Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

Strong convergence theorems for total quasi-ϕ-asymptotically nonexpansive multi-valued mappings in Banach spaces

Jinfang Tang[1] and Shih-sen Chang[2*]

* Correspondence: changss@yahoo.cn
[2]College of statistics and mathematics, Yunnan University of Finance and Economics, Kunming, Yunnan 650221, China
Full list of author information is available at the end of the article

Abstract

The main purpose of this article is to introduce the concept of total quasi-ϕ-asymptotically nonexpansive multi-valued mapping and prove the strong convergence theorem in a real uniformly smooth and strictly convex Banach space with Kadec-Klee property. In order to get the theorems, the hybrid algorithms are presented and are used to approximate the fixed point. The results presented in this article improve and extend some recent results announced by some authors.
AMS (MOS) Subject Classification: 47J06, 47J25.

Keywords: total quasi-ϕ-asymptotically nonexpansive multi-valued mappings, total quasi-ϕ-asymptotically nonexpansive mappings, quasi-ϕ-asymptotically non-expansive mappings.

1 Introduction

Throughout this article, we always assume that X is a real Banach space with the dual X^* and $J : X \to 2^X$ is the *normalized duality mapping* defined by

$$J(x) = \{f^* \in X^* :< x, f^* >= \|x\|^2 = \|f^*\|^2\}, \quad x \in E.$$

In the sequal, we use $F(T)$ to denote the set of fixed points of a mapping T, and use $\mathscr{R}$ and $\mathscr{R}^+$ to denote the set of all real numbers and the set of all nonnegative real numbers, respectively. We denote by $x_n \to x$ and $x_n \rightharpoonup x$ the strong convergence and weak convergence of a sequence $\{x_n\}$, respectively.

A Banach space X is said to be *strictly convex* if $\frac{\|x+y\|}{2} < 1$ for all $x, y \in U = \{z \in X : ||z|| = 1\}$ with $x \neq y$. X is said to be *uniformly convex* if, for each $\epsilon \in (0, 2]$, there exists $\delta > 0$ such that $\frac{\|x+y\|}{2} < 1 - \delta$ for all $x, y \in U$ with $||x - y|| \geq \epsilon$. X is said to be *smooth* if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for all $x, y \in U$. X is said to be *uniformly smooth* if the above limit is attained uniformly in $x, y \in U$.

Remark 1.1. The following basic properties of a Banach space X can be found in Cioranescu [1].

(i) If X is uniformly smooth, then J is uniformly continuous on each bounded subset of X;

(ii) If X is a reflexive and strictly convex Banach space, then J^{-1} is norm-weak-continuous;

(iii) If X is a smooth, strictly convex and reflexive Banach space, then J is single-valued, one-to-one and onto;

(iv) A Banach space X is uniformly smooth if and only if X^* is uniformly convex;

(v) Each uniformly convex Banach space X has the *Kadec-Klee property*, i.e., for any sequence $\{x_n\} \subset X$, if $x_n \rightharpoonup x \in X$ and $||x_n|| \to ||x||$, then $x_n \to x$.

Let X be a smooth Banach space. We always use $\phi : X \times X \to \mathscr{R}^+$ to denote the Lyapunov functional defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \quad \forall x, y \in X. \tag{1.1}$$

It is obvious from the definition of the function ϕ that

$$(\|x\| - \|y\|)^2 \le \phi(x, y) \le (\|x\| + \|y\|)^2, \quad \forall x, y \in X. \tag{1.2}$$

Following Alber [2], the *generalized projection* $\Pi_C : X \to C$ is defined by

$$\Pi_C(x) = \arg\inf_{y \in C} \phi(y, x), \quad \forall x \in X.$$

Lemma 1.2. [2] Let X be a smooth, strictly convex and reflexive Banach space and C be a nonempty closed convex subset of X. Then the following conclusions hold:

(a) $\phi(x, \Pi_C y) + \phi(\Pi_C y, y) \le \phi(x, y)$ for all $x \in C$ and $y \in X$;

(b) If $x \in X$ and $z \in C$, then

$$z = \Pi_C x \;\; iff \, \langle z - y, \; Jx - Jz\rangle \ge 0, \quad \forall y \in C;$$

(c) For $x, y \in X$, $\phi(x, y) = 0$ if and only if $x = y$.

Let X be a smooth, strictly convex and reflexive Banach space and C be a nonempty closed convex subset of X and $T : C \to C$ be a mapping. A point $p \in C$ is said to be an *asymptotic fixed point of* T if there exists a sequence $\{x_n\} \subset C$ such that $x_n \rightharpoonup p$ and $||x_n - Tx_n|| \to 0$. We denoted the set of all asymptotic fixed points of T by $\tilde{F}(T)$.

