RESEARCH ARTICLE

Some Newton-like methods with sharper error estimates for solving operator equations in Banach spaces

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Abstract

It is well known that the rate of convergence of *S*-iteration process introduced by Agarwal et al. (pp. 61-79) is faster than Picard iteration process for contraction operators. Following the ideas of *S*-iteration process, we introduce some Newton-like algorithms to solve the non-linear operator equation in Banach space setting. We study the semi-local as well as local convergence analysis of our algorithms. The rate of convergence of our algorithms are faster than the modified Newton method.

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1 Introduction

Let D be an open convex subset of a Banach space X and F be a Fréchet differentiable operator at each point of D with values in a Banach space Y. In the sequel, given any $x \in X$ and $r > 0$, $B_r[x]$ will designate the set $\{y \in X : ||y - x|| \le r\}$, $B_r(x)$ will designate the set $\{y \in X : || y - x || < r\}$, $B(Y, X)$ will designate the space of all bounded linear operators from Y to X and $\mathbb{N}_0$ will designate the set $\mathbb{N} \cup \{0\}$.

Many applied problems can be formulated to fit the model of the nonlinear operator equation

$$F(x) = 0, \tag{1.1}$$

where F is Fréchet differentiable operator at each point of D with values in a Banach space Y. A lot of problems about finding the solution of (1.1) are brought forward in many sciences and engineering (see [1]). Undoubtedly, Newton method is the most popular method for solving such problems. Starting with $x_0 \in X$, the famous Newton method is given by

$$x_{n+1} = x_n - F_{x_n}'^{-1} F(x_n), \quad n \in \mathbb{N}_0, \tag{1.2}$$

where F_x' denotes the Fréchet derivative of F at the point $x \in D$. There are numerous generalizations of Newton method for solving nonlinear operator Equation (1.1). Details can be found in Argyros [2], Wu and Zhao [3] and references therein.

In Newton method (1.2), functional value of inverse of derivative is required at each iteration. This bring us a natural question how to modify Newton iteration process (1.2), so that the computation of the inverse of derivative at each step in Newton method (1.2) can be avoided. Argyros [4], Bartle [5], Dennis [6] and Rheinboldt [7] discussed the modified Newton method

$$x_{n+1} = x_n - F_{x_0}'^{-1} F(x_n), \quad n \in \mathbb{N}_0. \tag{1.3}$$

In [8], Argyros proved the following theorem for semilocal convergence analysis of (1.3) to solve the operator Equation (1.1).

Theorem 1.1 *Let F be a Fréchet differentiable operator defined on an open convex subset D of a Banach space X with values in a Banach space Y. For some $x_0 \in D$, let $F_{x_0}'^{-1} \in B(Y, X)$. Assume that $F_{x_0}'^{-1}$ and F satisfy the following conditions:*

(i) $||F_{x_0}'^{-1}|| \leq \beta$, for some $\beta > 0$,

(ii) $||F_{x_0}'^{-1} F(x_0)|| \leq \eta$, for some $\eta > 0$,

(iii) $||F_x' - F_y'|| \leq K_0||x - y||$, $\forall x, y \in D$ and for some $K_0 > 0$.

Assume that $h = \eta\beta K_0 < \frac{1}{2}$ and $B_r[x_0] \subseteq D$, where $r = \frac{1-\sqrt{1-2h}}{h}\eta$. Then we have the following:

(a) The Equation (1.1) has a unique solution $x^* \in B_r[x_0]$.

(b) The sequence $\{x_n\}$ generated by (1.3) is in $B_r[x_0]$ and it converges to x^*.

(c) The following error estimate holds:

$$||x_{n+1} - x^*|| \leq \gamma^{n+1}||x_0 - x^*||, \quad \forall n \in \mathbb{N}_0, \tag{1.4}$$

where $\gamma = r\beta K_0$.

Let X be a Banach space and F a Fréchet differentiable operator on an open convex subset D of X with values in a Banach space Y. Let $x^* \in D$ be a solution of (1.1) such that $F_{x^*}'^{-1} \in B(Y, X)$. For some $x_0 \in D$, assume that $F_{x^*}'^{-1}$ and F satisfy the following:

$$||F_x' - F_{x_0}'|| \leq K_0||x - x_0||, \quad \forall x \in D \text{ and for some } K_0 > 0, \tag{1.5}$$

$$||F_{x^*}'^{-1}(F_x' - F_{x_0}')|| \leq K_1||x - x_0||, \quad \forall x \in D \text{ and for some } K_1 > 0 \tag{1.6}$$

and

$$||F_{x^*}'^{-1}(F_x' - F_{x^*}')|| \leq K_2||x - x^*||, \quad \forall x \in D \text{ and for some } K_2 > 0. \tag{1.7}$$

Ren and Argyros [9] studied the following local convergence analysis to solve the operator Equation (1.1).

Theorem 1.2 *Let F be a Fréchet differentiable operator defined on an open convex subset D of a Banach space X with values in a Banach space Y. Let $x^* \in D$ be a solution of (1.1) such that $F_{x^*}'^{-1} \in B(Y, X)$. For some $x_0 \in D$, let $F_{x^*}'^{-1}$ and F satisfy (1.6) and (1.7). Assume that $B_{r_1}(x^*) \subseteq D$, where $r_1 = \frac{2}{K_2}$. Then, for any initial point $x_0 \in B_r(x^*)$, where $r = \frac{2}{2K_2+3K_1}$, we have the following:*

(a) *The sequence $\{x_n\}$ generated by (1.3) is in $B_r(x_0)$ and it converges to the unique solution x^* in $B_{r_1}(x^*)$.*

(b) *The following error estimate holds*:

$$||x_{n+1} - x^*|| \leq (\delta_0)^{n+1} ||x_0 - x^*||, \quad \forall n \in \mathbb{N}_0, \tag{1.8}$$

where $\delta_0 = \frac{||x_0 - x^*||}{r}$.

Recently, Agarwal et al. [10] have introduced the *S*-iteration process as follows: Let X be a normed space, D a nonempty convex subset of X and $A : D \to D$ an operator. Then, for arbitrary $x_0 \in D$, the *S*-iteration process is defined by

$$\begin{cases} x_{n+1} = (1 - \alpha_n) Ax_n + \alpha_n Ay_n, \\ y_n = (1 - \beta_n) x_n + \beta_n Ax_n, \quad n \in \mathbb{N}_0, \end{cases} \tag{1.9}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in (0, 1).

In [11], motivated by *S*-iteration process, the first author has introduced the normal *S*-iteration process as follows: Let X be a normed space, D a nonempty convex subset of X and $A : D \to D$ an operator. Then, for arbitrary $x_0 \in D$, the normal *S*-iteration process is defined by

$$x_{n+1} = A((1 - \alpha_n) x_n + \alpha_n Ax_n), \quad n \in \mathbb{N}_0, \tag{1.10}$$

where $\{\alpha_n\}$ be a sequence in (0, 1). Noticing that the normal *S*-iteration process is applicable for finding solutions of constrained minimization problems and split feasibility problems (see Sahu [11]).

