n = x_2$ for all $n \in \mathbb{N}$. Then $\{z_n\}$ is Cauchy and (a) holds. Hence the proof is finished in this case. Suppose $x_2 \notin Tx_2$. Define $\kappa : (0, \infty) \to [0, 1)$ by $\kappa(t) = \frac{\varphi(t) + 1}{2}$. Then $\phi(t) < \kappa(t)$ and $0 < \kappa(t) < 1$ for all $t \in (0, \infty)$. By (3.3), there exists $x_3 \in Tx_2$ such that

$$p(x_2, x_3) < \kappa(p(x_1, x_2))p(x_1, x_2).$$

Since $x_2 \neq x_3$, $p(x_2, x_3) > 0$. By (3.2) again, we obtain

$$p(x_3, Tx_3) \leq \varphi(p(x_2, x_3))p(x_2, x_3).$$

If $x_3 \in Tx_3$, then, following a similar argument as above, we finish the proof. Otherwise, there exists $x_4 \in Tx_3$ such that

$$p(x_3, x_4) < \kappa(p(x_2, x_3))p(x_2, x_3).$$

By induction, we can obtain a sequence $\{x_n\}$ in X satisfying $x_{n+1} \in Tx_n$, $p(x_n, x_{n+1}) > 0$ and

$$p(x_{n+1}, x_{n+2}) < \kappa(p(x_n, x_{n+1}))p(x_n, x_{n+1}) \text{ for each } n \in \mathbb{N}. \tag{3.4}$$

Since $\kappa(t) < 1$ for all $t \in (0, \infty)$, the sequence $\{p(x_n, x_{n+1})\}$ is strictly decreasing in $(0, \infty)$. Then

$$\gamma := \lim_{n\to\infty} p(x_n, x_{n+1}) = \inf_{n\in\mathbb{N}} p(x_n, x_{n+1}) \geq 0 \text{ exists.}$$

We claim that $\gamma = 0$. Assume to the contrary that $\gamma > 0$. By $(\mathcal{R})$, we have $\limsup_{s\to\gamma^+} \varphi(s) < 1$. Applying Lemma 3.1,

$$0 \leq \sup_{n\in\mathbb{N}} \varphi(p(x_n, x_{n+1})) < 1.$$

By exploiting the last inequality we obtain

$$0 < \sup_{n\in\mathbb{N}} \kappa(p(x_n, x_{n+1})) = \frac{1}{2}\left[1 + \sup_{n\in\mathbb{N}} \varphi(p(x_n, x_{n+1}))\right] < 1.$$

Let $\lambda := \sup_{n\in\mathbb{N}} \kappa(p(x_n, x_{n+1}))$. So $\lambda \in (0, 1)$. It follows from (3.4) that

$$\begin{aligned} p(x_{n+1}, x_{n+2}) &< \kappa(p(x_n, x_{n+1}))p(x_n, x_{n+1}) \\ &\leq \lambda p(x_n, x_{n+1}) \\ &\leq \cdots \\ &\leq \lambda^n p(x_1, x_2) \text{ for each } n \in \mathbb{N}. \end{aligned}$$

Taking the limit in the last inequality as $n \to \infty$ yields $\lim_{n\to\infty} p(x_n, x_{n+1}) = 0$ which leads to a contradiction. Thus it must be

$$\gamma = \lim_{n\to\infty} p(x_n, x_{n+1}) = \inf_{n\in\mathbb{N}} p(x_n, x_{n+1}) = 0.$$