Definition 1.3. (1) A mapping $T : C \to C$ is said to be *relatively nonexpansive* [3] if $F(T) \neq \emptyset, F(T) = F(\tilde{T})$ and

$$\phi(p, Tx) \le \phi(p, x), \quad \forall x \in C, \quad p \in F(T).$$

(2) A mapping $T : C \to C$ is said to be *closed* if, for any sequence $\{x_n\} \subset C$ with $x_n \to x$ and $Tx_n \to y$, then $Tx = y$.

Definition 1.4. (1) A mapping $T : C \to C$ is said to be *quasi-ϕ-nonexpansive* if $F(T) \neq \emptyset$ and

$$\phi(p, Tx) \le \phi(p, x), \quad \forall x \in C, \quad p \in F(T).$$

(2) A mapping $T : C \to C$ is said to be *quasi-ϕ-asymptotically nonexpansive* if $F(T) \neq \emptyset$ and there exists a real sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ such that

$$\phi(p, T^n x) \leq k_n \phi(p, x), \quad \forall n \geq 1, \quad x \in C, \quad p \in F(T). \tag{1.3}$$

(3) A mapping $T : C \to C$ is said to be *total quasi-ϕ-asymptotically nonexpansive* if $F(T) \neq \emptyset$ and there exist nonnegative real sequences $\{v_n\}$, $\{\mu_n\}$ with $v_n \to 0$, $\mu_n \to 0$ (as $n \to \infty$) and a strictly increasing continuous function $\zeta : \mathscr{R}^+ \to \mathscr{R}^+$ with $\zeta(0) = 0$ such that for all $x \in C$, $p \in F(T)$

$$\phi(p, T^n x) \leq \phi(p, x) + v_n \zeta(\phi(p, x)) + \mu_n, \quad \forall n \geq 1. \tag{1.4}$$

Remark 1.5. From the definitions, it is easy to know that

(1) Taking $\zeta(t) = t$, $t \geq 0$, $v_n = k_n - 1$ and $\mu_n = 0$, then $v_n \to 0$(as $n \to \infty$) and (1.3) can be rewritten as

$$\phi(p, T^n x) \leq \phi(p, x) + v_n \zeta(\phi(p, x)) + \mu_n, \quad \forall n \geq 1, \quad x \in C, \quad p \in F(T). \tag{1.5}$$

This implies that the class of total quasi-ϕ-asymptotically nonexpansive mappings contains properly the class of quasi-ϕ-asymptotically nonexpansive mappings as a subclass, but the converse is not true.

(2) The class of quasi-ϕ-asymptotically nonexpansive mappings contains properly the class of quasi-ϕ-nonexpansive mappings as a subclass, but the converse is not true.

(3) The class of quasi-ϕ-nonexpansive mappings contains properly the class of relatively nonexpansive mappings as a subclass, but the converse is not true.

Let C be a nonempty closed convex subset of a Banach space X. Let $N(C)$ be the family of nonempty subsets of C.

Definition 1.6. (1) A multi-valued mapping $T : C \to N(C)$ is said to be *relatively nonexpansive* [3] if $F(T) \neq \emptyset, F(T) = F(\tilde{T})$ and

$$\phi(p, w) \leq \phi(p, x), \quad \forall x \in C, \quad w \in Tx, \quad p \in F(T).$$

(2) A multi-valued mapping $T : C \to N(C)$ is said to be *closed* if, for any sequence $\{x_n\} \subset C$ with $x_n \to x$ and $w_n \in T(x_n)$ with $w_n \to y$, then $y \in Tx$.

Definition 1.7. (1) A multi-valued mapping $T : C \to N(C)$ is said to be *quasi-ϕ-nonexpansive* if $F(T) \neq \emptyset$ and

$$\phi(p, w) \leq \phi(p, x), \quad \forall x \in C, \quad w \in Tx, \quad p \in F(T).$$

(2) A multi-valued mapping $T : C \to N(C)$ is said to be *quasi-ϕ-asymptotically nonexpansive* if $F(T) \neq \emptyset$ and there exists a real sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ such that

$$\phi(p, w_n) \leq k_n \phi(p, x), \quad \forall n \geq 1, \quad x \in C, \quad w_n \in T^n x, \quad p \in F(T). \tag{1.6}$$

(3) A multi-valued mapping $T : C \to N(C)$ is said to be *total quasi-ϕ-asymptotically nonexpansive* if $F(T) \neq \emptyset$ and there exist nonnegative real sequences $\{v_n\}$, $\{\mu_n\}$ with $v_n \to 0$, $\mu_n \to 0$(as $n \to \infty$) and a strictly increasing continuous function $\zeta : \mathscr{R}^+ \to \mathscr{R}^+$ with $\zeta(0) = 0$ such that for all $x \in C$, $p \in F(T)$

$$\phi(p, w_n) \leq \phi(p, x) + v_n \zeta(\phi(p, x)) + \mu_n, \quad \forall n \geq 1, \quad w_n \in T^n x. \tag{1.7}$$