Following [[11], Theorem 3.6], we remark that the normal *S*-iteration process is faster than the Picard and Mann iteration processes for contraction mappings.

In the present article, motivated by normal *S*-iteration process, we introduce the *S*-iteration processes of Newton-like for finding the solution of operator Equation (1.1).

Algorithm 1.3 *Let $\alpha \in (0, 1)$. Starting with $x_0 \in X$ and after $x_n \in X$ is defined, we define the next iterate x_{n+1} as follows*:

$$\begin{cases} x_{n+1} = z_n - F_{x_0}'^{-1} F(z_n), \\ z_n = (1 - \alpha) x_n + \alpha y_n, \\ y_n = x_n - F_{x_0}'^{-1} F(x_n), \quad n \in \mathbb{N}_0. \end{cases} \tag{1.11}$$

Algorithm 1.4 *Let $\alpha \in (0, 1)$. Starting with $x_0 \in X$ and after $x_n \in X$ is defined, we define the next iterate x_{n+1} as follows*:

$$\begin{cases} x_{n+1} = z_n - F_{z_0}'^{-1} F(z_n), \\ z_n = (1 - \alpha) x_n + \alpha y_n, \\ y_n = x_n - F_{x_0}'^{-1} F(x_n), \quad n \in \mathbb{N}_0. \end{cases} \tag{1.12}$$

The purpose of this article is to prove the semi-local as well as local convergence analysis of Algorithms 1.3 and 1.4. It is shown that the rate of convergence of (1.11) and (1.12) are faster than (1.3). Applications to initial value and boundary value problems are included.

2 Preliminaries

Definition 2.1 *Let C be a nonempty subset of normed space X. A mapping $T : C \to X$ is said to be*

(i) *Lipschitzian if there exists a constant* $L > 0$ *such that*

$$||T(x) - T(y)|| \leq L||x - y||, \quad \forall x, y \in C;$$

(ii) *contraction if there exists a constant* $L \in (0, 1)$ *such that*

$$||T(x) - T(y)|| \leq L||x - y||, \quad \forall x, y \in C;$$

(iii) *quasi-contraction* [12]*if there exists a constant* $L \in (0, 1)$ *and*

$$F(T) = \{x \in C : T(x) = x\} \neq \emptyset$$

such that

$$||T(x) - p|| \leq L||x - p||, \quad \forall x \in C \text{ and } p \in F(T).$$

Definition 2.2 [11]*Let C be a nonempty convex subset of a normed space X and T :* $C \to C$ *an operator. The operator* $G : C \to C$ *is said to be S-operator generated by* $\alpha \in (0, 1)$ *and T if*

$$G = T[(1 - \alpha)I + \alpha T],$$

where I is the identity operator.

Before presenting our main results we need the following technical lemmas.

Lemma 2.3 [4,13,14]*Let P be a bounded linear operator on a Banach space X. Then the following are equivalent:*

(a) *There is a bounded linear operator Q on X such that* Q^{-1} *exists, and* $||Q - P|| < \frac{1}{||Q^{-1}||}$.

(b) P^{-1} *exists.*

Further, if P^{-1} *exists, then*

$$||P^{-1}|| \leq \frac{||Q^{-1}||}{1 - ||1 - Q^{-1}P||} \leq \frac{||Q^{-1}||}{1 - ||Q^{-1}|| \, ||Q - P||}.$$

Lemma 2.4 [9]*Let F be a Fréchet differentiable operator defined on an open convex subset D of a Banach space X with values in a Banach space Y. Let* $x^* \in D$ *be a solution of (1.1) such that* $F_{x^*}'^{-1} \in B(Y, X)$ *and the operator F satisfies the conditions (1.7). Assume that* $B_r(x^*) \subseteq D$, *where* $r = \frac{1}{K_2}$. *Then, for any* $x \in B_r(x^*)$, F_x'*is invertible, and the following estimate holds:*

$$||\left(F_{x^*}'^{-1}F_x'\right)^{-1}|| \leq \frac{1}{1 - K_2||x - x^*||}.$$

Lemma 2.5 [15,16]*Let (X, d) be a complete metric space and* $F : X \to X$ *a contraction mapping. Then F has a unique fixed point in X.*

Lemma 2.6 [[17], Theorem 9.4.2] *Let F be a Fréchet differentiable operator defined on an open convex subset D of a Banach space X with values in a Banach space Y. Then, for all* $x, y \in D$, *we have*

$$F(x) - F(y) = \int_0^1 F_{y+t(x-y)}'(x - y)dt.$$

3 Convergence analysis for Algorithms 1.3 and 1.4

Before studying convergence analysis of Algorithm 1.3, we establish the following theorem for existence of a unique solution of operator Equation (1.1).

Theorem 3.1 *Let F be a Fréchet differentiable operator defined on an open convex subset D of a Banach space X with values in a Banach space Y. For some $x_0 \in D$, let $F_{x_0}'^{-1} \in B(Y, X)$ and the operator F satisfies (1.5) and the following conditions:*

(i) $||F_{x_0}'^{-1} F(x_0)|| \le \eta$, *for some* $\eta > 0$,

(ii) $||F_{x_0}'^{-1}|| \le \beta$, *for some* $\beta > 0$.

Assume that $h = \eta \beta K_0 < \frac{1}{2}$ *and* $B_r[x_0] \subseteq D$, *where* $r = \frac{1 - \sqrt{1 - 2h}}{h} \eta$. *Then, for fixed* $\alpha \in (0, 1)$, *we have the following:*

(a) The operator $A : B_r[x_0] \to X$ *defined by*

$$Ax = x - F_{x_0}'^{-1} F(x), \quad x \in B_r[x_0] \tag{3.1}$$

is a contraction self-operator on $B_r[x_0]$ with Lipschitz constant $\beta_r K_0$ and the operator Equation (1.1) has a unique solution in $B_r[x_0]$.

(b) The S-operator $A_\alpha : B_r[x_0] \to X$ generated by α and A is a contraction self-operator on $B_r[x_0]$ with Lipschitz constant $\beta r K_0 (1 - \alpha + \alpha \beta r K_0)$.

Proof: (a) Set $\gamma := \beta r K_0$. Note that $\gamma = 1 - \sqrt{1 - 2h} < 1$. For $x, y \in B_r[x_0]$, we have

$$
\begin{aligned}
||Ax - Ay|| &= ||x - y - F_{x_0}'^{-1}(F(x) - F(y))|| \\
&= \left\| F_{x_0}'^{-1} \left[\int_0^1 F_{y+t(x-y)}'(x-y)dt - \int_0^1 F_{x_0}'(x-y)\,dt \right] \right\| \\
&\le ||F_{x_0}'^{-1}|| \int_0^1 ||F_{y+t(x-y)}'(x-y) - F_{x_0}'(x-y)||dt \\
&\le \beta \int_0^1 K_0 ||y + t(x-y) - x_0|| \, ||x-y|| \, dt \\
&\le \beta r K_0 ||x - y|| \\
&\le \gamma ||x - y||.
\end{aligned}
$$

Therefore, the operator A is a contraction with Lipschitz constant γ.

