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A generalization of Geraghty's theorem in partially ordered metric spaces and applications to ordinary differential equations

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Abstract

The purpose of this article is to present some fixed point theorems for generalized contraction in partially ordered complete metric spaces. As an application, we give an existence and uniqueness for the solution of an initial-boundary-value problem.

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1. Introduction and preliminaries

Banach's contraction principle is one of the pivotal results of analysis. It is widely considered as the source of metric fixed point theory. Also, its significance lies in its vast applicability in a number of branches of mathematics. The existence of a fixed point, a common fixed point and a couple fixed point for some kinds of contraction type mappings in cone metric spaces, partially ordered metric spaces and fuzzy metric spaces has been considered recently by some authors [1-28] and, by using fixed point theorems, some of them have given some applications to matrix equations, ordinary differential equations, and integral equations are presented.

Let S denotes the class of the functions $\beta : [0, \infty) \to [0, 1)$ which satisfies the condition $\beta(t_n) \to 1$ implies $t_n \to 0$.

The following generalization of Banach's contraction principle is due to Geraghty [13].

Theorem 1.1. *Let (X, d) be a complete metric space and $f : X \to X$ be a mapping such that there exists $\beta \in S$ such that, for all $x, y \in X$,*

$$d\big(f(x), f(y)\big) \le \beta\big(d(x, y)\big)\, d(x, y).$$

Then f has a unique fixed point $z \in X$ and, for any choice of the initial point $x_0 \in X$, the sequence $\{x_n\}$ defined $x_n = f(x_{n-1}$ for each $n \ge 1$ converges to the point z.

Very recently, Amini-Harandi and Emami [3] proved the following existence theorem:

Theorem 1.2. *Let $(X, \le)$ be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space. Let $f : X \to X$ be an increasing mapping such that there exists $x_0 \in X$ with $x_0 \le f(x_0)$. Suppose that there exists β*

$\in$ S such that

$$d\left(f(x),\ f(y)\right) \le\ \beta\left(d(x,\ y)\right) d(x,\ y)$$

for all $x, y \in X$ with $x \ge y$. Assume that either f is continuous or X is such that if an increasing sequence $\{x_n\}$ converges to x, then $x_n \le x$ for each $n \ge 1$.

Besides, if

for all $x, y \in X$, there exists $z \in X$ which is comparable to x and y.

Then f has a unique fixed point in X.

In this article, we give a generalization of Theorem 1.2 in the context of partially ordered complete metric spaces. Moreover, by using our result, we show the existence of solution for the following initial-value problem:

$$\begin{cases} u_t(x,\ t)\ = u_{xx}(x,\ t)\ + F(x,\ t,\ u,\ u_x), & -\infty\ <\ x\ <\ \infty, 0 <\ t\ \le\ T, \\ u(x, 0)\ = \varphi\left(x\right), & -\infty\ <\ x\ <\ \infty, \end{cases}$$

where we assume that ϕ is continuously differentiable and ϕ, ϕ' are bounded and F : $\mathbb{R} \times I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ with $F(x, t, u, u_x)$ is a continuous function.

2. The main results

We begin with the following auxiliary lemma which is useful to prove some fixed point theorems in various spaces (see [25]):

Lemma 2.1. *Let (X, d) be a metric space and $\{x_n\}$ be a sequence in X such that $\{d(x_{n+1}, x_n)\}$ is decreasing and*

$$\lim_{n\to\infty} d\left(x_{n+1},\ x_n\right)\ =\ 0.$$

If $\{x_{2n}\}$ is not a Cauchy sequence, then there exist $\varepsilon > 0$ and two sequences $\{m_k\}$, $\{n_k\}$ of positive integers such that the following four sequences tend to ε as $k \to \infty$:

$$\{d(x_{2m_k}, x_{2n_k})\},\ \{d(x_{2m_k}, x_{2n_k+1})\},\ \{d(x_{2m_k-1}, x_{2n_k})\},\ \{d(x_{2m_k-1}, x_{2n_k+1})\}.$$

Let Ψ denotes the class of the functions $\psi : [0, \infty) \to [0, \infty)$ which satisfies the following conditions:

(a) ψ is nondecreasing;

(b) ψ is sub-additive, that is, $\psi(s + t) \le \psi(s) + \psi(t)$;

(c) ψ is continuous;

(d) $\psi(t) = 0 \Leftrightarrow t = 0$.