Now, we show that $\{x_n\}$ is indeed a Cauchy sequence in X. Let $\alpha_n = \frac{\lambda^{n-1}}{1-\lambda} p(x_1, x_2)$, $n \in \mathbb{N}$. For $m, n \in \mathbb{N}$ with $m > n$, we obtain

$$p(x_n, x_m) \le \sum_{j=n}^{m-1} p(x_j, x_{j+1}) < \alpha_n.$$

Since $\lambda \in (0, 1)$, $\lim_{n\to\infty} \alpha_n = 0$ and hence

$$\lim_{n\to\infty} \sup\{p(x_n, x_m) : m > n\} = 0.$$

Applying Lemma 1.3, we show that $\{x_n\}$ is a Cauchy sequence in X. Hence $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$. Since $\inf_{n\in\mathbb{N}} d(x_n, x_{n+1}) \le d(x_m, x_{m+1})$ for all $m \in \mathbb{N}$ and $\lim_{m\to\infty} d(x_m, x_{m+1}) = 0$, one also obtain

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = \inf_{n\in\mathbb{N}} d(x_n, x_{n+1}) = 0.$$

Since $x_{n+1} \in Tx_n$ for each $n \in \mathbb{N}$,

$$\inf_{x\in X} p(x, Tx) \le p(x_n, Tx_n) \le p(x_n, x_{n+1}) \tag{3.5}$$

and

$$\inf_{x\in X} d(x, Tx) \le d(x_n, Tx_n) \le d(x_n, x_{n+1}) \tag{3.6}$$

for all $n \in \mathbb{N}$. Since $\lim_{n\to\infty} p(x_n, x_{n+1}) = \lim_{n\to\infty} d(x_n, x_{n+1}) = 0$, by (3.5) and (3.6), we get

$$\inf_{x\in X} p(x, Tx) = \inf_{x\in X} d(x, Tx) = 0.$$

The proof is completed. □

Theorem 3.2. Let (X, d) be a metric space and $T : X \to \mathcal{N}(X)$ be a multivalued map. Suppose that

$(\mathcal{R}_d)$ there exists a function $\phi : (0, \infty) \to [0, 1)$ satisfying Reich's condition and for each $x \in X$ with $x \notin Tx$, it holds

$$d(y, Ty) \le \varphi(d(x, y))d(x, y) \quad \text{for all } y \in Tx.$$

Then the following statements hold.

(a) There exists a Cauchy sequence $\{x_n\}_{n\in\mathbb{N}}$ in X such that
  (i) $x_{n+1} \in Tx_n$ for each $n \in \mathbb{N}$;
  (ii) $\inf_{n\in\mathbb{N}} d(x_n, x_{n+1}) = \lim_{n\to\infty} d(x_n, x_{n+1}) = 0$.

(b) T have the approximate fixed point property in X.

Remark 3.1. **[23, Proposition 3.1]** is a special case of Theorems 3.1 and 3.2.

Theorem 3.3. Let (X, d) be a metric space, p be a τ^0-function, $\mathcal{D}_p$ be a τ^0-metric on $\mathcal{CB}(X)$ induced by $pT : X \to \mathcal{CB}(X)$ be a multivalued map and $h : X \times X \to [0, \infty)$ be a function. Suppose that

$(\mathcal{L})$ there exists a function $\phi : (0, \infty) \to [0, 1)$ satisfying Reich's condition and

$$\mathcal{D}_p(Tx, Ty) \le \varphi(p(x, y))p(x, y) + h(x, y)p(y, Tx) \quad \text{for all } x, y \in X \text{ with } x \ne y. \tag{3.7}$$

Then the following statements hold.

(a) There exists a Cauchy sequence $\{x_n\}_{n\in\mathbb{N}}$ in X such that
 (i) $x_{n+1} \in Tx_n$ for each $n \in \mathbb{N}$;
 (ii) $\inf_{n\in\mathbb{N}} p(x_n, x_{n+1}) = \lim_{n\to\infty} p(x_n, x_{n+1}) = \lim_{n\to\infty} d(x_n, x_{n+1}) = \inf_{n\in\mathbb{N}} d(x_n, x_{n+1}) = 0$.

(b) T have the p-approximate fixed point property and approximate fixed point property in X.

