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Common fixed point and approximation results for generalized (*f*, *g*)-weak contractions

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Abstract

The existence of common fixed points is established for three mappings where T is generalized (f, g)-weakly contractive mapping on a nonempty subset of a Banach space. As applications, the invariant approximation results are proved. Our results unify and improve several recent results in the literature.

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1. Introduction and preliminaries

We first review needed definitions. Let (X, d) be a metric space. A map $T : X \to X$ is called *weakly contractive* if, for each $x, y \in X$,

$$d(Tx, Ty) \leq d(x, y) - \phi(d(x, y)),$$

where φ: $[0, \infty) \to [0, \infty)$ is a lower semicontinuous function from right such that φ is positive on $(0, \infty)$ and $\varphi(0) = 0$.

A map $T : X \to X$ is called (f, g)-*weakly contractive* if, for each $x, y \in X$,

$$d(Tx, Ty) \leq d(fx, gy) - \phi(d(fx, gy)), \tag{1.1}$$

where $f, g : X \to X$ are self-mappings and φ: $[0, \infty) \to [0, \infty)$ is a lower semicontinuous function from right such that φ is positive on $(0, \infty)$ and $\varphi(0) = 0$. If $g = f$, then T is called f-*weakly contractive*. If $f = I$, the identity operator, then T is called *weakly contractive*. Note that if $g = f = I$ and φ is continuous nondecreasing, then the definition of (f, g)-weakly contractive maps is same as the one which appeared in [1,2]. Further if $f = I$ and $\varphi(t) = (1 - k)t$ for a constant k with $0 < k < 1$, then an f-weakly contractive mapping is called a *contraction*. Also note that if $f = g = I$ and φ is lower semicontinuous from the right, then $\psi(t) = t - \varphi(t)$ is upper semicontinuous from the right and the condition (1.1) is replaced by

$$d(Tx, Ty) \leq \psi(d(x, y)).$$

Therefore (f, g)-weakly contractive maps for which φ is lower semicontinuous from the right are of the type of Boyd and Wong [3]. And if we set $k(t) = 1 - \varphi(t)/t$ for $t > 0$ and $k(0) = 0$ together with $f = g = I$, then the condition (1.1) is replaced by

$$d\left(Tx, Ty\right) \leq k\left(d\left(x, y\right)\right) d\left(x, y\right).$$

Therefore (f, g)-weakly contractive maps are closely related to the maps studied by Mizoguchi and Takahashi [4].

If $\varphi(t) = (1 - k)t$ for a constant k with $0 < k < 1$, then an (f, g)-weakly contractive mapping is called a *(f, g)-contraction*, which has been investigated by Hussain and Jungck [5], Jungck and Hussain [6], Song [7] and many others.

The set of fixed points of T is denoted by $F(T)$. A point $x \in X$ is a coincidence point (common fixed point) of f and T if $fx = Tx$ ($x = fx = Tx$). The set of coincidence points of f and T is denoted by $C(f, T)$. The pair $\{f, T\}$ is called;

(1) *commuting* [8] if $T fx = fTx$ for all $x \in M$;

(2) *compatible* (see [6,9]) if $\lim_n d(Tfx_n, fTx_n) = 0$ whenever $\{x_n\}$ is a sequence such that $\lim_n Tx_n = \lim_n fx_n = t$ for some t in M;

(3) *weakly compatible* [10] if they commute at their coincidence points, i.e., if $fTx = Tfx$ whenever $fx = Tx$;

(4) *Banach operator pair*, if the set $F(f)$ is T-invariant, namely $T(F(f)) \subseteq F(f)$. Obviously, commuting pair (T, f) is a Banach operator pair but converse is not true in general; see [11-13]. If (T, f) is a Banach operator pair, then (f, T) need not be a Banach operator pair (cf. [[11], Example 1]).

The set M in a linear space X is called *q-starshaped* with $q \in M$, if the segment $[q, x] = \{(1 - k)q + kx : 0 \leq k \leq 1\}$ joining q to x is contained in M for all $x \in M$. The map f defined on a q-starshaped set M is called *affine* if

$$f\left((1-k)q + kx\right) = (1-k)fq + kfx, \quad \text{for all } x \in M.$$

Suppose that M is q-starshaped with $q \in F(f)$ and is both T- and f-invariant. Then T and f are called (5) *pointwise R-subweakly commuting* [14] if for given $x \in M$, there exists a real number $R > 0$ such that $||fTx - Tfx|| \leq R\text{dist}(fx, [q, Tx])$ (6) *R-subweakly commuting* on M (see [5]) if for all $x \in M$, there exists a real number $R > 0$ such that $||fTx - Tfx|| \leq R\text{dist}(fx, [q, Tx])$; (7) *$C_q$-commuting* (see [6,7] if $fTx = Tfx$ for all $x \in C_q(f, T)$, where $C_q(f, T) = \cup\{C(f, T_k): 0 \leq k \leq 1\}$ where $T_kx = (1 - k)q + kTx$.