(4) A total quasi-ϕ-asymptotically nonexpansive multi-valued mapping $T : C \to N(C)$ is said to be *uniformly L-Lipschitz continuous* if there exists a constant $L > 0$ such that

$$\|w_n - s_n\| \leq L \left\|x - y\right\|, \quad \forall x, y \in C, \quad w_n \in T^n x, \quad s_n \in T^n y, \quad n \geq 1.$$

Remark 1.8. From the definitions, it is easy to know that

(1) Taking $\zeta(t) = t$, $t \geq 0$, $\nu_n = k_n - 1$ and $\mu_n = 0$, then $\nu_n \to 0$ (as $n \to \infty$) and (1.6) can be rewritten as

$$\phi(p, w_n) \leq \phi(p, x) + \nu_n \zeta(\phi(p, x)) + \mu_n, \quad \forall n \geq 1, \quad x \in C, \quad w_n \in T^n x, \quad p \in F(T).$$

This implies that the class of total quasi-ϕ-asymptotically nonexpansive multi-valued mappings contains properly the class of quasi-ϕ-asymptotically nonexpansive multi-valued mappings as a subclass, but the converse is not true.

(2) The class of quasi-ϕ-asymptotically nonexpansive multi-valued mappings contains properly the class of quasi-ϕ-nonexpansive multi-valued mappings as a subclass, but the converse is not true.

(3) The class of quasi-ϕ-nonexpansive multi-valued mappings contains properly the class of relatively nonexpansive multi-valued mappings as a subclass, but the converse is not true.

In 2005, Matsushita and Takahashi [3] proved weak and strong convergence theorems to approximate a fixed point of a single relatively nonexpansive mapping in a uniformly convex and uniformly smooth Banach space X. In 2008, Plubtieng and Ungchittrakool [4] proved the strong convergence theorems to approximate a fixed point of two relatively nonexpansive mapping in a uniformly convex and uniformly smooth Banach space X. In 2010, Chang et al. [5] obtained the strong convergence theorem for an infinite family of quasi-ϕ-asymptotically nonexpansive mappings in a uniformly smooth and strictly convex Banach space X with Kadec-Klee property. In 2011, Chang et al. [6] proved some approximation theorems of common fixed points for countable families of total quasi-ϕ-asymptotically nonexpansive mappings in a uniformly smooth and strictly convex Banach space X with Kadec-Klee property. In 2011, Homaeipour and Razani [7] proved weak and strong convergence theorems for a single relatively nonexpansive multi-valued mapping in a uniformly convex and uniformly smooth Banach space X.

Motivated and inspired by the researches going on in this direction, the purpose of this article is first to introduce the concept of total quasi-ϕ-asymptotically nonexpansive multivalued mapping which contains many kinds of mappings as its special cases, and then by using the hybrid iterative algorithm to prove some strong convergence theorems in uniformly smooth and strictly convex Banach space with Kadec-Klee property. The results presented in the article improve and extend some recent results announced by some authors.

2 Preliminaries

Lemma 2.1. [6] Let X be a real uniformly smooth and strictly convex Banach space with Kadec-Klee property, and C be a nonempty closed convex set of X. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in C such that $x_n \to p$ and $\phi(x_n, y_n) \to 0$, where ϕ is the function defined by (1.1), then $y_n \to p$.

Lemma 2.2. Let X and C be as in Lemma 2.1. Let $T : C \to N(C)$ be a closed and total quasi-ϕ-asymptotically nonexpansive multi-valued mapping with nonnegative real sequences $\{\nu_n\}$, $\{\mu_n\}$ and a strictly increasing continuous function $\zeta : \mathscr{R}^+ \to \mathscr{R}^+$ such that $\nu_n \to 0$, $\mu_n \to 0$(as $n \to \infty$) and $\zeta(0) = 0$. If $\mu_1 = 0$, then the fixed point set $F(T)$ is a closed and convex subset of C.

Proof. Let $\{x_n\}$ be a sequence in $F(T)$ with $x_n \to p$ (as $n \to \infty$), we prove that $p \in F(T)$. In fact, by the assumption that T is total quasi-ϕ-asymptotically nonexpansive multi-valued mapping and $\mu_1 = 0$, we have

$$\phi(x_n, u) \le \phi(x_n, p) + \nu_1 \zeta(\phi(x_n, p)), \quad \forall u \in Tp.$$

Furthermore, we have

$$\begin{aligned}\phi(p, u) &= \lim_{n\to\infty} \phi(x_n, u)\\ &\le \lim_{n\to\infty} (\phi(x_n, p) + \nu_1 \zeta(\phi(x_n, p))) = 0, \quad \forall u \in Tp.\end{aligned}$$

By Lemma 1.2(c), $p = u$. Hence, $p \in Tp$. This implies that $p \in F(T)$, i.e., $F(T)$ is closed.