We are now ready to state and prove our main theorem.

Theorem 2.2. *Let $(X, \le)$ be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space. Let $f : X \to X$ be a nondecreasing mapping such that there exists $x_0 \in X$ with $x_0 \le f(x_0)$. Suppose that there exist $\beta \in S$ and $\psi \in \Psi$ such that*

$$\psi\left(d\left(f\left(x\right),\ f\left(y\right)\right)\right) \le\ \beta\left(\psi\left(d\left(x,\ y\right)\right)\right)\psi\left(d\left(x,\ y\right)\right) \tag{2.1}$$

for all $x, y \in X$ with $x \ge y$. Assume that either f is continuous or X is such that

$$\text{if an increasing sequence }\{x_n\}\text{ converges to } x,\text{ then } x_n \le\ x \text{ for each } n \ge 1. \tag{2.2}$$

Then f has a fixed point.

Proof. Since $x_0 \leq f(x_0)$ and f is a nondecreasing function, we obtain, by induction, that

$$x_0 \leq f(x_0) \leq f^2(x_0) \leq f^3(x_0) \leq \cdots \leq f^n(x_0) \leq f^{n+1}(x_0) \leq \cdots .$$

Put $x_n := f^n(x_0)$ for each $n \geq 1$. Since $x_n \leq x_{n+1}$ for each $n \geq 1$, by (2.1), we have

$$\begin{aligned}\psi(d(x_{n+1}, x_{n+2})) &= \psi\left(d\left(f^{n+1}(x_0), f^{n+2}(x_0)\right)\right)\\ &\leq \beta(\psi(d(x_n, x_{n+1})))\psi(d(x_n, x_{n+1}))\\ &\leq \psi(d(x_n, x_{n+1})).\end{aligned}$$

Thus it follows that $\{\psi(d(x_n, x_{n+1}))\}$ is a nonincreasing sequence and bounded below and so $\lim_{n\to\infty} \psi(d(x_n, x_{n+1})) = r$ exists. Let $\lim_{n\to\infty} \psi(d(x_n, x_{n+1})) = r \geq 0$. Assume $r > 0$. Then, from (2.1), we have

$$\frac{\psi(d(x_{n+1}, x_{n+2}))}{\psi(d(x_n, x_{n+1}))} \leq \beta(\psi(d(x_n, x_{n+1}))) \leq 1$$

for each $n \geq 1$, which yields that

$$\lim_{n\to\infty} \beta(\psi(d(x_n, x_{n+1}))) = 1.$$

On the other hand, since $\beta \in S$, we have $\lim_{n\to\infty} \psi(d(x_n, x_{n+1})) = 0$ and so $r = 0$.

Now, we show that $\{x_n\}$ is a Cauchy sequence. Suppose that $\{x_n\}$ is not a Cauchy sequence. Using Lemma 2.1, we know that there exist $\varepsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that the following four sequences tend to ε as $k \to \infty$:

$$\left\{d\left(x_{2m_k}, x_{2n_k}\right)\right\}, \ \left\{d\left(x_{2m_k}, x_{2n_k+1}\right)\right\}, \ \left\{d\left(x_{2m_k-1}, x_{2n_k}\right)\right\}, \ \left\{d\left(x_{2m_k-1}, x_{2n_k+1}\right)\right\}.$$

Putting, in the contractive condition, $x = x_{2m_k-1}$ and $y = x_{2n_k}$, it follows that

$$\begin{aligned}\psi\left(d\left(x_{2m_k}, x_{2n_k+1}\right)\right) &\leq \beta\left(\psi\left(d\left(x_{2m_k-1}, x_{2n_k}\right)\right)\right)\psi\left(d\left(x_{2m_k-1}, x_{2n_k}\right)\right)\\ &\leq \psi\left(d\left(x_{2m_k-1}, x_{2n_k}\right)\right)\end{aligned}$$

and so

$$\frac{\psi\left(d\left(x_{2m_k}, x_{2n_k+1}\right)\right)}{\psi\left(d\left(x_{2m_k-1}, x_{2n_k}\right)\right)} \leq \beta\left(\psi\left(d\left(x_{2m_k-1}, x_{2n_k}\right)\right)\right) \leq 1$$

and

$$\lim_{k\to\infty} \beta\left(\psi\left(d\left(x_{2m_k-1}, x_{2n_k}\right)\right)\right) = 1.$$