A Banach space X satisfies *Opial's condition* if, for every sequence $\{x_n\}$ in X weakly convergent to $x \in X$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \left\|x_n - y\right\|$$

holds for all $y \neq x$. Every Hilbert space and the space $l_p(1 < p < \infty)$ satisfy Opial's condition. The map $T : M \to X$ is said to be *demiclosed at* 0 if, for every sequence $\{x_n\}$ in M converging weakly to x and $\{Tx_n\}$ converges to $0 \in X$, then $0 = Tx$.

Let M be a subset of a normed space $(X, ||\cdot||)$. The set $P_M(u) = \{x \in M : ||x - u|| = \text{dist}(u, M)\}$ is called the set of best approximants to $u \in X$ out of M, where $\text{dist}(u, M) = \inf\{||y - u||: y \in M\}$. We denote by $\mathbb{N}$ and cl(M) (wcl(M)), the set of positive integers and the closure (weak closure) of a set M in X, respectively.

The concept of the weak contractive mapping has been defined by Alber and Guerre-Delabriere [1]. Actually, in [1], the authors proved the existence of fixed points for a single-valued weakly contractive mapping on Hilbert spaces. In 2001, Rhoades [[2], Theorem 2] obtained a generalization of Banach's contraction mapping principle [Note the weakly con-traction contains contraction as the special case ($\varphi(t) = (1\text{-}k)t$)].

Recently, Chen and Li [11] introduced the class of Banach operator pairs, as a new class of noncommuting maps and it has been further studied by Ciric et al. [15,16], Hussain [12,13], Hussain et al. [17], Khan and Akbar [18,19], Pathak and Hussain [20], Song and Xu [21] and Akbar and Khan [22].

In this article, we introduce the new concept of generalized (f, g)-weakly contractive map-pings, and consequently establish common fixed point and invariant best approximation results for the noncommuting generalized (f, g)-weakly contractive mapping. Our results improve and extend the recent common fixed point and invariant approximation results of Al-Thagafi [23], Al-Thagafi and Shahzad [24], Chen and Li [11], Habiniak [25], Hussain and Jungck [5], Jungck and Hussain [6], Jungck and Sessa [26], Pathak and Hussain [20], Sahab et al. [27], Singh [28,29], Song [7] and Song and Xu [21] to the class of (f, g)-weakly contractive maps. The applications of fixed point theorems are remarkable in diverse disciplines of mathematics, statistics, engineering and economics in dealing with the problems arising in approximation theory, potential theory, game theory, theory of differential equations, theory of integral equations and others (see [20,30,31]).

2. Results for *(f, g)*-weak contractions

The following result is a particular case of Song [[32], Theorem 3.1].

Lemma 2.1. Let M be a nonempty subset of a metric space (X, d), and T be a self-map of M. Assume that $\mathrm{cl}T(M) \subset M$, $\mathrm{cl}T(M)$ is complete, and T is weakly contractive mapping. Then $M \cap F(T)$ is singleton.

Theorem 2.2. Let M be a nonempty subset of a metric space (X, d), and T, f and g be self-maps of M. Assume that $F(f)\cap F(g)$ is nonempty, $\mathrm{cl}T(F(f)\cap F(g)) \subseteq F(f)\cap F(g)$, $\mathrm{cl}(T(M))$ is complete, and T is (f, g)-weakly contractive mapping. Then $M \cap F(T) \cap F(f) \cap F(g)$ is singleton.

Proof. $\mathrm{cl}(T(F(f) \cap F(g)))$ being subset of $\mathrm{cl}(T(M))$ is complete. Further, for all $x, y \in F(f) \cap F(g)$, we have by (f, g)-weak contractiveness of T,

$$d(Tx, Ty) \le d(fx, gy) - \phi(d(fx, gy)) = d(x, y) - \phi(d(x, y))$$

Hence T is weakly contractive mapping on $F(f)\cap F(g)$ and $\mathrm{cl}T(F(f)\cap F(g)) \subseteq F(f)\cap F(g)$. By Lemma 2.1, T has a unique fixed point z in $F(f) \cap F(g)$ and consequently, $M \cap F(T) \cap F(f) \cap F(g)$ is singleton.

Corollary 2.3. Let M be a nonempty subset of a metric space (X, d), and (T, f) and (T, g) be Banach operator pairs on M. Assume that $\mathrm{cl}(T(M))$ is complete, T is (f, g)-weakly contractive mapping and $F(f)\cap F(g)$ is nonempty and closed. Then $M\cap F(T)\cap F(f)\cap F(g)$ is singleton.