Next, we prove that $F(T)$ is convex. For any $x, y \in F(T)$, $t \in (0, 1)$, putting $q = tx + (1 - t)y$, we prove that $q \in F(T)$. Indeed, let $\{u_n\}$ be a sequence generated by

$$\begin{cases} u_1 \in Tq,\\ u_2 \in Tu_1 \subset T^2 q,\\ u_3 \in Tu_2 \subset T^3 q,\\ \vdots\\ u_n \in Tu_{n-1} \subset T^n q,\\ \vdots \end{cases} \tag{2.1}$$

In view of the definition of $\phi(x, y)$, for all $u_n \in Tu_{n-1} \subset T^n q$, we have

$$\begin{aligned}\phi(q, u_n) &= \|q\|^2 - 2\langle q, Ju_n\rangle + \|u_n\|^2\\ &= \|q\|^2 - 2t\langle x, Ju_n\rangle - 2(1-t)\langle y, Ju_n\rangle + \|u_n\|^2\\ &= \|q\|^2 + t\phi(x, u_n) + (1-t)\phi(y, u_n) - t\|x\|^2 - (1-t)\|y\|^2\end{aligned} \tag{2.2}$$

Since

$$\begin{aligned}&t\phi(x, u_n) + (1-t)\phi(y, u_n)\\ &\le t(\phi(x, q) + \nu_n \zeta(\phi(x, q)) + \mu_n) + (1-t)(\phi(y, q) + \nu_n \zeta(\phi(y, q)) + \mu_n)\\ &= t(\|x\|^2 - 2\langle x, Jq\rangle + \|q\|^2 + \nu_n \zeta(\phi(x, q)) + \mu_n)\\ &\quad + (1-t)(\|y\|^2 - 2\langle y, Jq\rangle + \|q\|^2 + \nu_n \zeta(\phi(y, q)) + \mu_n)\\ &= t\|x\|^2 + (1-t)\|y\|^2 - \|q\|^2 + t\nu_n \zeta(\phi(x, q)) + (1-t)\nu_n \zeta(\phi(y, q)) + \mu_n\end{aligned} \tag{2.3}$$

Substituting (2.2) into (2.1) and simplifying it we have

$$\phi(q, u_n) \le t\nu_n \zeta(\phi(x, q)) + (1-t)\nu_n \zeta(\phi(y, q)) + \mu_n \to 0 (n \to \infty).$$

By Lemma 2.1, we have $u_n \to q$ (as $n \to \infty$). This implies that $u_{n+1} \to q$ (as $n \to \infty$). Since T is closed, we have $q \in Tq$, i.e., $q \in F(T)$.

This completes the proof of Lemma2.2.

Lemma 2.3. [8] Let X be a uniformly convex Banach space, $r > 0$ be a positive number and $B_r(0)$ be a closed ball of X. Then, there exists a continuous, strictly increasing and convex function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that

$$\|\alpha x + \beta y\|^2 \le \alpha\|x\|^2 + \beta\|y\|^2 - \alpha\beta g(\|x - y\|), \tag{2.4}$$

for all $x, y \in B_r(0)$ and all $\alpha, \beta \in [0, 1]$ with $\alpha + \beta = 1$.

3 Main results

In this section, we shall use the hybrid iterative algorithm to study the iterative solutions of nonlinear operator equations with a closed and uniformly total quasi-ϕ-asymptotically nonexpansive multi-valued mapping in Banach space.

Theorem 3.1. Let X be a real uniformly smooth and strictly convex Banach space with Kadec-Klee property, and C be a nonempty closed and convex subset of X. Let $T : C \to N(C)$ be a closed and total quasi-ϕ-asymptotically nonexpansive multi-valued mapping with nonnegative real sequences $\{v_n\}$,$\{\mu_n\}$ and a strictly increasing continuous function $\zeta : \mathscr{R}^+ \to \mathscr{R}^+$ such that $\mu_1 = 0$, $v_n \to 0$, $\mu_n \to 0$ (as $n \to \infty$) and $\zeta(0) = 0$. Let $x_0 \in C$, $C_0 = C$ and $\{x_n\}$ be a sequence generated by

$$\begin{cases} y_n = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)Jz_n), \\ z_n = J^{-1}(\beta_n Jx_n + (1 - \beta_n)Jw_n), \\ C_{n+1} = \{v \in C_n : \phi(v, y_n) \le \phi(v, x_n) + \xi_n\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad \forall n \ge 0, \end{cases} \tag{3.1}$$

where $w_n \in T^n x_n$, $\forall n \ge 1$, $\xi_n = v_n \sup_{p \in F(T)} \zeta(\phi(p, x_n)) + \mu_n$, $\Pi_{C_{n+1}}$ is the generalized projection of X onto C_{n+1}, $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in [0,1] satisfies the following conditions:

(a) $\liminf_{n\to\infty} \beta_n(1 - \beta_n) > 0$;

(b) $0 \le \alpha_n \le \alpha < 1$ for some $\alpha \in (0, 1)$.

