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Fixed point results for multivalued contractive maps

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Abstract

Using the concept of *u*-distance, we prove a fixed point theorem for multivalued contractive maps. We also establish a multivalued version of the Caristi's fixed point theorem and common fixed point result. Consequently, several known fixed point results are either improved or generalized including the corresponding fixed point results of Caristi, Mizoguchi-Takahashi, Kada et al., Suzuki-Takahashi, Suzuki, and Ume.

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1 Introduction

Let X be a complete metric space with metric d. We denote the collection of nonempty subsets of X, nonempty closed subsets of X and nonempty closed bounded subsets of X by 2^X, $Cl(X)$, $CB(X)$, respectively Let H be the Hausdorff metric with respect to d, that is,

$$H(A, B) = \max\left\{\sup_{x\in A} d(x, B), \sup_{y\in B} d(y, A)\right\},$$

for every $A, B \in CB(X)$, where $d(x, B) = \inf_{y\in B} d(x, y)$.

A point $x \in X$ is called a fixed point of $T : X \to 2^X$ if $x \in T(x)$. A point $x \in X$ is called a common fixed point of $f : X \to X$ and T if $f(x) = x \in T(x)$.

A sequence $\{x_n\}$ in X is called an *orbit* of T at $x_0 \in X$ if $x_n \in T(x_{n-1})$ for all $n \geq 1$. A map $\varphi : X \to \mathbb{R}$ is called *lower semicontinuous* if for any sequence $\{x_n\} \subset X$ with $x_n \to x \in X$ imply that $\phi(x) \leq \liminf_{n\to\infty} \phi(x_n)$.

The well known Banach contraction principle, which asserts that "each single-valued contraction selfmap on a complete metric space has a unique fixed point" has been generalized in many different directions. Among these generalizations, the following Caristi's fixed point theorem [1] may be the most valuable one and has extensive applications in the field of mathematics.

Theorem 1.1. *Let X be a complete metric space and let $\psi : X \to (-\infty, \infty]$ be a proper, lower semicontinuous bounded below function. Let f be a single-valued selfmap of X. If for each $x \in X$*

$$d(x, f(x)) \leq \psi(x) - \psi(f(x)),$$

then f has a fixed point.

Investigations on the existence of fixed points for multivalued maps in the setting of metric spaces was initiated by Nadler [2]. Using the concept of Hausdorff metric, he generalized Banach contraction principle which states that each multivalued contraction map $T : X \to CB(X)$ has fixed point provided X is complete. Since then, many authors have used the Hausdorff metric to obtain fixed point results for multivalued maps. For example, see [3-6], and references therein.

Kada et al. [7] introduced the notion of w-distance on a metric space as follows:

A function $\omega : X \times X \to \mathbb{R}_+$ is called w-distance on X if it satisfies the following for $x, y, z \in X$:

(w_1) $\omega(x, z) \leq \omega(x, y) + \omega(y, z)$;

(w_2) the map $\omega(x,.) : X \to \mathbb{R}_+$ is lower semicontinuous; i.e., for $\{y_n\}$ in X with $y_n \to y \in X$, $\omega(x, y) \leq \liminf_{n\to\infty} \omega(x, y_n)$;

(w_3) for any $\varepsilon > 0$, there exists $\delta > 0$ such that $\omega(z, x) \leq \delta$ and $\omega(z, y) \leq \delta$ imply $d(x, y) \leq \varepsilon$.

Note that, in general for $x, y \in X$, $\omega(x, y) \neq \omega(y, x)$, and $\omega(x, y) = 0 \Leftrightarrow x = y$ does not necessarily hold. Clearly, the metric d is a w-distance on X. Examples and properties of a w-distance can be found in [7,8]. For single valued maps, Kada et al. [7] improved several classical results including the Caristi's fixed point theorem by replacing the involved metric with a generalized distance. Using this generalized distance, Suzuki and Takahashi [9] have introduced notions of single-valued and multivalued weakly contractive maps and proved fixed point results for such maps. Consequently, they generalized the Banach contraction principle and Nadler's fixed point result. Recent fixed point results concerning w-distance can be found [4,8,10-13].