Corollary 2.4. Let M be a nonempty subset of a metric space (X, d), and T, f and g be self-maps of M. Assume that $F(f) \cap F(g)$ is nonempty, $\mathrm{cl}T(F(f) \cap F(g)) \subseteq F(f) \cap F(g)$, $\mathrm{cl}(T(M))$ is complete. If T satisfies the following inequality for all $x, y \in M$,

$$d(Tx, Ty) \le \psi(d(fx, gy)) \tag{2.1}$$

where $\psi: [0, \infty) \to [0, \infty)$ is upper semicontinuous from right such that $\psi(0) = 0$ and $\psi(t) < t$ for each $t > 0$. Then $M \cap F(T) \cap F(f) \cap F(g)$ is singleton.

Proof. Set $\varphi(t) = t - \psi(t)$. Then inequality (2.1) implies

$$d(Tx, Ty) \leq d(fx, gy) - \phi(d(fx, gy)),$$

where φ: $[0, \infty) \to [0, \infty)$ is a lower semicontinuous function from right such that $\varphi(t) > 0$ for $t > 0$ and $\varphi(0) = 0$. The result follows from Theorem 2.2.

Corollary 2.5. Let M be a nonempty subset of a metric space (X, d), and T, f and g be self-maps of M. Assume that $F(f) \cap F(g)$ is nonempty, cl$T$ $(F(f) \cap F(g)) \subseteq F(f) \cap F(g)$, cl$(T(M))$ is complete. If T satisfies the following inequality for all $x, y \in M$,

$$d(Tx, Ty) \leq \alpha(d(fx, gy))\, d(fx, gy) \tag{2.2}$$

where α: $[0, \infty) \to (0, 1)$ is an upper semicontinuous from right. Then $M \cap F(T) \cap F(f) \cap F(g)$ is singleton.

Proof. Set $\varphi(t) = (1 - \alpha(t))t$, then inequality (2.2) implies

$$d(Tx, Ty) \leq d(fx, gy) - \phi(d(fx, gy)),$$

where φ: $[0, \infty) \to [0, \infty)$ is a lower semicontinuous function from right such that $\varphi(t) > 0$ for $t > 0$ and $\varphi(0) = 0$. The result follows from Theorem 2.2.

In Corollary 2.3, if $\varphi(t) = (1 - k)t$ for a constant k with $0 < k < 1$, and $f = g$, then we easily obtain the following result which improves Lemma 3.1 of Chen and Li [11].

Corollary 2.6. Let M be a nonempty subset of a metric space (X, d), and (T, f) be a Banach operator pair on M. Assume that cl$(T(M))$ is complete, T is f-contraction and $F(f)$ is nonempty and closed. Then $M \cap F(T) \cap F(f)$ is singleton.

The following result properly contains Theorems 3.2-3.3 of [11], Theorem 2.2 of [23], Theorem 4 of [25] and Theorem 6 of [26].

Theorem 2.7. Let M be a nonempty subset of a normed [resp. Banach] space X and T, f and g be self-maps of M. Suppose that $F(f) \cap F(g)$ is q-starshaped, cl$T(F(f) \cap F(g)) \subseteq F(f) \cap F(g)$ [resp. wcl$T$ $(F(f) \cap F(g)) \subseteq F(f) \cap F(g)$], cl$(T(M))$ is compact [resp. wcl$(T(M))$ is weakly compact], T is continuous on M [resp.$id - T$ is demiclosed at 0, where id stands for identity map] and

$$||Tx - Ty|| \leq \frac{||fx - gy||}{k} - \phi(||fx - gy||), \tag{2.3}$$

for all $k \in (0, 1)$ and $x, y \in M$ where φ: $[0, \infty) \to [0, \infty)$ is a lower semicontinuous function from right such that φ is positive on $(0, \infty)$ and $\varphi(0) = 0$. Then $M \cap F(T) \cap F(f) \cap F(g) \neq \emptyset$.

Proof. Define T_n: $F(f) \cap F(g) \to F(f) \cap F(g)$ by $T_n x = (1 - k_n)q + k_n Tx$ for all $x \in F(f) \cap F(g)$ and a fixed sequence of real numbers $k_n (0 < k_n < 1)$ converging to 1. Since $F(f) \cap F(g)$ is q-starshaped and cl$T$ $(F(f) \cap F(g)) \subseteq F(f) \cap F(g)$ [resp. wcl$T$ $(F(f) \cap F(g)) \subseteq F(f) \cap F(g)$], so cl$T_n(F(f) \cap F(g)) \subseteq F(f) \cap F(g)$] [resp. wcl$T_n(F(f) \cap F(g)) \subseteq F(f) \cap F(g)$] for each $n \geq 1$. Let $\varphi_n := k_n \varphi$. Then by (2.3),

$$\begin{aligned} ||T_n x - T_n y|| &= k_n ||Tx - Ty|| \\ &\leq k_n \left(\frac{||fx - gy||}{k_n} - \phi(||fx - gy||) \right) \\ &\leq ||fx - gy|| - k_n \phi(||fx - gy||) \\ &= ||fx - gy|| - \phi_n(||fx - gy||), \end{aligned}$$

for each $x, y \in F(f) \cap F(g)$ and for each $n \in \mathbb{N}$, $\varphi_n: [0, \infty) \to [0, \infty)$ is a lower semicontinuous function from right such that φ_n is positive on $(0, \infty)$ and $\varphi_n(0) = 0$.