If $F(T)$ is a nonempty and bounded subset of C, then the sequence $\{x_n\}$ converges strongly to $\Pi_{F(T)}x_0$.

Proof. We divide the proof of Theorem 3.1 into six steps.

(I) C_n is closed and convex for each $n \ge 0$.

In fact, by the assumption, $C_0 = C$ is closed and convex. Suppose that C_n is closed and convex for some $n \ge 1$. Since the condition $\phi(v, y_n) \le \phi(v, x_n) + \xi_n$ is equivalent to

$$2\langle v, Jx_n - Jy_n\rangle \le \|x_n\|^2 - \|y_n\|^2 + \xi_n, \quad n = 1, 2, \ldots,$$

hence the set

$$C_{n+1} = \{v \in C_n : 2\langle v, Jx_n - Jy_n\rangle \le \|x_n\|^2 - \|y_n\|^2 + \xi_n\}$$

is closed and convex. Therefore C_n is closed and convex for each $n \ge 0$.

(II) $\{x_n\}$ is bounded and $\{\phi(x_n, x_0)\}$ is a convergent sequence.

Indeed, it follows from (3.1) and Lemma 1.2(a) that for all $n \ge 0$, $u \in F(T)$

$$\phi(x_n, x_0) = \phi(\Pi_{C_n} x_0, x_0) \le \phi(u, x_0) - \phi(u, \Pi_{C_n} x_0) \le \phi(u, x_0).$$

This implies that $\{\phi(x_n, x_0)\}$ is bounded. By virtue of (1.2), we know that $\{x_n\}$ is bounded.

In view of structure of $\{C_n\}$, we have $C_{n+1} \subset C_n$, $C_{n+1} \subset C_n, x_n = \Pi_{C_n} x_0$ and $x_{n+1} = \Pi_{C_{n+1}} x_0$. This implies that $x_{n+1} \in C_n$ and

$$\phi(x_n, x_0) \le \phi(x_{n+1}, x_0), \quad \forall n \ge 0.$$

Therefore $\{\phi(x_n, x_0)\}$ is a convergent sequence.

(III) $F(T) \subset C_n$ *for all* $n \geq 0$.

It is obvious that $F(T) \subset C_0 = C$. Suppose that $F(T) \subset C_n$ for some $n \geq 1$. Since X is uniformly smooth, X^* is uniformly convex. For any given $u \in F(T) \subset C_n$ and $n \geq 1$ we have

$$\begin{aligned}\phi(u, y_n) &= \phi(u, J^{-1}(\alpha_n Jx_n + (1-\alpha_n)Jz_n)) \\ &= \|u\|^2 - 2\langle u, \alpha_n Jx_n + (1-\alpha_n)Jz_n\rangle + \left\|\alpha_n Jx_n + (1-\alpha_n)Jz_n\right\|^2 \\ &\leq \|u\|^2 - 2\alpha_n\langle u, Jx_n\rangle - 2(1-\alpha_n)\langle u, Jz_n\rangle + \alpha_n\|x_n\|^2 \\ &\quad + (1-\alpha_n)\|z_n\|^2 \\ &= \alpha_n\phi(u, x_n) + (1-\alpha_n)\phi(u, z_n).\end{aligned} \tag{3.2}$$

Furthermore, it follows from Lemma 2.3 that for any $u \in F(T)$, $w_n \in T^n x_n$ we have

$$\begin{aligned}\phi(u, z_n) &= \phi(u, J^{-1}(\beta_n Jx_n + (1-\beta_n)Jw_n)) \\ &= \|u\|^2 - 2\langle u, \beta_n Jx_n + (1-\beta_n)Jw_n\rangle + \left\|\beta_n Jx_n + (1-\beta_n)Jw_n\right\|^2 \\ &\leq \|u\|^2 - 2\beta_n\langle u, Jx_n\rangle - 2(1-\beta_n)\langle u, Jw_n\rangle + \beta_n\|x_n\|^2 \\ &\quad + (1-\beta_n)\|w_n\|^2 - \beta_n(1-\beta_n)g(\|Jx_n - Jw_n\|) \\ &= \beta_n\phi(u, x_n) + (1-\beta_n)\phi(u, w_n) - \beta_n(1-\beta_n)g(\|Jx_n - Jw_n\|) \\ &\leq \beta_n\phi(u, x_n) + (1-\beta_n)(\phi(u, x_n) + \nu_n\zeta(\phi(u, x_n)) + \mu_n) \\ &\quad - \beta_n(1-\beta_n)g(\|Jx_n - Jw_n\|) \\ &\leq \phi(u, x_n) + \nu_n \sup_{p\in F(T)} \zeta(\phi(p, x_n)) + \mu_n - \beta_n(1-\beta_n)g(\|Jx_n - Jw_n\|) \\ &= \phi(u, x_n) + \xi_n - \beta_n(1-\beta_n)g(\|Jx_n - Jw_n\|).\end{aligned} \tag{3.3}$$