Recently, Susuki [14] generalized the concept of w-distance by introducing the following notion of τ-distance on metric space (X, d).

A function $p : X \times X \to \mathbb{R}_+$ is a τ-distance on X if it satisfies the following conditions for any $x, y, z \in X$:

(τ_1) $p(x, z) \leq p(x, y) + p(y, z)$;

(τ_2) $\eta(x, 0) = 0$ and $\eta(x, t) \geq t$ for all $x \in X$ and $t \geq 0$, and η is concave and continuous in its second variable;

(τ_3) $\lim_n x_n = x$ and $\lim_n \sup\{\eta(z_n, p(z_n, x_m)) : m \geq n\} = 0$ imply $p(u, x) \leq \lim_n \inf p(u, x_n)$ for all $u \in X$;

(τ_4) $\lim_n \sup\{p(x_n, y_m) : m \geq n\} = 0$ and $\lim_n \eta(x_n, t_n) = 0$ imply $\lim_n \eta(y_n, t_n) = 0$;

(τ_5) $\lim_n \eta(z_n, p(z_n, x_n)) = 0$ and $\lim_n \eta(z_n, p(z_n, y_n)) = 0$ imply $\lim_n d(x_n, y_n) = 0$.

Examples and properties of τ-distance are given in [14]. In [14], Suzuki improved several classical results including the Caristi's fixed point theorem for single-valued maps with respect to τ-distance.

In the literature, several other kinds of distances and various versions of known results are appeared. For example, see [15-19], and references therein. Most recently, Ume [20] generalized the notion of τ-distance by introducing u-distance as follows:

A function $p : X \times X \to \mathbb{R}_+$ is called u-distance on X if there exists a function $\theta : X \times X \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ such that the following hold for $x, y, z \in X$:

(u_1) $p(x, z) \leq p(x, y) + p(y, z)$.

(u_2) $\theta(x, y, 0, 0) = 0$ and $\theta(x, y, s, t) \geq \min\{s, t\}$ for each $s, t \in \mathbb{R}_+$, and for every $\varepsilon > 0$, there exists $\delta > 0$ such that $| s - s_0 | < \delta$, $| t - t_0 | < \delta$, $s, s_0, t, t_0 \in R_+$ and $y \in X$ imply

$$\left|\theta\left(x, y, s, t\right) - \theta\left(x, y, s_0, t_0\right)\right| < \varepsilon. \tag{1}$$

(u_3) $\lim_{n\to\infty} x_n = x$

$$\lim_{n\to\infty} \sup\left\{\theta\left(w_n, z_n, p\left(w_n, x_m\right), p\left(z_n, x_m\right)\right) : m \geq n\right\} = 0 \tag{2}$$

imply

$$p\left(y, x\right), \liminf_{n\to\infty} p\left(y, x_n\right) \tag{3}$$

(u_4)

$$\begin{aligned}
&\lim_{n\to\infty} \sup\left\{p\left(x_n, w_m\right)\right) : m \geq n\right\} = 0,\\
&\lim_{n\to\infty} \sup\left\{p\left(y_n, z_{m'}\right)\right) : m \geq n\right\} = 0,\\
&\lim_{n\to\infty} \theta\left(x_n, w_n, s_n, t_n\right) = 0,\\
&\lim_{n\to\infty} \theta\left(y_n, z_n, s_n, t_n\right) = 0
\end{aligned} \tag{4}$$

imply

$$\lim_{n\to\infty} \theta\left(w_n, z_n, s_n, t_n\right) = 0 \tag{5}$$

or

$$\begin{aligned}
&\lim_{n\to\infty} \sup\left\{p\left(w_n, x_m\right)\right) : m \geq n\right\} = 0,\\
&\lim_{n\to\infty} \sup\left\{p\left(z_m, y_{n'}\right)\right) : m \geq n\right\} = 0,\\
&\lim_{n\to\infty} \theta\left(x_n, w_n, s_n, t_n\right) = 0,\\
&\lim_{n\to\infty} \theta\left(y_n, z_n, s_n, t_n\right) = 0
\end{aligned} \tag{6}$$