Proof. Let $x \in X$ with $x \notin Tx$ and let $y \in Tx$ be given. By Lemma 1.2, $p(y, Tx) = 0$ and hence (3.7) implies (3.2). Therefore the conclusion follows from Theorem 3.1. □

Theorem 3.4. Let (X, d) be a metric space, $T : X \to \mathcal{CB}(X)$ be a multivalued map and $h : X \times X \to [0, \infty)$ be a function. Suppose that

($\mathcal{L}_d$) there exists a function $\phi : (0, \infty) \to [0, 1)$ satisfying Reich's condition and

$$\mathcal{H}(Tx, Ty) \le \varphi(d(x,y))d(x,y) + h(x,y)d(y,Tx) \quad \text{for all } x, y \in X \text{ with } x \neq y.$$

Then the following statements hold.

(a) There exists a Cauchy sequence $\{x_n\}_{n\in\mathbb{N}}$ in X such that
 (i) $x_{n+1} \in Tx_n$ for each $n \in \mathbb{N}$;
 (ii) $\lim_{n\to\infty} d(x_n, x_{n+1}) = \inf_{n\in\mathbb{N}} d(x_n, x_{n+1}) = 0$.

(b) T have the approximate fixed point property in X.

Theorem 3.5. Let (X, d) be a metric space and $T : X \to \mathcal{CB}(X)$ be a multivalued map. Assume that one of the following conditions holds.

(1) T is a Berinde-Berinde's type contraction;
(2) T is a multivalued (θ, L)-almost contraction;
(3) T is a Mizoguchi-Takahashi's type contraction;
(4) T is a Nadler's type contraction.

Then the following statements hold.

(a) There exists a Cauchy sequence $\{x_n\}_{n\in\mathbb{N}}$ in X such that
 (i) $x_{n+1} \in Tx_n$ for each $n \in \mathbb{N}$;
 (ii) $\lim_{n\to\infty} d(x_n, x_{n+1}) = \inf_{n\in\mathbb{N}} d(x_n, x_{n+1}) = 0$.

(b) T have the approximate fixed point property in X.

Let Ω denote the class of functions $\mu : [0, \infty) \to [0, \infty)$ satisfying

- $\mu(0) = 0$;
- $0 < \mu(t) \le t$ for all $t > 0$;
- μ is l.s.c. from the right;
- $\limsup_{s\to 0^+} \dfrac{s}{\mu(s)} < \infty$.

Examples of such functions are $\mu(t) = \frac{t}{t+1}$, $\mu(t) = \ln(1+t)$ and $\mu(t) = ct$, where $c \in (0, 1)$, for all $t \geq 0$.

Theorem 3.6. Let (X, d) be a metric space, p be a τ^0-function and $T : X \to \mathcal{C}(X)$ be a multivalued map. Suppose that

(Δ) there exists $\mu \in \Omega$ such that for each $x \in X$ with $x \notin Tx$, it holds

$$p(y, Ty) \leq p(x, y) - \mu(p(x, y)) \quad \text{for all } y \in Tx. \tag{3.8}$$

Then the following statements hold.

(a) There exists a function α from $[0, \infty)$ into $[0, 1)$ such that α is a $\mathcal{R}$-function and $p(y, Ty) \leq \alpha(p(x, y))p(x, y)$ for all $y \in Tx$.

(b) There exists a Cauchy sequence $\{x_n\}_{n\in\mathbb{N}}$ in X such that

(i) $x_{n+1} \in Tx_n$ for each $n \in \mathbb{N}$;

(ii) $\inf_{n\in\mathbb{N}} p(x_n, x_{n+1}) = \lim_{n\to\infty} p(x_n, x_{n+1}) = \lim_{n\to\infty} d(x_n, x_{n+1}) = \inf_{n\in\mathbb{N}} d(x_n, x_{n+1}) = 0$.

(c) T have the p-approximate fixed point property and approximate fixed point property in X.