Since $\beta \in S$, it follows that $\lim_{k\to\infty} \psi\left(d\left(x_{2m_k-1}, x_{2n_k}\right)\right) = 0$. Since ψ is a continuous mapping, $\psi(\varepsilon) = 0$ and so $\varepsilon = 0$, which contradicts $\varepsilon > 0$. Therefore, $\{x_n\}$ is a Cauchy sequence in (X, d). Since (X, d) is a complete metric space, there exists $z \in X$ such that $\lim_{n\to\infty} x_n = z$.

Now, we show that z is a fixed point of f.

If f is continuous, then

$$z = \lim_{n\to\infty} f^n(x_0) = \lim_{n\to\infty} f^{n+1}(x_0) = f(\lim_{n\to\infty} f^n(x_0)) = f(z)$$

and hence $f(z) = z$.

If (2.2) holds, then we have

$$d(f(z), z) \leq d(f(z), f(x_n)) + d(f(x_n), z).$$

On the other hand, since ψ is nondecreasing and sub-additive, we have

$$\begin{aligned}\psi(d(f(z), z)) &\leq \psi(d(f(z), f(x_n))) + \psi(d(f(x_n), z))\\ &\leq \beta(\psi(d(z, x_n)))\psi(d(z, x_n)) + \psi(d(x_{n+1}, z))\\ &\leq \psi(d(z, x_n)) + \psi(d(x_{n+1}, z)).\end{aligned}$$

Since $d(z, x_n) \to 0$, $\psi(d(z, x_n)) \to 0$ and so

$$\psi(d(f(z), z)) = 0 \quad \Leftrightarrow \quad d(f(z), z) = 0.$$

Therefore, we get $f(z) = z$. this completes the proof. □

In the following, we give a sufficient condition for the uniqueness of the fixed point in Theorem 2.2. This condition is as follows:

$$\textit{every pair of elements in } X \textit{ has a lower bound or an upper bound.} \tag{2.3}$$

In [20], it is proved that the condition (2.3) is equivalent to the following:

$$\textit{for every } x, y \in X, \textit{ there exists } z \in X \textit{ which is comparable to } x \textit{ and } y. \tag{2.4}$$

Theorem 2.3. *Adding the condition* (2.4) *to the hypotheses of Theorem 2.2, the fixed point z of f is unique.*

Proof. Let y be another fixed point of f. From (2.4), there exists $x \in X$ which is comparable to y and z. The monotonicity implies that $f^n(x)$ is comparable to $f^n(y) = y$ and $f^n(z) = z$ for $n \geq 0$. Moreover, we have

$$\begin{aligned}\psi(d(z, f^n(x))) &= \psi(d(f^n(z), f^n(x)))\\ &= \psi(d(f(f^{n-1}(z)), f(f^{n-1}(x))))\\ &\leq \beta(\psi(d(f^{n-1}(z), f^{n-1}(x)))) \,.\, \psi(d(f^{n-1}(z), f^{n-1}(x)))\\ &\leq \psi(d(f^{n-1}(z), f^{n-1}(x)))\\ &= \psi(d(z, f^{n-1}(x))).\end{aligned} \tag{2.5}$$

Consequently, the sequence $\{\gamma_n\}$ defined by $\gamma_n = \psi(d(z, f^n(x)))$ is nonnegative and nonincreasing and so

$$\lim_{n\to\infty} \psi(d(z, f^n(x))) = \gamma \geq 0.$$

Now, we show that $\gamma = 0$. Assume that $\gamma > 0$. By passing to the subsequences, if necessary, we may assume that $\lim_{n\to\infty} \beta(\gamma_n) = \lambda$ exists. From (2.5), it follows that $\lambda\gamma = \gamma$ and so $\lambda = 1$. Since $\beta \in S$,

$$\gamma = \lim_{n\to\infty} \gamma_n = \lim_{n\to\infty} \psi(d(z, f^n(x))) = 0.$$

This is a contradiction and so $\gamma = 0$.