imply

$$\lim_{n\to\infty} \theta\left(w_n, z_n, s_n, t_n\right) = 0; \tag{7}$$

(u_5)

$$\begin{aligned}
&\lim_{n\to\infty} \theta\left(w_n, z_n, p\left(w_n, x_n\right), p\left(z_n, x_n\right)\right) = 0,\\
&\lim_{n\to\infty} \theta\left(w_n, z_n, p\left(w_n, y_n\right), p\left(z_n, y_n\right)\right) = 0
\end{aligned} \tag{8}$$

imply

$$\lim_{n\to\infty} d\left(x_n, y_n\right) = 0 \tag{9}$$

or

$$\begin{aligned}
&\lim_{n\to\infty} \theta\left(a_n, b_n, p\left(x_n, a_n\right), p\left(x_n, b_n\right)\right) = 0,\\
&\lim_{n\to\infty} \theta\left(a_n, b_n, p\left(y_n, a_n\right), p\left(y_n, b_n\right)\right) = 0
\end{aligned} \tag{10}$$

imply

$$\lim_{n\to\infty} d(x_n,\ y_n) = 0 \tag{11}$$

Remark 1.1. [20] (a) Suppose that θ from $X \times X \times \mathbb{R}_+ \times \mathbb{R}_+$ into $\mathbb{R}_+$ is a mapping satisfying (u2) ~ (u5). Then there exists a mapping η from $X \times X \times \mathbb{R}_+ \times \mathbb{R}_+$ into $\mathbb{R}_+$ such that η is nondecreasing in its third and fourth variable, respectively satisfying (u2)η ~ (u5)η, where (u2)η ~ (u5)η stand for substituting η for θ in (u2) ~ (u5), respectively

(b) In the light of (a), we may assume that θ is nondecreasing in its third and fourth variables, respectively, for a function θ from $X \times X \times \mathbb{R}_+ \times \mathbb{R}_+$ into $\mathbb{R}_+$ satisfying (u2) ~ (u5).

(c) Each τ-distance p on a metric space (X, d) is also a u-distance on X.

We present some examples of u-distance which are not τ-distance. (For the detail, see [20]).

Example 1.1. Let $X = \mathbb{R}_+$ with the usual metric. Define p: $X \times X \to \mathbb{R}_+$ by $p\left(x, y\right) = \left(\frac{1}{4}\right) x^2$. Then p is a u-distance on X but not a τ-distance on X.

Example 1.2. Let X be a normed space with norm $||.||$. Then a function p: $X \times X \to \mathbb{R}_+$ defined by $p(x, y) = ||x||$ for every $x, y \in X$ is a u-distance on X but not a τ-distance.

It follows from the above examples and Remark 1.1(c) that u-distance is a proper extension of τ-distance. Other useful examples are also given in [20]).

Let X be a metric space with a metric d and let p be a u-distance on X. Then a sequence $\{x_n\}$ in X is called p-Cauchy [20] if there exists a function θ from $X \times X \times \mathbb{R}_+ \times \mathbb{R}_+$ into $\mathbb{R}_+$ satisfying (u2) ~ (u5) and a sequence $\{z_n\}$ of X such that

$$\lim_{n\to\infty} \sup\left\{\theta\left(z_n, z_n, p\left(z_n, x_m\right), p\left(z_n, x_m\right)\right) : m \geq n\right\} = 0, \tag{12}$$

or

$$\lim_{n\to\infty} \sup\left\{\theta\left(z_n, z_n, p\left(x_m, z_n\right), p\left(x_m, z_n\right)\right) : m \geq n\right\} = 0. \tag{13}$$

The following lemmas concerning u-distance are crucial for the proofs of our results.