Proof. Set

$$\alpha(t) = \begin{cases} 1 - \frac{\mu(t)}{t}, & t > 0, \\ 0, & t = 0. \end{cases}$$

Since $0 < \mu(t) \leq t$ for all $t > 0$, we have $\alpha(t) \in [0, 1)$ for all $t \in [0, \infty)$. Hence α is a function from $[0, \infty)$ into $[0, 1)$. Let $x \in X$ with $x \notin Tx$ be given. Since p is a τ^0-function, $p(x, y) > 0$ for all $y \in Tx$. Hence (3.8) implies

$$p(y, Ty) \leq \varphi(p(x, y))p(x, y) \quad \text{for all } y \in Tx.$$

We claim that α is a $\mathcal{R}$-function. Indeed, by (Δ), the function $t \mapsto \frac{\mu(t)}{t}$ is l.s.c. from the right and hence

$$\limsup_{s\to t^+} \alpha(s) = 1 - \liminf_{s\to t^+} \frac{\mu(s)}{s} < 1 - \frac{\mu(t)}{t} < 1 \quad \text{for all } t > 0.$$

On the other hand, since $\frac{t}{\mu(t)} \geq 1$ for all $t > 0$ and $\limsup_{s\to 0^+} \frac{s}{\mu(s)} < \infty$, it follows that

$$\limsup_{s\to 0^+} \alpha(s) = 1 - \liminf_{s\to 0^+} \frac{\mu(s)}{s} = 1 - \frac{1}{\limsup_{s\to 0^+} \frac{s}{\mu(s)}} < 1.$$

So we prove $\limsup_{s\to t^+} \alpha(s) < 1$ for all $t \in [0, \infty)$ which say that $\alpha : [0, \infty) \to [0, 1)$ is a $\mathcal{R}$-function and (a) is true. The conclusions (b) and (c) follows from Theorem 3.1. □

Theorem 3.7. Let (X, d) be a metric space and $T : X \to \mathcal{C}(X)$ be a multivalued map. Suppose that

(Δ_d) there exists $\mu \in \Omega$ such that for each $x \in X$ with $x \notin Tx$, it holds

$$d(y, Ty) \leq d(x, y) - \mu(d(x, y)) \quad \text{for all } y \in Tx.$$

Then the following statements hold.

(a) There exists a function α from $[0, \infty)$ into $[0, 1)$ such that α is a $\mathcal{R}$-function and $d(y, Ty) \leq \alpha(d(x, y))d(x, y)$ for all $y \in Tx$.
(b) There exists a Cauchy sequence $\{x_n\}_{n\in\mathbb{N}}$ in X such that
  (i) $x_{n+1} \in Tx_n$ for each $n \in \mathbb{N}$;
  (ii) $\lim_{n\to\infty} d(x_n, x_{n+1}) = \inf_{n\in\mathbb{N}} d(x_n, x_{n+1}) = 0$.
(c) T have the approximate fixed point property in X.

4 Some applications in fixed point theory

The following existence theorem is a τ-function variant of generalized Ekeland's variational principle.

Lemma 4.1. Let (X, d) be a complete metric space, $f : X \to (-\infty, \infty]$ be a proper l.s.c. and bounded below function. Let p be a τ-function and $\varepsilon > 0$. Suppose that there exists $u \in X$ such that $p(u, \cdot)$ is l.s.c, $f(u) < \infty$ and $p(u, u) = 0$. Then there exists $v \in X$ such that

(a) $\varepsilon p(u, v) \leq f(u) - f(v)$;
(b) $\varepsilon p(v, x) > f(v) - f(x)$ for all $x \in X$ with $x \neq v$.