If $\mathrm{cl}(T(M))$ is compact, for each $n \in \mathbb{N}$, $\mathrm{cl}(T_n(F(f)\cap F(g)))$ is compact and hence complete. By Theorem 2.2, for each $n \in \mathbb{N}$ there exists $x_n \in F(f) \cap F(g)$ such that $x_n = fx_n = gx_n = T_n x_n$. The compactness of $\mathrm{cl}(T(M))$ implies that there exists a subsequence $\{Tx_m\}$ of $\{Tx_n\}$ such that $Tx_m \to z\ \mathrm{cl}(T(M))$ as $m \to \infty$. Since $\{Tx_m\}$ is a sequence in $T(F(f) \cap F(g))$ and $\mathrm{cl}T(F(f) \cap F(g)) \subseteq F(f) \cap F(g)$, therefore $z \in F(f) \cap F(g)$. Further, $x_m = T_m x_m = (1 - k_m)q + k_m Tx_m \to z$. By the continuity of T, we obtain $Tz = z$. Thus, $M \cap F(T) \cap F(f) \cap F(g) \neq \emptyset$ proves the first case.

The weak compactness of $\mathrm{wcl}(T(M))$ implies that $\mathrm{wcl}(T_n(F(f)\cap F(g)))$ is weakly compact and hence complete due to completeness of X. From Theorem 2.2, for each $n \geq 1$, there exists $x_n \in F(f) \cap F(g)$ such that $x_n = fx_n = gx_n = T_n x_n$. Moreover, we have $||x_n - Tx_n|| \to 0$ as $n \to \infty$. The weak compactness of $\mathrm{wcl}(T(M))$ implies that there is a subsequence $\{Tx_m\}$ of $\{Tx_n\}$ converging weakly to $y \in \mathrm{wcl}(T(M))$ as $m \to \infty$. Since $\{Tx_m\}$ is a sequence in $T(F(f)\cap F(g))$, therefore $y \in \mathrm{wcl}(T(F(f)\cap F(g))) \subseteq F(f)\cap F(g)$. Also we have, $x_m - Tx_m \to 0$ as $m \to \infty$. If $id - T$ is demiclosed at 0, then $y = Ty$. Thus $M \cap F(T) \cap F(f) \cap F(g) \neq \emptyset$.

Corollary 2.8. Let M be a nonempty subset of a normed [resp. Banach] space X and T, f and g be self-maps of M. Suppose that $F(f) \cap F(g)$ is q-starshaped and closed [resp. weakly closed], $\mathrm{cl}(T(M))$ is compact [resp. $\mathrm{wcl}(T(M))$ is weakly compact], T is continuous on M [resp.id-T is demiclosed at 0], (T, f) and (T, g) are Banach operator pairs and satisfy (2.3) for all $x, y \in M$. Then $M \cap F(T) \cap F(f) \cap F(g) \neq \emptyset$.

In Theorem 2.7 and Corollary 2.8, if $\phi(t) = (\frac{1}{k} - 1)t$ for any constant k with $0 < k < 1$, and $g = f$, then we easily obtain the following results.

Corollary 2.9. [[24], Theorem 2.4] Let M be a nonempty subset of a normed [resp. Banach] space X and T and f be self-maps of M. Suppose that $F(f)$ is q-starshaped, $\mathrm{cl}T(F(f)) \subseteq F(f)$ [resp. $\mathrm{wcl}T(F(f)) \subseteq F(f)$], $\mathrm{cl}(T(M))$ is compact [resp. $\mathrm{wcl}(T(M))$ is weakly compact and either id-T is demiclosed at 0 or X satisfies Opial's condition] and T is f-nonexpansive on M. Then $F(T) \cap F(I) \neq \emptyset$.

Corollary 2.10. [[11], Theorems 3.2-3.3] Let M be a nonempty subset of a normed [resp. Banach] space X and T, f be self-maps of M. Suppose that $F(f)$ is q-starshaped and closed [resp. weakly closed], $\mathrm{cl}(T(M))$ is compact [resp. $\mathrm{wcl}(T(M))$ is weakly compact and either $id - T$ is demiclosed at 0 or X satisfies Opial's condition], (T, f) is a Banach operator pair and T is f-nonexpansive on M. Then $M \cap F(T) \cap F(f) \neq \emptyset$.