Lemma 1.1. Let X be a metric space with a metric d and let p be a u-distance on X. If $\{x_n\}$ is a p-Cauchy sequence, then $\{x_n\}$ is a Cauchy sequence.

Lemma 1.2. Let X be a metric space with a metric d and let p be a u-distance on X.

(1) If sequences $\{x_n\}$ and $\{y_n\}$ of X satisfy $\lim_{n\to\infty} p(z, x_n) = 0$, and $\lim_{n\to\infty} p(z, y_n) = 0$ for some $z \in X$, then $\lim_{n\to\infty} d(x_n, y_n) = 0$.

(2) If $p(z, x) = 0$ and $p(z, y) = 0$, then $x = y$.

(3) Suppose that sequences $\{x_n\}$ and $\{y_n\}$ of X satisfy $\lim_{n\to\infty} p(x_n, z) = 0$, and $\lim_{n\to\infty} p(y_n, z) = 0$ for some $z \in X$. Then $\lim_{n\to\infty} d(x_n, y_n) = 0$.

(4) If $p(x, z) = 0$ and $p(y, z) = 0$, then $x = y$.

Lemma 1.3. Let X be a metric space with a metric d and let p be a u-distance on X. Suppose that a sequence $\{x_n\}$ of X satisfies

$$\lim_{n\to\infty} \sup\left\{p\left(x_n, x_m,\right)\right) : m > n\right\} = 0, \tag{14}$$

or

$$\lim_{n\to\infty} \sup\left\{p\left(x_m, x_n\right)\right) : m > n\right\} = 0. \tag{15}$$

Then $\{x_n\}$ is a p-Cauchy sequence.

Using u-distance, Ume [20] generalized Caristi's fixed point theorem as follows:

Theorem 1.2. *Let X be a complete metric space with metric d, let $\varphi : X \to (-\infty, \infty]$ be a proper lower semicontinuous function which is bounded from below. Let p be a u-distance on X. Suppose that f is a single-valued selfmap of X such that*

$$\phi\left(fx\right) + p\left(x, fx\right) \leq \phi\left(x\right),$$

for all $x \in X$. Then there exists $x_0 \in X$ such that $fx_0 = x_0$, and $p(x_0, x_0) = 0$.

We say a multivalued map $T : X \to 2^X$ is contractive with respect to u-distance p on X (in short, p-contractive) if there exist a u-distance p on X and a constant $r \in (0, 1)$ such that for any $x, y \in X$ and $u \in T(x)$, there is $v \in T(y)$ satisfying

$$p\left(u, v\right) \leq rp\left(x, y\right).$$

In particular, a single-valued map $g : X \to X$ is p-contractive if there exist a u-distance p on X and a constant $r \in (0, 1)$ such that for each $x, y \in X$

$$p\left(g\left(x\right), g\left(y\right)\right) \leq rp\left(x, y\right).$$

In this article, using the concept of u-distance, first we prove a useful lemma for multivalued mappings in metric spaces. Then using our lemma we prove a fixed point result for closed valued p-contraction mappings. Also, we prove multivalued version of the Caristi's fixed point theorem and then applying this result we establish common fixed point theorem. Consequently, several known fixed point results are either improved or generalized.

2 The results

Using Lemma 1.3, we prove the following key lemma in the setting of metric spaces.

Lemma 2.1. *Let X be a metric space with metric d. Let $T : X \to Cl(X)$ be a p-contractive map. Then, there exists an orbit $\{u_n\}$ of T at u_0 such that $\{u_n\}$ is a Cauchy sequence.*

Proof. Let u_0 be an arbitrary but fixed element of X and let $u_1 \in Tu_0$ be fixed. Since T is p-contractive, there exists $u_2 \in Tu_1$ such that

$$p\left(u_1, u_2\right) \leq rp\left(u_0, u_1\right),$$

where $r \in (0, 1)$. Continuing this process, we get a sequence $\{u_n\}$ in X such that $u_{n+1} \in Tu_n$ and

$$p\left(u_n, u_{n+1}\right) \leq rp\left(u_{n-1}, u_n\right),$$

for all $n \in \mathbb{N}$. Thus for any $n \in \mathbb{N}$, we have

$$p\left(u_n, u_{n+1}\right) \leq rp\left(u_{n-1}, u_n\right) \leq \cdots \leq r^n p\left(u_0, u_1\right)$$