Now we claim that $A(B_r[x_0]) \subseteq B_r[x_0]$. For $x \in B_r[x_0]$, we have

$$
\begin{aligned}
||Ax - x_0|| &\le ||Ax - Ax_0|| + ||Ax_0 - x_0|| \\
&\le ||F_{x_0}'^{-1}|| \int_0^1 ||F_{x_0}' - F_{x_0 + t(x-x_0)}'|| \, ||x - x_0|| dt + \eta \\
&\le \beta \int_0^1 K_0 ||t(x-x_0)|| \, ||x - x_0|| dt + \eta \\
&\le \beta K_0 ||x - x_0||^2 \int_0^1 t\,dt + \eta \\
&\le \frac{\beta r^2 K_0}{2} + \eta \\
&= \frac{\beta K_0}{2} \left[\frac{1 - \sqrt{1 - 2h}}{h} \eta \right]^2 + \eta \\
&= r.
\end{aligned}
$$

Hence, the operator A maps $B_r[x_0]$ into itself. By Banach contraction principle (see Lemma 2.5), A has a unique fixed point in $B_r[x_0]$.

(b) For $x, y \in B_r[x_0]$, we have

$$
\begin{aligned}
||A_\alpha x - A_\alpha y|| &= ||A[(1-\alpha)x + \alpha Ax] - A[(1-\alpha)y + \alpha Ay]|| \\
&\leq \gamma ||(1-\alpha)x + \alpha Ax - (1-\alpha)y - \alpha Ay|| \\
&\leq \gamma ||(1-\alpha)(x-y) + \alpha(Ax - Ay)|| \\
&\leq \gamma[(1-\alpha) + \alpha\gamma]||x-y||.
\end{aligned}
$$

Therefore, the S-operator A_α generated by α and A is a contraction operator on $B_r[x_0]$ with Lipschitz constant $\beta r K_0(1 - \alpha + \alpha\beta r K_0)$. □

Next we formulate that the operator A_λ defined by (3.2) is quasi-contraction.

Theorem 3.2 *Let F be a Fréchet differentiable operator defined on an open convex subset D of a Banach space X with values in a Banach space Y. Assume that $\lambda \in (0, 1]$ and $x^* \in D$ is a solution of (1.1) such that $F'^{-1}_{x^*} \in B(Y, X)$. For some $x_0 \in D$, let $F'^{-1}_{x^*}$ and F satisfy the conditions (1.6) and (1.7). Assume that $B_{r_1}(x^*) \subseteq D$, where $r_1 = \frac{2}{K_2}$. For $x_0 \in B_r(x^*)$ with $r = \frac{2}{2K_2 + 3K_1}$, let A_λ be an operator defined by*

$$
A_\lambda(x) = x - \lambda F'^{-1}_{x_0} F(x), \quad \forall x \in B_r(x^*). \tag{3.2}
$$

Then, we have the following
(a) For $x \in B_r(x^*)$, we have

$$
||A_\lambda(x) - x^*|| \leq (\lambda \delta_x + 1 - \lambda)||x - x^*||, \tag{3.3}
$$

where

$$
\delta_x = \frac{K_1}{2(1 - rK_2)}\left(||x - x^*|| + 2||x_0 - x^*||\right). \tag{3.4}
$$

(b) A_λ is a quasi-contraction and self-operator on $B_r(x^*)$ with constant $1 - (1 - \delta)\lambda$, where $\delta = \sup\limits_{x \in B_r(x^*)} \{\delta_x\}$.

Proof: (a) For $x \in B_r(x^*)$ with $x \neq x^*$, we have

$$
\begin{aligned}
||A_\lambda x - x^*|| &= ||x - \lambda F'^{-1}_{x_0} F(x) - x^*|| \\
&= ||\lambda(x - x^*) - \lambda F'^{-1}_{x_0}(F(x) - F(x^*)) - (1-\lambda)(x - x^*)|| \\
&\leq \lambda ||F'^{-1}_{x_0}[F(x) - F(x^*) - F'_{x_0}(x - x^*)]|| + (1-\lambda)||x - x^*|| \\
&= \lambda \left\| (F'^{-1}_{x^*} F'_{x_0})^{-1} \int_0^1 F'^{-1}_{x^*}(F'_{tx+(1-t)x^*}(x - x^*) - F'_{x_0}(x - x^*))\, dt \right\| + (1-\lambda)||x - x^*|| \\
&\leq \lambda ||(F'^{-1}_{x^*} F'_{x_0})^{-1}|| \int_0^1 ||F'^{-1}_{x^*}(F'_{tx+(1-t)x^*} - F'_{x_0})|| ||x - x^*|| \, dt + (1-\lambda)||x - x^*||.
\end{aligned}
$$

By (1.6) and Lemma 2.4, we have

$$
\begin{aligned}
||A_\lambda x - x^*|| &\leq \frac{\lambda K_1}{1 - K_2||x_0 - x^*||} \int_0^1 ||t(x - x^*) + (x^* - x_0)|| ||x - x^*|| \, dt + (1-\lambda)||x - x^*|| \\
&\leq \frac{\lambda K_1}{1 - K_2||x_0 - x^*||} \int_0^1 (t||x - x^*|| + ||x_0 - x^*||)||x - x^*|| \, dt + (1-\lambda)||x - x^*|| \\
&= \frac{\lambda K_1}{1 - K_2||x_0 - x^*||} \frac{(||(x - x^*)|| + ||x_0 - x^*||)^2 - ||x_0 - x^*||^2}{2} + (1-\lambda)||x - x^*|| \\
&\leq \frac{\lambda K_1}{2(1 - rK_2)}(||x - x^*|| + 2||x_0 - x^*||)||x - x^*|| + (1-\lambda)||x - x^*|| \\
&\leq (\lambda \delta_x + 1 - \lambda)||x - x^*||.
\end{aligned}
$$

(b) The operator A_λ is a quasi-contraction with constant $1 - (1 - \delta)\lambda$. Indeed,

$$
\begin{aligned}
\sup_{x \in B_r(x^*)} \{\delta_x\} &= \frac{K_1}{2(1 - rK_2)} \left(\sup_{x \in B_r(x^*)} ||x - x^*|| + 2||x_0 - x^*|| \right) \\
&\leq \frac{K_1}{2(1 - rK_2)} \left(r + 2||x_0 - x^*|| \right) \\
&< \frac{3rK_1}{2(1 - rK_2)} \\
&= 1.
\end{aligned}
$$

This completes the proof. □

Corollary 3.3 *Let F be a Fréchet differentiable operator defined on an open convex subset D of a Banach space X with values in a Banach space Y. Let $\alpha \in (0, 1)$ and $x^* \lfloor D$ be a solution of (1.1) such that $F_{x^*}'^{-1} \in B(Y, X)$. For $x_0 \in D$, let $F_{x^*}'^{-1}$ and F satisfy the conditions (1.6) and (1.7). Assume that $B_{r_1}(x^*) \subseteq D$, where $r_1 = \frac{2}{K_2}$. For x_0 in $B_r(x^*)$ with $r = \frac{2}{2K_2 + 3K_1}$, let A be an operator defined by (3.1) and let A_α be the S-operator generated by α and A. Then, the following hold:*

$$||A(x) - x^*|| \leq \delta_x ||x - x^*||, \quad \forall x \in B_r(x^*), \tag{3.5}$$

$$||A_\alpha(x) - x^*|| \leq \delta_x (1 - \alpha + \alpha\delta_x) ||x - x^*||, \quad \forall x \in B_r(x^*), \tag{3.6}$$

where δ_x is defined in (3.4).