Substituting (3.3) into (3.2) and simplifying it, we have

$$\phi(u, y_n) \leq \phi(u, x_n) + (1-\alpha_n)\xi_n \leq \phi(u, x_n) + \xi_n, \quad \forall u \in F(T), \tag{3.4}$$

i.e., $u \in C_{n+1}$ and so $F(T) \subset C_{n+1}$ for all $n \geq 0$.

By the way, in view of the assumption on $\{\nu_n\}$, $\{\mu_n\}$ we have

$$\xi_n = \nu_n \sup_{p\in F(T)} \zeta(\phi(p, x_n)) + \mu_n \to 0 (n \to \infty).$$

(IV) $\{x_n\}$ *converges strongly to some point* $p^* \in C$.

In fact, since $\{x_n\}$ is bounded and X is reflexive, there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $x_{n_i} \rightharpoonup p^*$ (some point in C). Since C_n is closed and convex and $C_{n+1} \subset C_n$, this implies that C_n is weakly closed and $p^* \in C_n$ for each $n \geq 0$. In view of $x_{n_i} = \Pi_{C_{n_i}} x_0$, we have

$$\phi(x_{n_i}, x_0) \leq \phi(p^*, x_0), \quad \forall n_i \geq 0.$$

Since the norm $||\cdot||$ is weakly lower semi-continuous, we have

$$\begin{aligned}\liminf_{n_i\to\infty} \phi(x_{n_i}, x_0) &= \liminf_{n_i\to\infty}(\left\|x_{n_i}\right\|^2 - 2\langle x_{n_i}, Jx_0\rangle + \|x_0\|^2) \\ &\geq \left\|p^*\right\|^2 - 2\langle p^*, Jx_0\rangle + \|x_0\|^2 = \phi(p^*, x_0),\end{aligned}$$

and so

$$\phi(p^*, x_0) \le \liminf_{n_i \to \infty} \phi(x_{n_i}, x_0) \le \limsup_{n_i \to \infty} \phi(x_{n_i}, x_0) \le \phi(p^*, x_0).$$

This implies that $\lim_{n_i \to \infty} \phi(x_{n_i}, x_0) = \phi(p^*, x_0)$, and so $\|x_{n_i}\| \to \|p^*\|$. Since $x_{n_i} \rightharpoonup p^*$, by virtue of Kadec-Klee property of X, we obtain that

$$\lim_{n_i \to \infty} x_{n_i} = p^*.$$

Since $\{\phi(x_n, x_0)\}$ is convergent, this together with $\lim_{n_i \to \infty} \phi(x_{n_i}, x_0) = \phi(p^*, x_0)$, which shows that $\lim_{n \to \infty} \phi(x_n, x_0) = \phi(p^*, x_0)$. If there exists some sequence $\{x_{n_j}\} \subset \{x_n\}$ such that $x_{n_j} \to q$, then from Lemma 1.2(a) we have that

$$\begin{aligned}
\phi(p^*, q) &= \lim_{n_i, n_j \to \infty} \phi(x_{n_i}, x_{n_j}) = \lim_{n_i, n_j \to \infty} \phi(x_{n_i}, \Pi_{C_{n_j}} x_0) \\
&\le \lim_{n_i, n_j \to \infty} (\phi(x_{n_i}, x_0) - \phi(\Pi_{C_{n_j}} x_0, x_0)) \\
&= \lim_{n_i, n_j \to \infty} (\phi(x_{n_i}, x_0) - \phi(x_{n_j}, x_0)) \\
&= \phi(p^*, x_0) - \phi(p^*, x_0) = 0.
\end{aligned}$$

This implies that $p^* = q$ and

$$\lim_{n \to \infty} x_n = p^*. \tag{3.5}$$

(V) Now we prove that $p^* \in F(T)$.