Now, for any $n, m \in \mathbb{N}$ with $m > n$,

$$\begin{aligned} p\left(u_n, u_m\right) &\leq p\left(u_n, u_{n+1}\right) + p\left(u_{n+1}, u_{n+2}\right) + \cdots + p\left(u_{m-1}, u_m\right) \\ &\leq r^n \left[1 + r + r^2 + \cdots + r^{m-n-1}\right] p\left(u_0, u_1\right) \\ &\leq \frac{r^n}{1-r} p\left(u_0, u_1\right), \end{aligned}$$

and hence

$$\lim_{n\to\infty} \sup \left\{p\left(u_n, u_m\right) : m > n\right\} = 0.$$

By Lemma 1.3, $\{u_n\}$ is a p-Cauchy sequence and hence by Lemma 1.1, $\{u_n\}$ is a Cauchy sequence.

Now, applying Lemma 2.1 we prove the following fixed point result for multivalued p-contractive maps.

Theorem 2.2. *Let X be a complete metric space with metric d and let $T : X \to Cl(X)$ be p-contractive map. Then there exists $x_0 \in X$ such that $x_0 \in Tx_0$ and $p(x_0, x_0) = 0$.*

Proof. By Lemma 2.1, there exists a Cauchy sequence $\{u_n\}$ in X such that $u_n \in Tu_{n-1}$ for each $n \in \mathbb{N}$. Since X is complete, $\{u_n\}$ converges to some $v_0 \in X$. For $n \in \mathbb{N}$, from (u_3) and the proof of Lemma 2.1, we have

$$p\left(u_n, v_0\right) \le \liminf_{m\to\infty} p\left(u_n, u_m\right) \le \frac{r^n}{1-r} p\left(u_0, u_1\right)$$

Since $u_n \in Tu_{n-1}$ and T is p-contractive, there exist $w_n \in Tv_0$ such that

$$p\left(u_n, w_n\right) \le rp\left(u_{n-1}, v_0\right).$$

Thus for any $n \in \mathbb{N}$

$$p\left(u_n, w_n\right) \le rp\left(u_{n-1}, v_0\right) \le \frac{r^n}{1-r} p\left(u_0, u_1\right),$$

and so $\lim_{n\to\infty} p\left(u_n, w_n\right) = 0$. Now, since $\lim_{n\to\infty} p\left(u_n, v_0\right) = 0$ it follows from Lemma 1.2 that

$$\lim_{n\to\infty} d\left(w_n, v_0\right) = 0$$

Since the sequence $\{w_n\} \subset Tv_0$ and Tv_0 is closed, we get $v_0 \in Tv_0$. Since T is p-contractive map so for such v_0 there is $v_1 \in Tv_0$ such that

$$p\left(v_0, v_1\right) \le rp\left(v_0, v_0\right)$$

Thus, we also have a sequence $\{v_n\}$ in X such that $v_{n+1} \in Tv_n$ and

$$p\left(v_0, v_{n+1}\right) \le rp\left(v_0, v_n\right),$$

for all $n \in \mathbb{N}$. Now, as in the proof of Lemma 2.1 we get v_n is a p-Cauchy sequence in X and thus it converges to some $x_0 \in X$. Moreover, we have $p\left(v_0, x_0\right) \le \liminf_{n\to\infty} p\left(v_0, v_n\right) \le 0$, which implies $p(v_0, x_0) = 0$. So for any $n \in \mathbb{N}$ we have

$$p\left(u_n, x_0\right) \le p\left(u_n, v_0\right) + p\left(v_0, x_0\right) \le \frac{r^n}{1-r} p\left(u_0, u_1\right)$$

Now, since $\lim_{n\to\infty} p\left(u_n, v_0\right) = 0$ and $\lim_{n\to\infty} p\left(u_n, x_0\right) = 0$, so by Lemma 1.2, it follows that $d(x_0, v_0) = 0$. Hence we get $x_0 = v_0$ and $p(v_0, v_0) = 0$.