Corollary 3.4 *Let F be a Fréchet differentiable operator defined on an open convex subset D of a Banach space X with values in a Banach space Y. Let $x^* \in D$ be a solution of (1.1) such that $F_{x^*}'^{-1} \in B(Y, X)$. For $x_0 \in D$ and $\alpha \in (0, 1)$, let $F_{x^*}'^{-1}$ and F satisfy the conditions (1.6) and (1.7). Assume that $B_{r_1}(x^*) \subseteq D$, where $r_1 = \frac{2}{K_2}$. For x_0 in $B_r(x^*)$ with $r = \frac{2}{2K_2 + 3K_1}$, let U_α be an operator defined by*

$$U_\alpha(x) = x - \alpha F_{x_0}'^{-1} F(x), \quad \forall x \in B_r(x^*). \tag{3.7}$$

Then U_α is a quasi-contraction, self-operator on $B_r(x^*)$ and the following holds:

$$||U_\alpha(x) - x^*|| \leq (\alpha\delta_x + 1 - \alpha) ||x - x^*||, \quad \forall x \in B_r(x^*). \tag{3.8}$$

Now, we ready to study the semilocal convergence analysis of Algorithm 1.3.

Theorem 3.5 *Let F be a Fréchet differentiable operator defined on an open convex subset D of a Banach space X with values in a Banach space Y. For some $x_0 \in D$, let $F_{x_0}'^{-1} \in B(Y, X)$. Assume that $F_{x_0}'^{-1}$ and F satisfy (1.5) with the following conditions:*

(i) $||F_{x_0}'^{-1} F(x_0)|| \leq \eta$, for some $\eta > 0$,

(ii) $||F_{x_0}'^{-1}|| \leq \beta$, for some $\beta > 0$.

Assume that $\alpha \in (0, 1)$, $h = \eta\beta K_0 < \frac{1}{2}$ and $B_r[x_0] \subseteq D$, where $r = \frac{1 - \sqrt{1 - 2h}}{h}\eta$. Then we have the following:

(a) The Equation (1.1) has a unique solution $x^* \in B_r[x_0]$.

(b) The sequence $\{x_n\}$ generated by Algorithm 1.3 is in $B_r[x_0]$ and it converges strongly to x^*.

(c) The following error estimate holds:

$$||x_{n+1} - x^*|| \leq \rho^{n+1}||x_0 - x^*||, \quad \forall n \in \mathbb{N}_0, \tag{3.9}$$

where $\rho = \gamma (1 - \alpha + \alpha\gamma)$ and $\gamma = \beta r K_0$.

Proof : (a) It follows from Theorem 3.1.

(b) From Algorithm 1.3, we have

$$\begin{aligned}
||x_{n+1} - x^*|| &= ||A_\alpha (x_n) - A_\alpha (x^*)|| \\
&\leq \gamma [(1 - \alpha) + \alpha\gamma]||x_n - x^*|| \\
&\leq \gamma^{n+1}[(1 - \alpha) + \alpha\gamma]^{n+1}||x_0 - x^*||, \quad \forall n \in \mathbb{N}_0.
\end{aligned} \tag{3.10}$$

Therefore, $x_n \to x^*$ as $n \to \infty$.

(c) It follows from (3.10).

Remark 3.6 *The condition* (1.5) *of Theorem 3.5 is weaker assumption than assumption (iii) of Theorem 1.1. Also one can observe from* (1.4) *and* (3.9) *that*

$$\rho = \gamma (1 - \alpha + \alpha\gamma) < \gamma. \tag{3.11}$$

The strict inequality (3.11) *shows that the error estimate in Theorem 3.5 is sharper than that of Theorem 1.1.*

Now, we study the local convergence analysis for Algorithm 1.3.

Theorem 3.7 *Let F be a Fréchet differentiable operator defined on an open convex subset D of a Banach space X with values in a Banach space Y. Let $\alpha \in (0, 1)$ and let $x^* \lfloor D$ be a solution of* (1.1) *such that $F_{x^*}'^{-1} \in B(Y, X)$. For $x_0 \in D$, let $F_{x^*}'^{-1}$ and F satisfy the conditions* (1.6) *and* (1.7). *Assume that $B_{r_1} (x^*) \subseteq D$, where $r_1 = \frac{2}{K_2}$. Then, we have the following:*

(a) For initial $x_0 \in B_r(x^*)$ with $r = \frac{2}{2K_2+3K_1}$, the sequence $\{x_n\}$ generated by Algorithm 1.3 is in $B_r(x^*)$ and it converges strongly to the unique solution x^* in $B_{r_1} (x^*)$.

(b) The following error estimate holds:

$$||x_{n+1} - x^*|| \leq (\rho\prime)^{n+1}||x_0 - x^*||, \quad \forall n \in \mathbb{N}_0, \tag{3.12}$$

where $\rho' = \delta_0(1 - \alpha + \alpha\delta_0)$ and $\delta_0 = \frac{||x_0 - x^*||}{r}$.

Proof : (a) First we show that x^* is unique solution of (1.1) in $B_{r_1} (x^*)$. For contradiction, suppose that y^* is another solution of (1.1) in $B_{r_1} (x^*)$. Then, we have

$$0 = F(x^*) - F(y^*) = \int_0^1 F_{y^*+t(x^*-y^*)}' (x^* - y^*)dt.$$

Define an operator L by

$$L(h) = \int_0^1 F_{y^*+t(x^*-y^*)}' h\, dt, \quad \forall h \in X.$$

Consequently, we have

$$
\begin{aligned}
||I - F_{x^*}'^{-1}L|| &= \left\| \int\limits_0^1 F_{x^*}'^{-1} \left(F_{x^*}' - F_{y^*+t(x^*-y^*)}' \right) dt \right\| \\
&\leq \frac{K_2}{2} ||x^* - y^*|| \\
&< \frac{r_1 K_2}{2} = 1.
\end{aligned}
$$

It follows from Lemma 2.3 that the operator L is invertible and hence, $x^* = y^*$, a contradiction. Thus, x^* is the unique solution of (1.1) in $B_{r_1}(x^*)$.