Proof. Let

$$Y = \{x \in X : \varepsilon p(u, x) \leq f(u) - f(x)\}.$$

Clearly, $u \in Y$. By the completeness of X and the lower semicontinuity of f and $p(u, \cdot)$, we know that (Y, d) is a nonempty complete metric space. Applying a generalization version of Ekeland's variational principle due to Lin and Du (see, for instance, [26,28]), there exists $v \in Y$ such that $\varepsilon p(v, x) > f(v) - f(x)$ for all $x \in Y$ with $x \neq v$. Hence (a) holds from $v \in Y$. For any $x \in X \setminus Y$, since

$$\begin{aligned} \varepsilon[p(u, v) + p(v, x)] &\geq \varepsilon p(u, x) \\ &> f(u) - f(x) \\ &\geq \varepsilon p(u, v) + f(v) - f(x), \end{aligned}$$

it follows that $\varepsilon p(v, x) > f(v) - f(x)$ for all $x \in X \setminus Y$. Therefore $\varepsilon p(v, x) > \phi(f(v))(f(v) - f(x))$ for all $x \in X$ with $x \neq v$. The proof is completed. □

Theorem 4.1. Let L be a nonempty closed subset of a complete metric space (X, d), p be a τ^0-function and $T : L \to \mathcal{C}(X)$ be a (p, c, η, φ)-**DHC**. Suppose that

(i) there exist $u \in L$ and $\delta > 0$ such that $p(u, \cdot)$ is l.s.c,

$$p(u, Tu) \leq c\delta, \tag{4.1}$$

and

$$\sup_{t\in(0,\delta)} (\phi(t) - \eta(t)) < 0, \tag{4.2}$$

(ii) the function $f : L \to [0, \infty)$ defined by $f(x) = p(x, Tx)$ is l.s.c.

Then $\mathcal{F}(T) \cap L \neq \emptyset$.

Proof. Since L is a nonempty closed subset in X, (L, d) is also a complete metric space. By (4.1), $f(u) \leq c\delta < \infty$. From (4.2), there exists $\gamma > 0$ such that

$$\sup_{t\in(0,\delta)} (\phi(t) - \eta(t)) \leq -\gamma. \tag{4.3}$$

Applying Lemma 4.1 for u and $\frac{\gamma}{2}$, there exists $v \in L$, such that

$$\frac{\gamma}{2}p(u, v) \leq f(u) - f(v); \tag{4.4}$$

$$\frac{\gamma}{2}p(v, x) > f(v) - f(x) \quad \text{for all } x \in L \text{ with } x \neq v. \tag{4.5}$$

So $f(v) \leq f(u)$ from (4.4). We claim that $v \in Tv$, or equivalent, $p(v, Tv) = 0$. On the contrary, suppose that $f(v) = p(v, Tv) > 0$. Since T is a (p, c, η, φ)-**DHC**, there exists $y_v \in L \setminus \{v\}$ and $z_v \in Tv$ such that

$$p(z_v, Ty_v) \leq \phi(p(v, y_v))p(v, y_v) \tag{4.6}$$

and

$$\eta(p(v, y_v))p(v, y_v) + p(y_v, z_v) \leq f(v). \tag{4.7}$$

Since $y_v \neq v$, $c < \eta(p(v, y_v))$ and $f(v) \leq f(u)$, by (4.1) and (4.7), we have

$$0 < p(v, y_v) < c^{-1}f(v) \leq c^{-1}f(u) \leq \delta. \tag{4.8}$$

Combining (4.3) and (4.8), we get

$$\phi(p(v, y_v)) - \eta(p(v, y_v)) \leq -\gamma. \tag{4.9}$$

By Lemma 1.1, (4.6), (4.7), and (4.9), one obtains

$$\begin{aligned} f(y_v) = p(y_v, Ty_v) &\leq p(y_v, z_v) + p(z_v, Ty_v) \\ &\leq f(v) + [\phi(p(v, y_v)) - \eta(p(v, y_v))]p(v, y_v) \\ &\leq f(v) - \gamma p(v, y_v). \end{aligned}$$

On the other hand, since $y_v \in L \setminus \{v\}$, it follows from (4.5) and the last inequality that

$$\begin{aligned} f(v) &< f(y_v) + \frac{\gamma}{2}p(v, y_v) \\ &\leq f(v) + \left(\frac{\gamma}{2} - \gamma\right) p(v, y_v) \\ &= f(v) - \frac{\gamma}{2}p(v, y_v) \\ &< f(v), \end{aligned}$$

which yields a contradiction. Hence it must be $f(v) = p(v, Tv) = 0$. Since Tv is closed, by Lemma 1.2, we get $v \in Tv$ which means that $v \in \mathcal{F}(T) \cap L$. The proof is completed. □

Remark 4.1.