Similarly, we can prove that

$$\lim_{n\to\infty} \psi(d(y, f^n(x))) = 0.$$

Finally, from $d(z, y) \leq d(z, f^n(x)) + d(f^n(x), y)$, it follows that

$$\psi(d(z, y)) \leq \psi(d(z, f^n(x))) + \psi(d(f^n(x), y))$$

since ψ is nondecreasing and sub-additive. Therefore, taking $n \to \infty$, we have $\psi(d(z, y)) = 0$.

It follows that $d(z, y) = 0$ and so $z = y$. This completes the proof. □

3. Applications

In this section, we show the existence of solution for the following initial-value problem by using Theorems 2.2 and 2.3:

$$\begin{cases} u_t(x, t) = u_{xx}(x, t) + F(x, t, u, u_x), & -\infty < x < \infty, \ 0 < t \leq T, \\ u(x, 0) = \varphi(x), & -\infty < x < \infty, \end{cases} \tag{3.1}$$

where we assume that ϕ is continuously differentiable and that ϕ and ϕ' are bounded and $F(x, t, u, u_x)$ is a continuous function.

Definition 3.1. By a *solution* of an initial-boundary-value problem for any $u_t = u_{xx} + F(x, t, u, u_x)$ in $\mathbb{R} \times I$, where $I = [0, T]$, we mean a function $u = u(x, t)$ defined in $\mathbb{R} \times I$ such that

(a) $u \in C(\mathbb{R} \times I)$,

(b) u_t, u_x and $u_{xx} \in C(\mathbb{R} \times I)$,

(c) u and u_x are bounded in $\mathbb{R} \times I$,

(d) $u_t(x, t) = u_{xx}(x, t) + F(x, t, u(x, t), u_x(x, t))$ for all $(x, t) \in \mathbb{R} \times I$.

Now, we consider the space

$$\Omega = \{v(x, t) : \quad v, v_x \in C(\mathbb{R} \times I \text{ and } ||v|| < \infty\},$$

where

$$||v|| = \sup_{x\in\mathbb{R}, t\in I} |v(x, t)| + \sup_{x\in\mathbb{R}, t\in I} |v_x(x, t)|.$$

The set Ω with the norm $||\cdot||$ is a Banach space. Obviously, the space with the metric given by

$$d(u, v) = \sup_{x\in\mathbb{R}, t\in I} |u(x, t) - v(x, t)| + \sup_{x\in\mathbb{R}, t\in I} |u_x(x, t) - v_x(x, t)|$$

is a complete metric space. The set Ω can also equipped with a partial order given by

$$u, v \in \Omega, \quad u \leq v \quad \Leftrightarrow \quad u(x,t) \leq v(x,t), \ u_x(x,t) \leq v_x(x,t)$$

for any $x \in \mathbb{R}$ and $t \in I$. Obviously, $(\Omega, \leq)$ satisfies the condition (2.4) since, for any $u, v \in \Omega$, the functions $\max\{u, v\}$ and $\min\{u, v\}$ are the least upper and greatest lower bounds of u and v, respectively.

Taking a monotone nondecreasing sequence $\{v_n\} \subseteq \Omega$ converging to v in Ω, for any $x \in \mathbb{R}$ and $t \in I$,

$$v_1(x,t) \leq v_2(x,t) \leq v_3(x,t) \leq \cdots \leq v_n(x,t) \leq \cdots$$

and

$$\nu_{1_x}(x,t) \leq \nu_{2_x}(x,t) \leq \nu_{3_x}(x,\ t) \ \leq \cdots \ \leq \nu_{n_x}(x,t) \leq \cdots .$$

Further, since the sequences $\{\nu_n(x,\ t)\}$ and $\{\nu_{n_x}(x,t)\}$ of real numbers converge to $\nu(x,\ t)$ and $\nu_x(x,\ t)$, respectively, it follows that, for all $x \in \mathbb{R}$, $t \in I$ and $n \geq 1$,

$$\nu_n(x,t) \leq \nu(x,t)$$

and

$$\nu_{n_x}(x,t) \leq \nu_x(x,t).$$

Therefore, $\nu_n \leq \nu$ for all $n \geq 1$ and so $(\Omega, \leq)$ with the above mentioned metric satisfies the condition (2.2).