Next, we show that $\{x_n\}$ converges to x^*. By Corollary 3.3, the operator A_α is a quasi-contraction self-operator on $B_r(x^*)$. Thus, $x_n \in B_r(x^*)$, $\forall n \in \mathbb{N}_0$. Now, we have

$$
\begin{aligned}
||x_{n+1} - x^*|| &= ||A_\alpha(x_n) - x^*|| \\
&\leq \delta_{x_n} \left[(1 - \alpha) + \alpha \delta_{x_n} \right] ||x_n - x^*||, \quad \forall n \in \mathbb{N}_0,
\end{aligned}
\tag{3.13}
$$

where δ_x is defined by (3.4). Since $\delta_{x_n} < 1, \forall n \in \mathbb{N}_0$, we have

$$
0 \leq ||x_{n+1} - x^*|| \leq ||x_n - x^*|| \leq \cdots \leq ||x_0 - x^*||, \quad \forall n \in \mathbb{N}_0.
$$

By definition of δ_x, we have

$$
\begin{aligned}
\delta_{x_n} &= \frac{K_1}{2(1 - rK_2)} \left(||x_n - x^*|| + 2||x_0 - x^*|| \right) \\
&\leq \frac{3K_1 ||x_0 - x^*||}{2(1 - rK_2)} \\
&\leq \frac{||x_0 - x^*||}{r} \\
&= \delta_0, \quad \forall n \in \mathbb{N}_0.
\end{aligned}
$$

Thus, by (3.13), we have

$$
||x_{n+1} - x^*|| \leq [\delta_0(1 - \alpha + \alpha \delta_0)]^{n+1} ||x_0 - x^*||, \quad \forall n \in \mathbb{N}_0,
\tag{3.14}
$$

which implies $x_n \to x^*$ as $n \to \infty$.

(b) By (3.14), we get the error estimates. □

Remark 3.8 *One can observe from* (1.8) *and* (3.12) *that*

$$
\rho' = \delta_0(1 - \alpha + \alpha \delta_0) < \delta_0.
\tag{3.15}
$$

The strict inequality (3.15) *shows that the error estimate in Theorem 3.7 is sharper than that of Theorem 1.2.*

Before presenting local convergence result for Algorithm 1.4, we need the following theorem:

Theorem 3.9 *Let F be a Fréchet differentiable operator defined on an open convex subset D of a Banach space X with values in a Banach space Y. Let $x^* \in D$ be a solution of* (1.1) *such that $F_{x^*}'^{-1} \in B(Y, X)$. For $x_0 \in D$ and $\alpha \in (0, 1)$, let $F_{x^*}'^{-1}$ and F satisfy the conditions* (1.6) *and* (1.7)*. Let $B_{r_1}(x^*) \subseteq D$ and $y_0 \in B_r(x^*)$, where $r_1 = \frac{2}{K_2}$ and $r = \frac{2}{2K_2+3K_1}$. Assume that*

$$||F_{x^*}'^{-1}\left(F_x' - F_{\gamma_0}'\right)|| \le K_3||x - y_0||, \quad \forall x \in D \text{ and for some } K_3 > 0. \tag{3.16}$$

Define an operator V by

$$V(x) = x - F_{\gamma_0}'^{-1}F(x), \quad \forall x \in B_r(x^*). \tag{3.17}$$

Then, we have the following:

(a) For $x \in B_r(x^*)$, we have

$$||V(x) - x^*|| \le \delta_x'||x - x^*||,$$

where $\delta_x' = \dfrac{K_3}{2(1 - rK_2)}\left(||x - x^*|| + 2||y_0 - x^*||\right), \quad \forall x \in B_r(x^*)$.

(b) If $K_3 \le K_1$, then V is a quasi-contraction and self-operator on $B_r(x^*)$ with constant δ', where $\delta' = \sup_{x \in B_r(x^*)} \{\delta_x'\}$.

Proof: (a) For $x \in B_r(x^*)$, we have

$$
\begin{aligned}
||Vx - x^*|| &= ||x - F_{\gamma_0}'^{-1}F(x) - x^*|| \\
&= ||F_{\gamma_0}'^{-1}\left(F(x) - F(x^*) - F_{\gamma_0}'(x - x^*)\right)|| \\
&= \left\|\left(F_{x^*}'^{-1}F_{\gamma_0}'\right)^{-1}\int_0^1 F_{x^*}'^{-1}(F_{x^*+t(x-x^*)}'(x - x^*) - F_{\gamma_0}'(x - x^*))dt\right\| \\
&\le \frac{K_3}{2\left(1 - K_2||y_0 - x^*||\right)}\left[\left(||x - x^*|| + ||y_0 - x^*||\right)^2 - ||y_0 - x^*||^2\right] \\
&\le \frac{K_3}{2(1 - rK_2)}\left(||x - x^*|| + 2||y_0 - x^*||\right)||x - x^*|| \\
&= \delta_x'||x - x^*||.
\end{aligned}
$$

(b) The operator V is a quasi-contraction with constant δ'. Indeed,

$$
\begin{aligned}
\sup_{x \in B_r(x^*)} \{\delta_x'\} &= \frac{K_3}{2(1 - rK_2)}\left(\sup_{x \in B_r(x^*)}||x - x^*|| + 2||y_0 - x^*||\right) \\
&\le \frac{K_3}{2(1 - rK_2)}\left(r + 2||y_0 - x^*||\right) \\
&< \frac{3rK_1}{2(1 - rK_2)} \\
&= 1.
\end{aligned}
$$

That completes the proof. □

Now, we ready to study the local convergence analysis for Algorithm 1.4.

Theorem 3.10 *Let F be a Fréchet differentiable operator defined on an open convex subset D of a Banach space X with values in a Banach space Y. Let $\alpha \in (0, 1)$ and let $x^* \in D$ be a solution of (1.1) such that $F_{x^*}'^{-1} \in B(Y, X)$. Assume that $B_{r_1}(x^*) \subseteq D$, where $r_1 = \frac{2}{K_2}$. For any $x_0 \in B_r(x^*)$, where $r = \frac{2}{2K_2+3K_1}$, assume that $F_{x^*}'^{-1}$ and F satisfy the conditions (1.6), (1.7) and (3.16) and that $K_3 \le K_1$. Then we have the following:*

(a) The sequence $\{x_n\}$ generated by Algorithm 1.4 is in $B_r(x^*)$ and it converges strongly to the unique solution x^* in $B_{r_1}(x^*)$.

(b) The following error estimate holds:

$$||x_{n+1} - x^*|| \leq (\rho'')^{n+1} ||x_0 - x^*||, \quad \forall n \in \mathbb{N}_0, \qquad (3.18)$$

where $\rho'' = (\alpha \delta_0 + 1 - \alpha) \delta_0'$ *and* $\delta_0' = \frac{3K_3}{2(1-rK_2)} ||x_0 - x^*||$.