(a) Let K be a nonempty subset of a metric space (X, d) and $T : X \to \mathcal{C}(X)$ be u.s.c. Then the function $f : K \to [0, \infty)$ defined by $f(x) = d(x, Tx)$ is l.s.c. For more detail, one can see, e.g., [31, Lemma 3.1] and [32, Lemma 2].
(b) [23, Theorem 2.1] is a special case of Theorem 4.1.

Theorem 4.2. Let L be a nonempty closed subset of a complete metric space (X, d) and $T : L \to \mathcal{C}(X)$ be a (c, η, φ)-**DHC**. Suppose that

(i) there exist $u \in L$ and $\delta > 0$ such that

$$d(u, Tu) \geq c\delta,$$

and

$$\sup_{t\in(0,\delta)} (\phi(t) - \eta(t)) < 0,$$

(ii) the function $f : L \to [0, \infty)$ defined by $f(x) = d(x, Tx)$ is l.s.c.

Then $\mathcal{F}(T) \cap L \neq \emptyset$.

Theorem 4.3. Let L be a nonempty closed subset of a complete metric space (X, d) and p be a τ^0-function. Let $T : L \to \mathcal{C}(X)$ be a (p, c, η, φ)-**DHC** satisfying

$$\limsup_{s\to 0^+} \phi(s) < \liminf_{s\to 0^+} \eta(s), \tag{4.10}$$

and it has the p-approximate fixed point property in L. Suppose that there exists $u \in X$ such that $p(u, \cdot)$ is l.s.c. and the function $f : L \to [0, \infty)$ defined by $f(x) = p(x, Tx)$ is l.s.c, then $\mathcal{F}(T) \cap L \neq \emptyset$.

Proof. First, we note that (4.10) implies that the existences of $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$\sup_{t\in(0,\delta_1)} \phi(t) < \inf_{t\in(0,\delta_2)} \eta(t). \tag{4.11}$$

Let $\delta = \min\{\delta_1, \delta_2\} > 0$. Thus (4.11) implies

$$\sup_{t\in(0,\delta)} (\phi(t) - \eta(t)) \leq \sup_{t\in(0,\delta)} \phi(t) - \inf_{t\in(0,\delta)} \eta(t) \leq \sup_{t\in(0,\delta_1)} \phi(t) - \inf_{t\in(0,\delta_2)} \eta(t) < 0.$$

Since T has the p-approximate fixed point property in L, we have $\inf_{x\in L} p(x, Tx) = 0 < c\delta$ and hence there exists $u \in L$ such that $p(u, Tu) < c\delta$. So all the hypotheses of Theorem 4.1 are fulfilled. It is therefore possible to apply Theorem 4.1 to get the thesis. □

Theorem 4.4. Let L be a nonempty closed subset of a complete metric space (X, d). Let $T : L \to \mathcal{C}(X)$ be a (c, η, φ)-**DHC** satisfying

$$\limsup_{s\to 0^+} \phi(s) < \liminf_{s\to 0^+} \eta(s),$$

and it has the approximate fixed point property in L. Suppose that the function f : $L \to [0, \infty)$ defined by $f(x) = d(x, Tx)$ is l.s.c, then $\mathcal{F}(T) \cap L \neq \emptyset$.