In fact, since $x_{n+1} \in C_{n+1} \subset C_n$, it follows from (3.1) and (3.5) that

$$\phi(x_{n+1}, y_n) \le \phi(x_{n+1}, x_n) + \xi_n \to 0 (n \to \infty).$$

Since $x_n \to p^*$, by the virtue of Lemma 2.1

$$\lim_{n \to \infty} y_n = p^*. \tag{3.6}$$

From (3.2) and (3.3), for any $u \in F(T)$ and $w_n \in T^n x_n$, we have

$$\phi(u, y_n) \le \phi(u, x_n) + \xi_n - (1 - \alpha_n)\beta_n(1 - \beta_n) g(\|Jx_n - Jw_n\|),$$

i.e.,

$$(1 - \alpha_n)\beta_n(1 - \beta_n) g(\|Jx_n - Jw_n\|) \le \phi(u, x_n) + \xi_n - \phi(u, y_n) \to 0 (n \to \infty).$$

By conditions (a) and (b) it shows that $\lim_{n \to \infty} g(||Jx_n - Jw_n||) = 0$. In view of property of g, we have

$$\|Jx_n - Jw_n\| \to 0 (n \to \infty).$$

Since $Jx_n \to Jp^*$, this implies that $Jw_n \to Jp^*$. From Remark 1.1 (ii) it yields

$$w_n \rightharpoonup p^* (n \to \infty). \tag{3.7}$$

Again since

$$\left| \|w_n\| - \|p^*\| \right| = \left| \|Jw_n\| - \|Jp^*\| \right| \le \|Jw_n - Jp^*\| \to 0 (n \to \infty),$$

this together with (3.7) and the Kadec-Klee property of X shows that

$$\lim_{n\to\infty} w_n = p^*. \tag{3.8}$$

Let $\{s_n\}$ be a sequence generated by

$$\begin{cases} s_2 \in Tw_1 \subset T^2x_1, \\ s_3 \in Tw_2 \subset T^3x_2, \\ \vdots \\ s_{n+1} \in Tw_n \subset T^{n+1}x_n, \\ \vdots \end{cases}$$

By the assumption that T is uniformly L-Lipschitz continuous, hence for any $w_n \in T^n x_n$ and $s_{n+1} \in Tw_n \subset T^{n+1}x_n$ we have

$$\begin{aligned} \|s_{n+1} - w_n\| &\le \|s_{n+1} - w_{n+1}\| + \|w_{n+1} - x_{n+1}\| + \|x_{n+1} - x_n\| + \|x_n - w_n\| \\ &\le (L+1)\|x_{n+1} - x_n\| + \|w_{n+1} - x_{n+1}\| + \|x_n - w_n\|. \end{aligned} \tag{3.9}$$

This together with (3.5) and (3.8) shows that $\lim_{n\to\infty} ||s_{n+1} - w_n|| = 0$ and $\lim_{n\to\infty} s_{n+1} = p^*$. In view of the closeness of T, it yields that $p^* \in Tp^*$, i.e.,

$$p^* \in F(T).$$

(VI) we prove that $x_n \to p^* = \Pi_{F(T)}x_0$.

Let $t = \Pi_{F(T)}x_0$. Since $t \in F(T) \subset C_n$ and $x_n = \Pi_{C_n}x_0$, we have

$$\phi(x_n, x_0) \le \phi(t, x_0), \quad \forall n \ge 0.$$

This implies that

$$\phi(p^*, x_0) = \lim_{n\to\infty} \phi(x_n, x_0) \le \phi(t, x_0). \tag{3.10}$$

In view of the definition of $\Pi_{F(T)}x_0$, from (3.10) we have $p^* = t$. Therefore, $x_n \to p^* = \Pi_{F(T)}x_0$.

This completes the proof of Theorem 3.1.

From Remark 1.8, the following theorems can be obtained from Theorem 3.1 immediately.

Theorem 3.2. Let X be a real uniformly smooth and strictly convex Banach space with Kadec-Klee property, and C be a nonempty closed and convex subset of X. Let $T : C \to N(C)$ be a closed and quasi-ϕ-asymptotically nonexpansive multi-valued mapping with a real sequences $\{k_n\} \subset [1, \infty)$ and $k_n \to 1(n \to (\infty)$. Let $x_0 \in C$, $C_0 = C$ and $\{x_n\}$ be a sequence generated by

$$\begin{cases} y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)Jz_n), \\ z_n = J^{-1}(\beta_n Jx_n + (1-\beta_n)Jw_n), \quad w_n \in T^n x_n, \\ C_{n+1} = \{v \in C_n : \phi(v, y_n) \le \phi(v, x_n) + \xi_n\}, \\ x_{n+1} = \Pi_{C_{n+1}}x_0, \quad \forall n \ge 0, \end{cases}$$

where $\xi_n = (k_n - 1)\sup_{p\in F(T)}(\phi(p, x_n))$, $\Pi_{C_{n+1}}$ is the generalized projection of X onto C_{n+1}, $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in [0,1] satisfies the following conditions:

(a) $\liminf_{n\to\infty} \beta_n(1 - \beta_n) > 0$;

(b) $0 \le \alpha_n \le \alpha < 1$ for some $\alpha \in (0, 1)$.

If $F(T)$ is a nonempty and bounded subset of C, then the sequence $\{x_n\}$ converges strongly to $\Pi_{F(T)}x_0$.