Proof: (a) Similarly, as in Theorem 3.7, x^* is the unique solution of (1.1) in $B_{r_1}(x^*)$. Again, Corollary 3.4 shows that the operator U_α defined by (3.7) is a self-operator on $B_r[x^*]$. Hence, from (1.12), we have

$$\gamma_0 = U_\alpha x_0 \in B_r(x^*).$$

It follows from Theorem 3.9 that the operator V defined by (3.17) is a self operator on $B_r[x^*]$. Thus, 1.4 can be written as

$$x_{n+1} = VU_\alpha(x_n), \quad \forall n \in \mathbb{N}_0.$$

Since,

$$\begin{aligned} ||x_{n+1} - x^*|| &= ||VU_\alpha(x_n) - x^*|| \\ &\leq \delta_{\gamma_n}' ||U_\alpha(x_n) - x^*|| \\ &\leq (\alpha \delta_{x_n} + 1 - \alpha) \delta_{\gamma_n}' ||x_n - x^*||, \quad \forall n \in \mathbb{N}_0. \end{aligned} \qquad (3.19)$$

By definition of δ_x', we have

$$\begin{aligned} \delta_{\gamma_n}' &= \frac{K_3}{2(1-rK_2)} \left(||U_\alpha(x_n) - x^*|| + 2||x_0 - x^*|| \right) \\ &\leq \frac{K_3}{2(1-rK_2)} \left((\alpha \delta_{x_n} + 1 - \alpha) ||x_n - x^*|| + 2||x_0 - x^*|| \right) \\ &\leq \frac{K_3}{2(1-rK_2)} \left(||x_0 - x^*|| + 2||x_0 - x^*|| \right) \\ &= \frac{3K_3}{2(1-rK_2)} ||x_0 - x^*|| \\ &= \delta_0', \quad \forall n \in \mathbb{N}_0. \end{aligned}$$

By (3.19), we have

$$||x_{n+1} - x^*|| \leq \left((\alpha \delta_0 + 1 - \alpha) \delta_0' \right)^{n+1} ||x_0 - x^*||, \quad \forall n \in \mathbb{N}_0, \qquad (3.20)$$

which implies $x_n \to x^*$ as $n \to \infty$.
(b) From (3.20), we get the error estimates. □
Remark 3.11 *One can observe from (1.8) and (3.18) that*

$$\begin{aligned} \rho'' &= (\alpha \delta_0 + 1 - \alpha) \delta_0' \\ &= \frac{3K_3(\alpha \delta_0 + 1 - \alpha)}{2(1-rK_2)} ||x_0 - x^*|| \\ &\leq \frac{3K_1(\alpha \delta_0 + 1 - \alpha)}{2(1-rK_2)} ||x_0 - x^*|| \\ &\leq \delta_0(\alpha \delta_0 + 1 - \alpha) \\ &= \rho' < \delta_0. \end{aligned} \qquad (3.21)$$

The strict inequality (3.21) shows that the error estimate in Theorem 3.10 is sharper than that of Theorems 1.2 and 3.7.

4 Application to initial and boundary value problems

Throughout this section, let $C0[1]$ be the space of real-valued continuous functions defined on the interval $0[1]$ with norm

$$||x|| = \max_{0 \le t \le 1} |x(t)|.$$

4.1 Initial value problem

Consider initial value problem

$$\frac{dy(s)}{ds} = f(s, y(s)), y(0) = 0, \quad 0 \le s \le 1. \tag{4.1}$$

Let $f_2'(s, y(s)) = \frac{\partial}{\partial y} f(s, y(s))$, the partial derivative of f with respect to second component. Assume that $f_2'(s, y(s))$ exists for all $(s, y(s)) \in S \subseteq 0[1] \times \mathbb{R}$ and

$$||f_2'(s, y(s)) - f_2'(s, y(s))|| \le K_0 ||y - y_0||, \quad \forall y \in C^1[0, 1] \text{ and for some } K_0 > 0, \tag{4.2}$$

where $y_0 \in C^1 0[1]$ and $C^1 0[1]$ is the space of all real-valued continuously differentiable functions defined on $0[1]$.

Consider the operator $F : C^1 0[1] \to C0[1]$ defined by

$$F(y)(s) = \frac{dy(s)}{ds} - f(s, y(s)). \tag{4.3}$$

Then, solving problem (4.1) is equivalent to solving the Equation (1.1). One can observe that the operator F defined by (4.3) is Fréchet differetiable and its Fréchet derivative is given by

$$F_y' h(s) = \frac{dh(s)}{ds} - f_2'(s, y(s)) h(s), \quad \forall h \in C^1[0, 1].$$

Theorem 4.1 *Let* $F : C^1 0[1] \to C0[1]$ *be an operator defined by (4.3). For some* $y_0 \in C^1 0[1]$, *assume that* $F_{y_0}'^{-1}$ *exists and* $f_2'(s, y(s))$ *satisfies (4.2). Suppose that*

$h = \dfrac{K_0 ||F(y_0)||}{(1 - \theta_1)^2} < \dfrac{1}{2}$, *where* $\theta_1 = \sup_{s \in [0, 1]} |f_2'(s, y_0(s))|$. *Then, we have the following:*

(a) The initial value problem (4.1) has a unique solution y^* *in* $B_r[y_0]$, *where* $r = \dfrac{1 - \sqrt{1 - 2h}}{h} \eta$.

(b) For $x_0 = y_0$, *the sequence* $\{x_n\}$ *generated by Algorithm 1.3 is in* $B_r[y_0]$ *and it converges strongly to* y^*.

Proof: Our goal is to find an upper bound for $F_{y_0}'^{-1}$. Set

$$F_{y_0}' h(s) = \frac{dh(s)}{ds} - f_2'(s, y_0(s)) h(s) = u(s).$$

Since $F_{y_0}'^{-1}$ exists, we can immediately write $h(s) = F_{y_0}'^{-1} u(s)$ and arrive at the first order linear initial value problem

$$\begin{cases} \dfrac{dh(s)}{ds} = u(s) + f_2'\left(s, \, y_0(s)\right) h(s); \\ h(0) = 0. \end{cases} \tag{4.4}$$

It should be noted that problem (4.4) is equivalent to the integral equation of Voltera type of second kind (see [18])

$$h(s) = \int\limits_0^s \left[u(\tau) + f_2'\left(\tau, y_0(\tau)\right) h(\tau) \right] d\tau.$$

Again consider a operator L defined by

$$Lh(s) = \int\limits_0^s f_2'\left(\tau, \, y_0(\tau)\right) h(\tau) \, d\tau.$$

Clearly, the operator L is linear. We can write

$$(I - L)(h)(s) = \int\limits_0^s u(\tau) \, d\tau.$$

Using max-norm in $C0[1]$, the operator L is bounded and satisfies $||L|| \le \theta_1$, where $\theta_1 = \sup_{s \in [0,1]} |f_2'\left(\tau, \, y_0(\tau)\right)|$. Consequently, if $\theta_1 < 1$, then by Lemma 2.3, the inverse $(I - L)^{-1}$ exists and $||(I - L)^{-1}|| \le \dfrac{1}{1 - \theta_1}$. Therefore, $||F_{y_0}'^{-1}|| \le \dfrac{1}{1 - \theta_1}$. Indeed

$$||F_{y_0}'^{-1} u|| = \max_{s \in [0,1]} |F_{y_0}'^{-1} u(s)| = \max_{s \in [0,1]} \left| (I - L)^{-1} \int\limits_0^s u(\tau) d\tau \right| \le \dfrac{1}{1 - \theta_1} ||u||.$$