Definition 3.2. A *lower solution* of the initial-value problem (3.1) is a function $u \in \Omega$ such that

$$\begin{cases} u_t \leq u_{xx} + F(x,t,u,u_x), & -\infty < x < \infty,\ 0 < t \leq T, \\ u(x,\ 0) \leq \varphi(x), & -\infty < x < \infty, \end{cases}$$

where we assume that ϕ is continuously differentiable ϕ and ϕ' are bounded, the set Ω is defined in above and $F\ (x,\ t,\ u,\ u_x)$ is a continuous function.

Theorem 3.3. *Consider the problem* (*3.1*) *with* $F : \mathbb{R} \times I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *continuous and assume the following:*

(1) *for any* $c > 0$ *with* $|s| < c$ *and* $|p| < c$, *the function* $F\ (x,\ t,\ s,\ p)$ *is uniformly Hölder continuous in X and t for each compact subset of* $\mathbb{R} \times I$;

(2) *there exists a constant* $c_F \leq (T + 2\pi^{\frac{-1}{2}} T^{\frac{1}{2}})^{-1}$ *such that*

$$0 \leq F(x,t,s_2,p_2) - F(x,t,s_1,p_1) \leq c_F(\ln(s_2 - s_1 + p_2 - p_1 + 1))$$

for all $(s_1,\ p_1)$ *and* $(s_2,\ p_2)$ *in* $\mathbb{R} \times \mathbb{R}$ *with* $s_1 \leq s_2$ *and* $p_1 \leq p_2$;

(3) *F is bounded for bounded s and p.*

Then the existence of a lower solution for the initial-value problem (*3.1*) *provides the existence of the unique solution of the problem* (*3.1*).

Proof. The problem (3.1) is equivalent to the integral equation

$$u(x,t) = \int_{-\infty}^{\infty} k(x-\xi,\ t)\varphi(\xi)d\xi + \int_0^t \int_{-\infty}^{\infty} k(x-\xi,\ t-\tau)F(\xi,\ \tau,\ u(\xi,\ \tau),\ u_x(\xi,\ \tau))d\xi d\tau$$

for all $x \in \mathbb{R}$ and $0 < t \leq T$, where

$$k(x,\ t) = \frac{1}{\sqrt{4\pi t}} \exp\left\{\frac{-x^2}{4t}\right\}$$

for all $x \in \mathbb{R}$ and $t > 0$. The initial-value problem (3.1) possesses a unique solution if and only if the above integral-differential equation possesses a unique solution u such that u and u_x are continuous and bounded for all $x \in \mathbb{R}$ and $0 < t \leq T$.

Define a mapping F: $\Omega \to \Omega$ by

$$(Fu)\,(x,t) = \int_{-\infty}^{\infty} k(x-\xi,t)\varphi(\xi)d\xi + \int_0^t \int_{-\infty}^{\infty} k(x-\xi,\ t-\tau)F(\xi,\tau,u(\xi,\ \tau),\ u_x(\xi,\ \tau))d\xi d\tau$$

for all $x \in \mathbb{R}$ and $t \in I$. Note that, if $u \in \Omega$ is a fixed point of **F**, then u is a solution of the problem (3.1).

Now, we show that the hypothesis in Theorems 2.2 and 2.3 are satisfied. The mapping F is nondecreasing since, by the hypothesis, for $u \geq v$,

$$F(x,\ t,\ u(x,\ t),\ u_x(x,\ t)) \geq F(x,\ t,\ v(x,\ t),\ v_x(x,\ t)).$$

By using that $k(x, t) > 0$ for all $(x, t) \in \mathbb{R} \times (0, T\,]$, we conclude that

$$\begin{aligned}(Fu)(x,\ t) &= \int_{-\infty}^{\infty} k(x-\xi,\ t)\varphi(\xi)d\xi \\ &\quad + \int_0^t \int_{-\infty}^{\infty} k(x-\xi,\ t-\tau)F(\xi,\ \tau,\ u(\xi,\ \tau),\ u_x(\xi,\ \tau))d\xi d\tau \\ &\geq \int_{-\infty}^{\infty} k(x-\xi,\ t)\varphi(\xi)d\xi + \int_0^t \int_{-\infty}^{\infty} k(x-\xi,\ t-\tau)F(\xi,\ \tau,\ v(\xi,\ \tau),\ v_x(\xi,\ \tau))d\xi d\tau \\ &= (Fv)\,(x,\ t)\end{aligned}$$