Corollary 2.11. [[23], Theorem 2.1] Let M be a nonempty closed and q-starshaped subset of a normed space X and T and f be self-maps of M such that $T(M) \subseteq f(M)$. Suppose that T commutes with f and $q \in F(f)$. If $\mathrm{cl}(T(M))$ is compact, f is continuous and linear and T is f-nonexpansive on M, then $M \cap F(T) \cap F(f) \neq \emptyset$.

Let $C = P_M(u) \cap C_M^{f,g}(u)$, where $C_M^{f,g}(u) = C_M^f(u) \cap C_M^g(u)$ and $C_M^f(u) = \{x \in M : fx \in P_M(u)\}$.

Corollary 2.12. Let X be a normed [resp. Banach] space X and T, f and g be self-maps of X. If $u \in X$, $D \subseteq C$, $D_0 := D \cap F(f) \cap F(g)$ is q-starshaped, $\mathrm{cl}(T(D_0)) \subseteq D_0$ [resp. $\mathrm{wcl}(T(D_0)) \subseteq D_0$], $\mathrm{cl}(T(D))$ is compact [resp. $\mathrm{wcl}(T(D))$ is weakly compact], T is continuous on D [resp.$id - T$ is demiclosed at 0] and (2.3) holds for all $x, y \in D$, then $P_M(u) \cap F(T) \cap F(f) \cap F(g) \neq \emptyset$.

Corollary 2.13. Let X be a normed [resp. Banach] space X and T, f and g be self-maps of X. If $u \in X$, $D \subseteq P_M(u)$, $D_0 := D \cap F(f) \cap F(g)$ is q-starshaped, $\mathrm{cl}(T(D_0)) \subseteq D_0$ [resp. $\mathrm{wcl}(T(D_0)) \subseteq D_0$], $\mathrm{cl}(T(D))$ is compact [resp. $\mathrm{wcl}(T(D))$ is weakly compact], T is continuous on D[resp.id - T is demiclosed at 0] and (2.3) holds for all $x, y \in D$, then $P_M(u) \cap F(T) \cap F(f) \cap F(g) \neq \emptyset$.

Remark 2.14. Corollary 2.5 of [24], and Theorems 4.1 and 4.2 of Chen and Li [11] and the corresponding results in [23,25-29] are particular cases of Corollaries 2.12 and 2.13.

We denote by $\Im_0$ the class of closed convex subsets of X containing 0. For $M \in \Im_0$, we define $M_u = \{x \in M : ||x|| \leq 2\ ||u||\}$. It is clear that $P_M(u) \subset M_u \in \Im_0$ (see [5,23]).

Theorem 2.15. Let f, g, T be self-maps of a normed [resp. Banach] space X. If $u \in X$ and $M \in \Im_0$ such that $T(M_u) \subseteq M$, $\mathrm{cl}(T(M_u))$ is compact [resp. $\mathrm{wcl}(T(M_u))$ is weakly compact] and $||Tx - u|| \leq ||x - u||$ for all $x \in M_u$, then $P_M(u)$ is nonempty, closed and convex with $T(P_M(u)) \subseteq P_M(u)$. If, in addition, $D \subseteq P_M(u)$, $D_0 := D \cap F(f) \cap F(g)$ is q-starshaped, $\mathrm{cl}(T(D_0)) \subseteq D_0$ [resp. $\mathrm{wcl}(T(D_0)) \subseteq D_0$], T is continuous on D [resp.id - T is demiclosed at 0] and (2.3) holds for all $x, y \in D$, then $P_M(u) \cap F(T) \cap F(f) \cap F(g) \neq \emptyset$.

Proof. We may assume that $u \notin M$. If $x \in M \backslash M_u$, then $||x|| > 2\ ||u||$. Note that

$$\|x - u\| \geq \|x\| - \|u\| > ||u|| \geq \mathrm{dist}\,(u, M).$$

Thus, $\mathrm{dist}(u, M_u) = \mathrm{dist}(u, M) \leq ||u||$. If $\mathrm{cl}(T(M_u))$ is compact, then by the continuity of norm, we get $||z - u|| = \mathrm{dist}(u, \mathrm{cl}(T(M_u)))$ for some $z \in \mathrm{cl}(T(M_u))$.

If we assume that $\mathrm{wcl}(T(M_u))$ is weakly compact, using Lemma 5.5 of [[33], p. 192] we can show the existence of a $z \in \mathrm{wcl}(T(M_u))$ such that $\mathrm{dist}(u, \mathrm{wcl}(T(M_u))) = ||z - u||$.