Theorem 4.5. Let (X, d) be a complete metric space and p be a τ^0-function and $T : X \to \mathcal{C}(X)$ be a multivalued map. Suppose that

$(\mathcal{V})$ there exists a $\mathcal{R}$-function $\alpha : [0, \infty) \to [0, 1)$ such that for each $x \in X$ with $x \notin Tx$, it holds

$$p(y, Ty) \leq \alpha(p(x, y))p(x, y) \text{ for all } y \in Tx.$$

If there exists $u \in X$ such that $p(u, \cdot)$ is l.s.c. and the function $f : X \to [0, \infty)$ defined by $f(x) = p(x, Tx)$ is l.s.c, then $\mathcal{F}(T) \neq \emptyset$.

Proof. First, we observe that the condition $(\mathcal{V})$ implies the condition $(\mathcal{P})$ as in Theorem 2.1. So we can apply Theorem 2.1 to know that there exist $c \in (0, 1)$ and functions $\eta : [0, \infty) \to (c, 1]$ and $\varphi : [0, \infty) \to [0, 1)$ such that

(a) $\limsup_{s \to 0^+} \phi(s) < \liminf_{s \to 0^+} \eta(s)$;

(b) T is a (p, c, η, φ)-**DHC**.

On the other hand, the condition $(\mathcal{V})$ also implies the condition $(\mathcal{R})$ as in Theorem 3.1. Hence T have the p-approximate fixed point property by using Theorem 3.1. Therefore the thesis follows from Theorem 4.3. □

Theorem 4.6. Let (X, d) be a complete metric space and $T : X \to \mathcal{C}(X)$ be a multivalued map. Suppose that

$(\mathcal{V}_d)$ there exists a $\mathcal{R}$-function $\alpha : [0, \infty) \to [0, 1)$ such that for each $x \in X$ with $x \notin Tx$, it holds

$$d(y, Ty) \leq \alpha(d(x, y))d(x, y) \text{ for all } y \in Tx.$$

If the function $f : X \to [0, \infty)$ defined by $f(x) = d(x, Tx)$ is l.s.c, then $\mathcal{F}(T) \neq \emptyset$.

Theorem 4.7. Let (X, d) be a complete metric space, p be a τ^0-function, $\mathcal{D}_p$ be a τ^0-metric on $\mathcal{CB}(X)$ induced by $p, T : X \to \mathcal{CB}(X)$ be a multivalued map and $h : X \times X \to [0, \infty)$ be a function. Suppose that

$(\mathcal{W})$ there exists a $\mathcal{R}$-function $\alpha : [0, \infty) \to [0, 1)$ such that

$$\mathcal{D}_p(Tx, Ty) \leq \alpha(p(x, y))p(x, y) + h(x, y)p(y, Tx) \quad \text{for all } x, y \in X.$$

If there exists $u \in X$ such that $p(u, \cdot)$ is l.s.c. and the function $f : X \to [0, \infty)$ defined by $f(x) = p(x, Tx)$ is l.s.c, then $\mathcal{F}(T) \neq \emptyset$.

The following result is a generalization of Berinde-Berinde's fixed point theorem. It is worth observing that the following generalized Berinde-Berinde's fixed point theorem does not require the lower semicontinuity assumption on the function $f(x) = d(x, Tx)$.

Theorem 4.8. Let (X, d) be a complete metric space, $T : X \to \mathcal{CB}(X)$ be a multivalued map and $g : X \to [0, \infty)$ be a function. Suppose that there exists a $\mathcal{R}$-function α : $[0, \infty) \to [0, 1)$ such that

$$\mathcal{H}(Tx, Ty) \leq \alpha(d(x, y))d(x, y) + g(y)d(y, Tx) \quad \text{for all } x, y \in X. \tag{4.12}$$

Then $\mathcal{F}(T) \neq \emptyset$.

Proof. Observe that the condition (4.12) implies that for each $x \in X$ with $x \notin Tx$, it holds

$$d(y, Ty) \leq \varphi(d(x, y))d(x, y) \quad \text{for all } y \in Tx.$$

It is therefore possible to apply Theorem 3.2 to obtain a Cauchy sequence $\{x_n\}_{n\in\mathbb{N}}$ in X satisfying

- $x_{n+1} \in Tx_n$, $n \in \mathbb{N}$,
- $\lim_{n\to\infty} d(x_n, x_{n+1}) = \inf_{n\in\mathbb{N}} d(x_n, x_{n+1}) = 0$.