Next, by (4.2), we have

$$||F_\gamma' - F_{y_0}'|| \le K_0 ||\gamma - y_0||, \quad \forall \gamma \in C^1[0, 1].$$

Clearly, $\eta = \dfrac{||F(y_0)||}{1 - \theta_1}$ and $\beta = \dfrac{1}{1 - \theta_1}$. By assumption, $h = \eta \beta K_0 < \dfrac{1}{2}$. Thus, all the assumption of Theorem 3.5 are satisfied. Therefore, Theorem 4.1 follows from Theorem 3.5. □

4.2 Boundary value problem
Consider the second-order boundary value problem

$$\dfrac{d^2 \gamma(s)}{ds^2} = g\left(s, \, \gamma(s)\right), \quad \gamma(0) = 0, \quad \gamma(1) = 0. \tag{4.5}$$

Assume that $g_2'\left(s, \, \gamma(s)\right)$ exists for all $(s, y(s)) \in S \subseteq [0, 1] \times \mathbb{R}$ and

$$||g_2'\left(s, \, \gamma(s)\right) - g_2'\left(s, \, y_0(s)\right)|| \le K_0 ||\gamma - y_0||, \tag{4.6}$$

where $y_0 \in C^2 0[1]$ and $C^2 0[1]$ is the space of all real-valued second time continuously differentiable functions defined on $0[1]$. Consider the operator $F : C^2 0[1] \to C0[1]$ defined by

$$F(y)(s) = \frac{d^2 y(s)}{ds^2} - g(s, y(s)).$$
(4.7)

Then, solving problem (4.5) is equivalent to solving the (1.1). As in the case of initial value problem, one can observe that the operator F defined by (4.7) is Fréchet differentiable and its Fréchet derivative at each $y \in C^2 0[1]$ is given by

$$F'_y h(s) = \frac{d^2 h(s)}{ds^2} - g'_2(s, y(s)) h(s), \quad \forall h \in C^2[0, 1], \quad \forall s \in [0, 1].$$

Theorem 4.2 Let $F : C^2 0[1] \to C0[1]$ be an operator defined by (4.7). For some $y_0 \in C^2 0[1]$, assume that $F'^{-1}_{y_0}$ exists and $g'_2(s, y(s))$ satisfies (4.6). Suppose that $h = \frac{K_0 \|F(y_0)\|}{(8 - \theta_2)^2} < \frac{1}{2}$, where $\theta_2 = \sup_{s \in [0,1]} |g'_2(s, y_0(s))|$. Then, we have the following:

(a) The boundary value problem (4.5) has a unique solution y^* in $B_r[y_0]$, where $r = \frac{1 - \sqrt{1 - 2h}}{h} \eta$.

(b) For $x_0 = y_0$, the sequence $\{x_n\}$ generated by Algorithm 1.3 is in $B_r[y_0]$ and it converges strongly to y^*.

Proof: To prove the theorem it is sufficient to find an upper bound of $\|F'^{-1}_{y_0}\|$. Set

$$F'_{y_0} h(s) = \frac{d^2 h(s)}{ds^2} - g'_2(s, y_0(s)) h(s) = u(s),$$

then $h(s) = F'^{-1}_{y_0} w(s)$ and we arrive at the linear boundary value problem

$$\begin{cases} \dfrac{d^2 h(s)}{ds^2} = u(s) + g'_2(s, y_0(s)) h(s); \\ u(0) = u(1) = 0. \end{cases}$$
(4.8)

Problem (4.8) may be written in the form of the integral equation of Fredholm type of second kind (see [18]) as

$$h(s) = -\int_0^1 G(s, \tau)[u(\tau) + g'_2(\tau, y_0(\tau)) h(\tau)] d\tau,$$

where

$$G(s, \tau) = \begin{cases} \tau(1 - s), & s \geq \tau; \\ s(1 - \tau), & s \leq \tau. \end{cases}$$

Consider the operator L defined by

$$L(h)(s) = -\int_0^1 G(s, \tau) g'_2(\tau, y_0(\tau)) h(\tau) d\tau$$

and consequently, we have

$$(I - L)(h)(s) = -\int_0^1 G(s, \tau) u(\tau) d\tau.$$

Using the max norm in $C0[1]$, the operator L is bounded and

$$\|L\| \leq \frac{\theta_2}{8}, \text{ where } \theta_2 = \sup_{s \in [0,1]} |g_2'(s, \gamma(s))|.$$

By Lemma 2.3, there exists $(I - L)^{-1}$ if $\theta_2 < 8$ and

$$\left\| (I - L)^{-1} \right\| \leq \frac{8}{8 - \theta_2}.$$

Observe that

$$\left\| F_{\gamma_0}'^{-1} \right\| \leq \frac{1}{8 - \theta_2} \text{ and } \left\| F_{\gamma_0}'^{-1} F(\gamma_0) \right\| \leq \frac{\|F(\gamma_0)\|}{8 - \theta_2}.$$

Finally, by (4.6), we have

$$\left\| F_\gamma' - F_{\gamma_0}' \right\| \leq K_0 \left\| \gamma - \gamma_0 \right\|, \quad \forall \gamma \in C^2[0, 1].$$

Clearly, $\eta = \frac{\|F(\gamma_0)\|}{8 - \theta_2}$ and $\beta = \frac{1}{8 - \theta_2}$. By assumption, $h = \eta \beta K_0 < \frac{1}{2}$. Thus, all the assumption of Theorem 3.5 are satisfied. Therefore, Theorem 4.2 follows from Theorem 3.5. □

5 Numerical examples

First, we derive the following corollary from Theorem 3.5.

Corollary 5.1 *For $N \in \mathbb{N}$, let $F : \mathbb{R}^N \to \mathbb{R}^N$ be a Fréchet differentiable operator at each point of an open convex subset D of $\mathbb{R}^N$ defined by*

$$F(x) = (f_1(x), f_2(x), \ldots, f_N(x)), \forall x = (x_1, x_2, \ldots, x_N) \in D,$$

where $f_i : \mathbb{R}^N \to \mathbb{R}$, for $i = 1, 2, \ldots, N$. For some $x_0 \in D$, assume that the Jacobian matrix $[J_F(x_0)]$ of F at x_0 defined by

$$[J_F(x_0)] = \begin{cases} \dfrac{\partial f_1(x_0)}{\partial x_1} & \dfrac{\partial f_1(x_0)}{\partial x_2} & \cdots & \dfrac{\partial f_1(x_0)}{\partial x_N} \\ \dfrac{\partial f_2(x_0)}{\partial x_1} & \dfrac{\partial f_2(x_0)}{\partial x_2} & \cdots & \dfrac{\partial f_2(x_0)}{\partial x_N} \\ \cdots & \cdots & \cdots & \cdots \\ \dfrac{\partial f_N(x_0)}{\partial x_1} & \dfrac{\partial f_N(x_0)}{\partial x_2} & \cdots & \dfrac{\partial f_N(x_0)}{\partial x_N} \end{cases}$$

is invertible. Suppose that the inverse matrix $[J_F(x_0)]^{-1}$ and F satisfy the following conditions:

(i) $||[J_F(x_0)]^{-1} F(x_0)|| \leq \eta$, for some $\eta > 0$,

(ii) $|| [J_F(x_0)]^{-1} || \leq \beta$, for some $\beta > 0$,

(iii) $|| [J_F(x)] - [J_F(x_0)]|| \leq K_0 ||x - x_0||$, $\forall x \in D$ and for some $K_0 > 0$.