Theorem 3.3. Let X be a real uniformly smooth and strictly convex Banach space with Kadec-Klee property, and C be a nonempty closed and convex subset of X. Let $T : C \to N(C)$ be a closed and quasi-ϕ nonexpansive multi-valued mapping. Let $x_0 \in C$, $C_0 = C$ and $\{x_n\}$ be a sequence generated by

$$\begin{cases} y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)Jz_n), \\ z_n = J^{-1}(\beta_n Jx_n + (1-\beta_n)Jw_n), \quad w_n \in Tx_n, \\ C_{n+1} = \{v \in C_n : \phi(v, y_n) \le \phi(v, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}}x_0, \quad \forall n \ge 0, \end{cases}$$

where $\Pi_{C_{n+1}}$ is the generalized projection of X onto C_{n+1}, $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in [0,1] satisfies the following conditions:

(a) $\liminf_{n\to\infty} \beta_n(1 - \beta_n) > 0$;

(b) $0 \le \alpha_n \le \alpha < 1$ for some $\alpha \in (0, 1)$.

If $F(T)$ is a nonempty and bounded subset of C, then the sequence $\{x_n\}$ converges strongly to $\Pi_{F(T)}x_0$.

Theorem 3.4. Let X be a real uniformly smooth and strictly convex Banach space with Kadec-Klee property, and C be a nonempty closed and convex subset of X. Let $T : C \to N(C)$ be a closed and relatively nonexpansive multi-valued mapping. Let $x_0 \in C$, $C_0 = C$ and $\{x_n\}$ be a sequence generated by

$$\begin{cases} y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)Jz_n), \\ z_n = J^{-1}(\beta_n Jx_n + (1-\beta_n)Jw_n), \quad w_n \in Tx_n, \\ C_{n+1} = \{v \in C_n : \phi(v, y_n) \le \phi(v, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}}x_0, \quad \forall n \ge 0, \end{cases}$$

where $\Pi_{C_{n+1}}$ is the generalized projection of X onto C_{n+1}, $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in [0,1] satisfies the following conditions:

(a) $\liminf_{n\to\infty} \beta_n(1 - \beta_n) > 0$;

(b) $0 \le \alpha_n \le \alpha < 1$ for some $\alpha \in (0, 1)$.

If $F(T)$ is a nonempty and bounded subset of C, then the sequence $\{x_n\}$ converges strongly to $\Pi_{F(T)}x_0$.

Author details

[1]Department of Mathematics, Yibin University, Yibin, Sichuan 644007, China [2]College of statistics and mathematics, Yunnan University of Finance and Economics, Kunming, Yunnan 650221, China

Authors' contributions

All the authors contributed equally to the writing of the present article. And they also read and approved the final manuscript.

Competing interests

The authors declare that they have no competing interests.

Received: 6 December 2011 Accepted: 19 April 2012 Published: 19 April 2012

References

1. Cioranescu, I: Geometry of Banach spaces, Duality Mappings and Nonlinear Problems. Kluwer Academic Publishers, Dordrecht (1990)
2. Alber, YI: Metric and generalized projection operators in Banach space: properties and application. In: Kartosator AG (ed.) Theory and Applications of Nonlinear Operators of Accretive and Monotone Type. pp. 15–50. Marcel Dekker, New York (1996)

3. Matsushita, S, Takahashi, W: A strong convergence theorem for relatively nonexpansive mappings in Banach spaces. J Approx Theory. **134**, 257–266 (2005)
4. Plubtieng, S, Ungchittrakool, K: Hybrid iterative method for convex feasibility problems and fixed point problems of relatively nonexpansive mappings in Banach spaces. Fixed Point Theory Appl **2008**, 19 (2008). (Article ID 583082) doi:10.1155/2008/58308
5. Chang, SS, Kim, JK, Wang, XR: Modified block iterative algorithm for solving convex feasibility problems in Banach spaces. J Inequal Appl **2010**, 14 (2010). (Article ID 869684) doi:10.1155/2010/869684
6. Chang, SS, Joseph Lee, HW, Chan, CK, Yang, : Approximation theorems for total quasi-ϕ-asymptotically nonexpansive mappings with applications. Appl Math Comput. **218**, 2921–2931 (2011)
7. Homaeipour, S, Razani, A: Weak and strong convergence theorems for relatively nonex-pansive multi-valued mappings in Banach spaces. Fixed Point Theorem Appl **73** (2011). doi:10.1186/1687-1812
8. Xu, HK: Inequalities in Banach spaces with applications. Nonlinear Anal. **16**, 1127–1138 (1991)

doi:10.1186/1687-1812-2012-63
Cite this article as: Tang and Chang: **Strong convergence theorems for total quasi-φ-asymptotically nonexpansive multi-valued mappings in Banach spaces.** *Fixed Point Theory and Applications* 2012 **2012**:63.