for all $x \in \mathbb{R}$ and $t \in I$. Besides, we have

$$\begin{aligned}&|(Fu)(x,t) - (Fv)\,(x,\ t)| \\ &\leq \int_0^t \int_{-\infty}^{\infty} k(x-\xi,t-\tau)|F(\xi,\tau,u(\xi,\tau),u_x(\xi,\tau)) - F(\xi,\tau,v(\xi,\tau),v_x(\xi,\tau))|d\xi d\tau \\ &\leq \int_0^t \int_{-\infty}^{\infty} k(x-\xi,\ t-\tau)\ \cdot c_F \ln(u(\xi,\ \tau) - v(\xi,\ \tau) + u_x(\xi,\ \tau) - v_x(\xi,\ \tau) + 1)d\xi d\tau \qquad (3.2) \\ &\leq c_F \cdot \ln(d(u,v)+1) \int_0^t \int_{-\infty}^{\infty} k(x-\xi,t-\tau)d\xi d\tau \\ &\leq c_F \cdot \ln(d(u,\ v)+1)\ \cdot\ T\end{aligned}$$

for all $u \geq v$. Similarly, we have

$$\begin{aligned}\left|\frac{\partial Fu}{\partial x}(x,\ t) - \frac{\partial Fv}{\partial x}(x,\ t)\right| &\leq c_F \cdot \ln(d(u,\ v)+1) \int_0^t \int_{-\infty}^{\infty} |\frac{\partial k}{\partial x}(x-\xi,\ t-\tau)|\,d\xi d\tau \\ &\leq c_F \cdot \ln(d(u,\ v)+1)\ \cdot\ 2\pi^{\frac{-1}{2}}T^{\frac{1}{2}}.\end{aligned} \qquad (3.3)$$

Combining (3.2) with (3.3), we obtain

$$d(Fu,\ Fv) \leq c_F(T + 2\pi^{\frac{-1}{2}}T^{\frac{1}{2}})\ln(d(u,v)+1) \leq \ln(d(u,\ v)+1),$$

which implies that

$$\begin{aligned}\ln(d(Fu,\ Fv)+1) &\leq \ln(\ln(d(u,\ v)+1)+1) \\ &= \frac{\ln(\ln(d(u,v)+1)+1)}{\ln(d(u,v)+1)}.\ln(d(u,\ v)+1).\end{aligned}$$

Put $\psi(x) = \ln(x+1)$ and $\beta(x) = \frac{\psi(x)}{x}$. Obviously, $\psi : [0, \infty) \to [0, \infty)$ is continuous, sub-additive, nondecreasing $(\psi'(x) = \frac{1}{x+1} > 0)$ and ψ is positive in $(0, \infty)$ with $\psi(0) = 0$ and also $\psi(x) < x$ for any $x > 0$ and $\beta \in S$.

Finally, let $\alpha(x, t)$ be a lower solution for (3.1). Then we show that $\alpha \leq F\alpha$. Integrating the following:

$$(\alpha(\xi, \tau)k(x-\xi, t-\tau))_\tau - (\alpha_\xi(\xi, \tau)k(x-\xi, t-\tau))_\xi + (\alpha(\xi, \tau)k_\xi(x-\xi, t-\tau))_\xi \\ \leq F(\xi, \tau, \alpha(\xi, \tau), \alpha_\xi(\xi, \tau))k(x-\xi, t-\tau)$$

for $-\infty < \xi < \infty$ and $0 < \tau < t$, we obtain the following:

$$\alpha(x, t) \leq \int_{-\infty}^{\infty} k(x-\xi, t)\varphi(\xi)d\xi + \int_0^t \int_{-\infty}^{\infty} k(x-\xi, t-\tau)F(\xi, \tau, \alpha(\xi, \tau), \alpha_\xi(\xi, \tau))d\xi d\tau \\ = (F\alpha)(x, t)$$

for all $x \in \mathbb{R}$ and $t \in (0, T]$. Therefore, by Theorems 2.2 and 2.3, F has a unique fixed point. This completes the proof. □

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Authors' contributions
All authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

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