Thus, in both cases, we have

$$\mathrm{dist}\,(u, M_u) \leq \mathrm{dist}\,(u,\ \mathrm{cl}T\,(M_u)) \leq \mathrm{dist}\,(u, T\,(M_u)) \leq \|Tx - u\| \leq \|x - u\|,$$

for all $x \in M_u$. Hence $||z - u|| = \mathrm{dist}(u, M)$ and so $P_M(u)$ is nonempty, closed and convex with $T(P_M(u)) \subseteq P_M(u)$. The compactness of $\mathrm{cl}(T(M_u))$ [resp. weak compactness of $\mathrm{wcl}(T(M_u))$] implies that $\mathrm{cl}(T(D))$ is compact [resp. $\mathrm{wcl}(T(D))$ is weakly compact]. The result now follows from Corollary 2.13.

Remark 2.16. Theorem 2.15 extends Theorems 4.1 and 4.2 of [23], Theorem 2.6 of [24], and Theorem 8 of [25].

3. Results for generalized (*f*, *g*)-weak contractions

Definition 3.1. A map $T : X \to X$ is called *generalized weak contraction* [34] if, for each $x, y \in X$,

$$d\left(Tx, Ty\right) \leq M\left(x, y\right) - \phi\left(M\left(x, y\right)\right), \tag{3.1}$$

where φ: $[0, \infty) \to [0, \infty)$ is a lower semicontinuous function from right such that φ is positive on $(0, \infty)$, $\varphi(0) = 0$ and

$$M\left(x, y\right) = \max\left\{d\left(x, y\right), d\left(Tx, x\right), d\left(Ty, y\right), \frac{1}{2}\left[d\left(Tx, y\right) + d\left(Ty,\ x\right)\right]\right\}$$

In (3.1), if we change $M(x, y)$ by

$$m(x, y) := \max\left\{d(fx, gy), d(Tx, fx), d(Ty, gy), \frac{1}{2}\left[d(Tx, gy) + d(Ty, fx)\right]\right\}$$

then T is called *generalized* (f, g)*-weak contraction*. If

$$d(Tx, Ty) \leq m(x, y), \tag{3.2}$$

then T is called *generalized* (f, g)*-contraction* (see [7]). Notice that $m(x, y)$ coincides with $M(x, y)$ on $F(f) \cap F(g)$.

The following result is a particular case of Theorem 2.1 of Zhang and Song [34].

Lemma 3.2. Let M be a nonempty subset of a metric space (X, d), and T be a self-map of M. Assume that $\mathrm{cl}T(M) \subset M$, $\mathrm{cl}T(M)$ is complete, and T is a generalized weak contraction. Then $M \cap F(T)$ is singleton.

Theorem 3.3. Let M be a nonempty subset of a metric space (X, d), and T, f and g be self-maps of M. Assume that $F(f) \cap F(g)$ is nonempty, $\mathrm{cl}T(F(f) \cap F(g)) \subseteq F(f) \cap F(g)$, $\mathrm{cl}(T(M))$ is complete, and T is generalized (f, g)-weak contraction. Then $M \cap F(T) \cap F(f) \cap F(g)$ is singleton.

Proof. $\mathrm{cl}(T(F(f) \cap F(g)))$ being subset of $\mathrm{cl}(T(M))$ is complete. Further, for all $x, y \in F(f) \cap F(g)$, we have by generalized (f, g)-weak contractiveness of T,

$$d(Tx, Ty) \leq m(x, y) - \phi(m(x, y)) = M(x, y) - \phi(M(x, y))$$

Hence T is generalized weak contraction mapping on $F(f) \cap F(g)$ and $\mathrm{cl}T(F(f) \cap F(g)) \subseteq F(f) \cap F(g)$. By Lemma 3.2, T has a unique fixed point z in $F(f) \cap F(g)$ and consequently, $M \cap F(T) \cap F(f) \cap F(g)$ is singleton.

Corollary 3.4. Let M be a nonempty subset of a metric space (X, d), and (T, f) and (T, g) be Banach operator pairs on M. Assume that $\mathrm{cl}(T(M))$ is complete, T is generalized (f, g)-weakly contractive mapping and $F(f) \cap F(g)$ is nonempty and closed. Then $M \cap F(T) \cap F(f) \cap F(g)$ is singleton.

Corollary 3.5. Let M be a nonempty subset of a metric space (X, d), and T, f and g be self-maps of M. Assume that $F(f) \cap F(g)$ is nonempty, $\mathrm{cl}T(F(f) \cap F(g)) \subseteq F(f) \cap F(g)$, $\mathrm{cl}(T(M))$ is complete. If T satisfies the following inequality for all $x, y \in M$,

$$d(Tx, Ty) \leq \psi(m(x, y)) \tag{3.3}$$

where $\psi: [0, \infty) \to [0, \infty)$ is upper semicontinuous from right such that $\psi(0) = 0$ and $\psi(t) < t$ for each $t > 0$, then $M \cap F(T) \cap F(f) \cap F(g)$ is singleton.