By the completeness of X, there exists $v \in X$ such that $x_n \to v$ as $n \to \infty$. It follows from (4.12) again that

$$\begin{aligned}\lim_{n\to\infty} d(x_{n+1}, Tv) &\leq \lim_{n\to\infty} \mathcal{H}(Tx_n, Tv) \\ &\leq \lim_{n\to\infty} \{\varphi(d(x_n, v))d(x_n, v) + h(v)d(v, x_{n+1})\} = 0,\end{aligned}$$

which implies $\lim_{n\to\infty} d(x_n, Tv) = 0$. By the continuity of $d(\cdot, Tv)$ and $x_n \to v$ as $n \to \infty$, $d(v, Tv) = 0$. By the closedness of Tv, we get $v \in Tv$ or $v \in \mathcal{F}(T)$. □

Remark 4.2.

(a) Theorem 4.8 generalizes [7, Theorem 2.6], Berinde-Berinde's fixed point theorem, Mizoguchi-Takahashi's fixed point theorem and references therein.

(b) In [7, Theorem 2.6], the authors shown that a generalized multivalued almost contraction T in a metric space (X, d) have $\mathcal{F}(T) \neq \emptyset$ provided either (X, d) is compact and the function $f(x) = d(x, Tx)$ is l.s.c. or T is closed and compact. But reviewing Theorem 4.8, we know that the conditions in [7, Theorem 2.6] are redundant.

Corollary 4.1. (M. Berinde and V. Berinde [5]). Let (X, d) be a complete metric space, $T : X \to \mathcal{CB}(X)$ be a multivalued map and $L \geq 0$. Suppose that there exists a $\mathcal{R}$-function $\alpha : [0, \infty) \to [0, 1)$ such that

$$\mathcal{H}(Tx, Ty) \geq \alpha(d(x, y))d(x, y) + Ld(y, Tx) \quad \text{for all } x, y \in X.$$

Then $\mathcal{F}(T) \neq \emptyset$

Corollary 4.2 [3]. Let (X, d) be a complete metric space and $T : X \to \mathcal{CB}(X)$ be a multivalued map. Suppose that there exists a $\mathcal{R}$-function $\alpha : [0, \infty) \to [0, 1)$ such that

$$\mathcal{H}(Tx, Ty) \leq \alpha(d(x, y))d(x, y) \quad \text{for all } x, y \in X.$$

Then $\mathcal{F}(T) \neq \emptyset$.

Theorem 4.9. Let (X, d) be a complete metric space, p be a τ^0-function and $T : X \to \mathcal{C}(X)$ be a multivalued map. Suppose that

(Δ) there exists $\mu \in \Omega$ such that for each $x \in X$ with $x \notin Tx$, it holds

$$p(y, Ty) \leq p(x, y) - \mu(p(x, y)) \quad \text{for all } y \in Tx.$$

and further assume that there exists $u \in X$ such that $p(u, \cdot)$ is l.s.c. and the function $f : X \to [0, \infty)$ defined by $f(x) = p(x, Tx)$ is l.s.c, then $\mathcal{F}(T) \neq \emptyset$.

Proof. The conclusion follows from Theorems 3.6 and 4.5. □

Theorem 4.10. Let (X, d) be a complete metric space and $T : X \to \mathcal{C}(X)$ be a multi-valued map. Suppose that

(Δ_d) there exists $\mu \in \Omega$ such that for each $x \in X$ with $x \notin Tx$, it holds

$$d(y, Ty) \leq d(x, y) - \mu(d(x, y)) \quad \text{for all } y \in Tx.$$

and further assume that the function $f : X \to [0, \infty)$ defined by $f(x) = d(x, Tx)$ is l.s.c, then $\mathcal{F}(T) \neq \emptyset$.

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