A direct consequence of Theorem 2.2 is the following generalization of the Banach contraction principle.

Corollary 2.3. *Let X be a complete metric space with metric d. If a single-valued map $T : X \to X$ is p-contractive, then T has a unique fixed point $x_0 \in X$. Further, such x_0 satisfies $p(x_0, x_0) = 0$.*

Proof. By Theorem 2.2, it follows that there exists $x_0 \in X$ with $Tx_0 = x_0$ and $p(x_0, x_0) = 0$ For the uniqueness of x_0 we let $y_0 = Ty_0$. Then by the definition of T there exist $r \in (0, 1)$ such that $p(x_0, y_0) = p(Tx_0, Ty_0) \le rp(x_0, y_0)$, and $p(y_0, y_0) = p(Ty_0, Ty_0) \le rp(y_0, y_0)$. Thus

$$p\left(x_0, y_0\right) = p\left(y_0, y_0\right) = 0,$$

and hence by Lemma 1.2, we have $x_0 = y_0$.

Remark 2.4. Since w-distance and τ-distance are u-distance, Theorem 2.2 is a generalization of [[9], Theorem 1], while Corollary 2.3 contains [[9], Theorem 2] and [[14], Theorem 2].

We now prove a multivalued version of the Caristi's fixed point theorem with respect to u-distance.

Theorem 2.5. *Let X be a complete metric space and let $\varphi : X \to (-\infty, \infty]$ be proper, lower semicontinuous bounded below function. Let $T : X \to 2^X$. Assume that there exists a u-distance p on X such that for every $x \in X$, there exists $y \in Tx$ satisfying*

$$\phi\left(y\right) + p\left(x, y\right) \le \phi\left(x\right).$$

Then T has a fixed point $x_0 \in X$ such that $p(x_0, x_0) = 0$.

Proof. For each $x \in X$, we put $f(x) = y$, where $y \in T(x) \subset X$ and $\varphi(y) + p(x, y) \le \varphi(x)$. Note that f is a selfmap of X satisfying

$$\phi\left(f\left(x\right)\right) + p\left(x, f\left(x\right)\right) \le \phi\left(x\right),$$

for every $x \in X$. Since the map φ is proper, there exists $u \in X$ with $\varphi(u) < \infty$ and so we get $p(u, u) = 0$. Put

$$M = \left\{x \in X : \phi\left(x\right) \le \phi\left(u\right) - p\left(u, x\right)\right\},$$

and assume that for a sequence $\{x_n\}$ in M either $\lim\limits_{n\to\infty} \sup\left\{p\left(x_n, x_m\right) : m > n\right\} = 0$ or $\lim\limits_{n\to\infty} \sup\left\{p\left(x_m, x_n\right) : m > n\right\} = 0$. Note that M is nonempty because $u \in M$. Now, we show that the set M is closed. Let $\{x_n\}$ be a sequence in M which converges to some $x \in X$. Then $\{x_n\}$ is a p-Cauchy sequence and thus it follows from (u_3) that

$$p\left(u, x\right) \le \liminf_{n\to\infty} p\left(u, x_n\right). \tag{16}$$

Using the lower semicontinuity φ it is easy to show that the set M is closed in X. Thus M is a complete metric space. Now, we show that the set M is invariant under f. Note that for each $x \in M$, we have

$$\phi\left(f\left(x\right)\right) + p\left(x, f\left(x\right)\right) \le \phi\left(x\right) \le \phi\left(u\right) - p\left(u, x\right)$$

and thus

$$\phi\left(f\left(x\right)\right) \le \phi\left(u\right) - \left\{p\left(u, x\right) + p\left(x, f\left(x\right)\right)\right\} \le \phi\left(s\right) - p\left(s, f\left(x\right)\right).$$

It follows that $f(x) \in M$ and hence f is a selfmap of M. Applying Theorem 1.2, there exists $x_0 \in M$ such that $f(x_0) = x_0 \in T(x_0)$ and $p(x_0, x_0) = 0$.