Proof. Set $\varphi(t) = t - \psi(t)$. Then inequality (3.3) implies

$$d(Tx, Ty) \leq m(x, y) - \phi(m(x, y)),$$

where $\varphi: [0, \infty) \to [0, \infty)$ is lower semicontinuous function from right such that $\varphi(t) > 0$ for $t > 0$ and $\varphi(0) = 0$. The result follows from Theorem 3.3.

In Theorem 3.3 and Corollary 3.4, if $\varphi(t) = (1 - k)t$ for a constant k with $0 < k < 1$, then we easily obtain the following results which improve Lemma 3.1 of Chen and Li [11] and provide the conclusions about common fixed points in Theorem 2.1 and Corollaries 2.2 and 2.3 for different classes of maps.

Corollary 3.6. Let M be a nonempty subset of a metric space (X, d), and T, f and g be self-maps of M. Assume that $F(f) \cap F(g)$ is nonempty, $\mathrm{cl}T(F(f) \cap F(g)) \subseteq F(f) \cap F(g)$, $\mathrm{cl}(T(M))$ is complete, and T is generalized (f, g)-contraction. Then $M \cap F(T) \cap F(f) \cap F(g)$ is singleton.

Corollary 3.7. Let M be a nonempty subset of a metric space (X, d), and (T, f) and (T, g) are Banach operator pairs on M. Assume that $\mathrm{cl}(T(M))$ is complete, T is generalized (f, g)-contraction and $F(f) \cap F(g)$ is nonempty and closed. Then $M \cap F(T) \cap F(f) \cap F(g)$ is singleton.

Theorem 3.8. Let M be a nonempty subset of a normed [resp. Banach] space X and T, f and g be self-maps of M. Suppose that $F(f) \cap F(g)$ is q-starshaped, $\mathrm{cl}T(F(f) \cap F(g)) \subseteq F(f) \cap F(g)$ [resp. $\mathrm{wcl}T\,(F(f) \cap F(g)) \subseteq F(f) \cap F(g)$], $\mathrm{cl}(T(M))$ is compact [resp. $\mathrm{wcl}(T(M))$ is weakly compact], T is continuous on M [resp. $id - T$ is demiclosed at 0, where id stands for identity map] and

$$\left\|Tx - Ty\right\| \le \frac{n(x, y)}{k} - \phi\left(n(x, y)\right), \tag{3.4}$$

for all $k \in (0, 1)$ and $x, y \in M$ where φ: $[0, \infty) \to [0, \infty)$ is a lower semicontinuous function from right such that φ is positive on $(0, \infty)$, $\varphi(0) = 0$ and

$$n(x, y) = \max\left\{\left\|fx - gy\right\|,\ \mathrm{dist}\left(fx, [q, Tx]\right),\ \mathrm{dist}\left(gy, [q, Ty]\right)\right.$$
$$\left.\frac{1}{2}\left[\mathrm{dist}\left(gy, [q, Tx]\right) + \mathrm{dist}\left(fx, [q, Ty]\right)\right]\right\}.$$

Then $M \cap F(T) \cap F(f) \cap F(g) \neq \emptyset$.

Proof. We utilize Theorem 3.3 instead of Theorem 2.2 in the proof of Theorem 2.7.

Corollary 3.9. Let M be a nonempty subset of a normed [resp. Banach] space X and T, f and g be self-maps of M. Suppose that $F(f) \cap F(g)$ is q-starshaped and closed [resp. weakly closed], $\mathrm{cl}(T(M))$ is compact [resp. $\mathrm{wcl}(T(M))$ is weakly compact], T is continuous on M [resp. id-T is demiclosed at 0], (T, f) and (T, g) are Banach operator pairs and satisfy (3.4) for all $x, y \in M$. Then $M \cap F(T) \cap F(f) \cap F(g) \neq \emptyset$.

In Theorem 3.8, if $\phi(t) = (\frac{1}{k} - 1)t$ for any constant k with $0 < k < 1$, then (3.4) changes into

$$\left\|Tx - Ty\right\| \le n(x, y) \tag{3.5}$$

Such a map T is called *generalized (f, g)-nonexpansive* (see [7]).

Corollary 3.10. Let M be a nonempty subset of a normed [resp. Banach] space X and T, f and g be self-maps of M. Suppose that $F(f) \cap F(g)$ is q-starshaped, $\mathrm{cl}T(F(f) \cap F(g)) \subseteq F(f) \cap F(g)$ [resp. $\mathrm{wcl}T\,(F(f) \cap F(g)) \subseteq F(f) \cap F(g)$], $\mathrm{cl}(T(M))$ is compact [resp. $\mathrm{wcl}(T(M))$ is weakly compact], T is continuous on M [resp. id-T is demiclosed at 0] and T is generalized (f, g)-nonexpansive. Then $M \cap F(T) \cap F(f) \cap F(g) \neq \emptyset$.

Remark 3.11. (1) By comparing Theorem 2.2(i) of Hussain and Jungck [5] with the first case of Corollary 3.10, their assumptions " M is complete, q-starshaped, f and g are affine and continuous on M, $T(M) \subseteq f(M) \cap g(M)$, $q \in F(f) \cap F(g)$ and (T, f) and (T, g) are R-subweakly commuting on M" are replaced with " M is a nonempty subset, $F(f) \cap F(g)$ is q-starshaped, $\mathrm{cl}T\,(F(f) \cap F(g)) \subseteq F(f) \cap F(g)$".

(2) By comparing Theorem 2.2(ii) of Hussain and Jungck [5] with the second case of Corollary 3.10, their assumptions " M is weakly compact, q-starshaped, f and g are affine and continuous on M, $T(M) \subseteq f(M) \cap g(M)$, $q \in F(f) \cap F(g)$, f - T is demiclosed at 0 and (T, f) and (T, g) are R-subweakly commuting on M" are replaced with " wcl$(T(M))$ is weakly compact, $F(f) \cap F(g)$ is q-starshaped, wcl$T(F(f) \cap F(g)) \subseteq F(f) \cap F(g)$, id - T is demiclosed at 0".

(3) By comparing Theorem 2.13 of Hussain and Jungck [5] with the first case of Corollary 3.10 with $g = f$, their assumptions " M is complete, q-starshaped, $f(M) = M$, f is continuous on M, the pair (T, f) is compatible, $ffv = fv$ for $v \in C(f, T)$" are replaced with " M is a nonempty subset, $F(f)$ is q-starshaped, cl$T(F(f)) \subseteq F(f)$".

(4) By comparing Theorem 2.4 of Song [7] with the first case of Corollary 3.10, his assumptions " M is nonempty, q-starshaped, f g are continuous and affine with $q \in F(f) \cap F(g)$, cl$T(M) \subset f(M) \cap g(M)$ and (T, f) and (T, g) are C_q-commuting on M" are replaced with " M is a nonempty subset, $F(f) \cap F(g)$ is q-starshaped, cl$T(F(f) \cap F(g)) \subseteq F(f) \cap F(g)$".

Corollary 3.12. Let X be a normed [resp. Banach] space X and T, f and g be self-maps of X. If u X, $D \subseteq C$, $D_0 := D \cap F(f) \cap F(g)$ is q-starshaped, cl$(T(D_0)) \subseteq D_0$ [resp. wcl$(T(D_0)) \subseteq D_0$], cl$(T(D))$ is compact [resp. wcl$(T(D))$ is weakly compact], T is continuous on D[resp.id - T is demiclosed at 0] and (3.4) holds for all $x, y \in D$, then $P_M(u) \cap F(T) \cap F(f) \cap F(g) \neq \emptyset$.

Corollary 3.13. Let X be a normed [resp. Banach] space X and T, f and g be self-maps of X. If $u \in X$, $D \subseteq P_M(u)$, $D_0 := D \cap F(f) \cap F(g)$ is q-starshaped, cl$(T(D_0)) \subseteq D_0$ [resp. wcl$(T(D_0)) \subseteq D_0$], cl$(T(D))$ is compact [resp. wcl$(T(D))$ is weakly compact], T is continuous on D[resp.id - T is demiclosed at 0] and (3.4) holds for all $x, y \in D$, then $P_M(u) \cap F(T) \cap F(f) \cap F(g) \neq \emptyset$.

Remark 3.14. (1) Corollaries 3.12 and 3.13 improve and develop Theorems 2.8-2.11 of Hussain and Jungck [5] and Theorems 3.1-3.4 of Song [7].

Theorem 3.15. Let f, g, T be self-maps of a normed [resp. Banach] space X. If $u \in X$ and $M \in \Im_0$ such that $T(M_u) \subseteq M$, cl$(T(M_u))$ is compact [resp. wcl$(T(M_u))$ is weakly compact] and $||Tx - u|| \leq ||x - u||$ for all $x \in M_u$, then $P_M(u)$ is nonempty, closed and convex with $T(P_M(u)) \subseteq P_M(u)$. If, in addition, $D \subseteq P_M(u)$, $D_0 := D \cap F(f) \cap F(g)$ is q-starshaped, cl$(T(D_0)) \subseteq D_0$ [resp. wcl$(T(D_0)) \subseteq D_0$], T is continuous on D [resp.id - T is demiclosed at 0] and (3.4) holds for all $x, y \in D$, then $P_M(u) \cap F(T) \cap F(f) \cap F(g) \neq \emptyset$.

Proof. We utilize Corollary 3.13 instead of Corollary 2.13 in the proof of Theorem 2.15.

Remark 3.16 Theorem 3.15 extends Theorem 4.1 and 4.2 of [23], Theorem 2.6 of [24], Theorem 8 of [25], Theorem 2.14 of [5], and Theorem 2.12 of [6].

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