Remark 2.6. Theorem 2.5 is a multivalued version of Theorem 1.2 due to Ume [20] and generalizes a fixed point result due to Mizoguchi and Takahashi [[5], Theorem 1].

Further, Theorem 2.5 contains [[7], Theorem 2] and [[14], Theorem 3] which are single-valued generalizations of the Caristi's fixed point theorem.

Theorem 2.7. Let X be a complete metric space, f be a single-valued selfmap of X with $f(X) = M$ complete and let $T : X \to 2^X$ be such that $T(X) \subset M$. Assume that there exists a u-distance p on X such that for every $x \in X$, there exists $y \in Tx$ satisfying

$$p\left(x, f\left(y\right)\right) \leq \phi\left(x\right) - \phi\left(f\left(y\right)\right),$$

where $\varphi : M \to (-\infty, \infty]$ is proper, lower semicontinuous, and bounded from below. Then, there exits a point $x_0 \in M$ such that $x_0 \in fT(x_0)$.

Proof. For each $y \in M$, define

$$J\left(y\right) = fT\left(y\right) = \bigcup_{x \in T(y)} \left\{f\left(x\right)\right\}.$$

Clearly, J carries M into 2^M. Now, for each $s \in J(y)$, there exists some $t \in T(y)$ with $s = f(t)$ and $p(y, f(t)) \leq \varphi(y) - \varphi(f(t))$, that is; $p(y, s) \leq \varphi(y) - \varphi(s)$. Since φ is proper, there exists $z \in M$ with $\varphi(z) < +\infty$.

Let

$$Y = \left\{y \in M : \phi\left(y\right) \leq \phi\left(z\right) - p\left(z, y\right)\right\},$$

and assume that for a sequence $\{x_n\}$ in Y either $\lim_{n\to\infty} \sup\{p(x_n, x_m) : m > n\} = 0$ or $\lim_{n\to\infty} \sup\{p(x_m, x_n) : m > n\} = 0$. Note that Y is nonempty closed subset of a complete space M. Thus Y is a complete metric space. Now we show that Y is invariant under the map J. Now, let $s \in J(y)$, $y \in Y$. By definition of J, there exists $t \in T(y)$ such that $s = f(t)$, and

$$\phi\left(s\right) + p\left(y, s\right) \leq \phi\left(y\right) \leq \phi\left(z\right) - p\left(z, y\right)$$

and hence

$$\phi\left(s\right) \leq \phi\left(z\right) - p\left(z, s\right),$$

proving that $s \in Y$ and hence $J(y) \subset Y$ for all $y \in Y$. Now, Theorem 2.5 guarantees that there exits $x_0 \in M$ such that $x_0 \in J(x_0) = fT(x_0)$.

Finally, we obtain a common fixed point result.

Theorem 2.8. Suppose that X, M, f, and T satisfy the assumptions of Theorem 2.7. Moreover, the following conditions hold:

(a) f and T commute weakly.

(b) $x \notin \mathrm{Fix}(f)$ implies $x \notin fT(x)$.

Then T and f have a common fixed point in M.

Proof. As in the proof of Theorem 2.7, there exits $x_0 \in M$ such that $x_0 \in fT(x_0)$. Using conditions (a) and (b), we obtain

$$x_0 = f\left(x_0\right) \in fT\left(x_0\right) \subseteq Tf\left(x_0\right) = T\left(x_0\right)$$

Thus, x_0 must be a common fixed point of f and T.

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Authors' contributions
All authors participated in the design of this work and performed equally. All authors read and approved the final manuscript.

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The authors declare that they have no competing interests.

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