Let $\alpha \in (0, 1)$, $h = \eta \beta K_0 < \frac{1}{2}$ and $B_r[x_0] \subseteq D$, where $r = \frac{1 - \sqrt{1 - 2h}}{h} \eta$. Then we have the following:

(a) The Equation (1.1) has a unique solution $x^* \in B_r[x_0]$.

(b) *The sequence $\{x_n\}$ generated by*

$$\begin{cases} x_{n+1} = z_n - [J_F(x_0)]^{-1} F(z_n), \\ z_n = (1 - \alpha) x_n + \alpha \gamma_n, \\ \gamma_n = x_n - [J_F(x_0)]^{-1} F(x_n), \quad n \in \mathbb{N}_0 \end{cases} \tag{5.1}$$

is in $B_r[x_0]$ and it converges to x^*.

(c) *The following error estimate holds:*

$$\left\| x_{n+1} - x^* \right\| \le \rho^{n+1} \left\| x_0 - x^* \right\|, \quad \forall n \in \mathbb{N}_0,$$

where $\rho = \gamma (1 - \alpha + \alpha \gamma)$ and $\gamma = \beta r K_0$.

Following example shows numerically that (5.1) is faster than the modified Newton method defined by (1.3).

Example 5.2 *Let $X = \mathbb{R}$, $D = (-1, 1)$ and $F : D \to \mathbb{R}$ an operator defined by*

$$F(x) = e^x - 1, \quad \forall x \in D.$$

Then F is Fréchet differentiable and its Fréchet derivative F'_x at any point $x \in D$ is given by

$$F'_x = e^x.$$

For $x_0 = 0.26$, we have

$$F'_{x_0}{}^{-1} = \frac{1}{e^{0.26}}.$$

Set $\beta = 0.771051585803566$, $\eta = 0.228948414196434$ and $K_0 = 2.718281828459046$, we have $h = \eta \beta K_0 < \frac{1}{2}$ and

$$\begin{aligned} \left\| F'_{x_0}{}^{-1} \right\| &\le \beta_0, \\ \left\| F'_{x_0}{}^{-1} F(x_0) \right\| &\le \eta_0, \\ \left\| F'_x - F'_{x_0} \right\| &\le K_0 \left\| x - x_0 \right\|. \end{aligned}$$

Hence, all the condition of Corollary 5.1 are satisfied. Therefore, the sequence $\{x_n\}$ generated by (5.1) is in $B_r[x_0]$ and it converges to a unique $x^* \in B_r[x_0]$. The Figure 1 and Table 1 show that sequence $\{x_n\}$ generated by (5.1) is faster than the modified Newton method defined by (1.3).

For $N = 2$ in Corollary 5.1, the following example shows numerically the convergence of (5.1).

Example 5.3 *Let $D = X = Y = \mathbb{R}^2$ under the norm*

$$\| x \| = \sqrt{x^2 + y^2}, \quad \forall x = (x, y) \in \mathbb{R}^2$$

and induced matrix norm. Consider an operator $F : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$F(x) = \left(-x^2 + \frac{1}{3}, -y^2 + \frac{1}{3} \right), \quad \forall x = (x, y) \in \mathbb{R}^2.$$
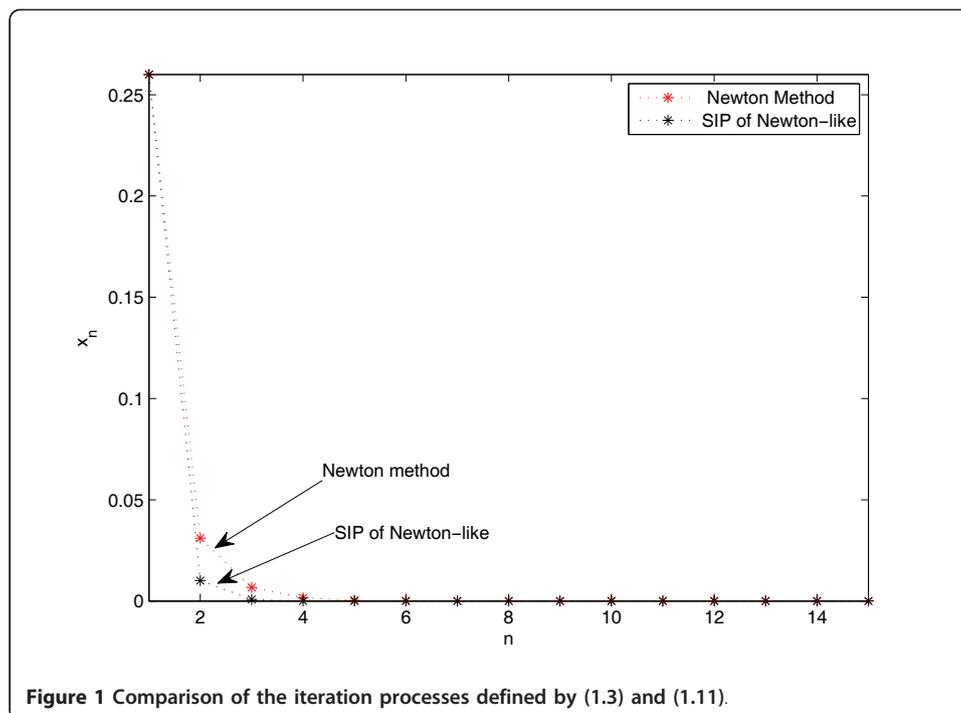
Figure 1 Comparison of the iteration processes defined by (1.3) and (1.11).

Clearly, the point $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ is the zero of F in D. It can be seen that F is Fréchet differentiable at each point of D and its Jacobian matrix $[J_F(x)]$ at any point $x = (x, y) \in \mathbb{R}^2$ is given by

$$[J_F(x)] = \begin{bmatrix} -2x & 0 \\ 0 & -2y \end{bmatrix}.$$

Now, for any $x, x_0 \in D$, we have

$$\left\| F'_x - F'_{x_0} \right\| \leq 2 \left\| x - x_0 \right\|.$$

For $x_0 = (1, 1)$, we get

$$[J_F(x_0)]^{-1} = \begin{bmatrix} -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}.$$

Therefore, for $\beta = \frac{1}{\sqrt{2}}$, $\eta = \frac{\sqrt{2}}{3}$ and $K_0 = 2$, we have $h = \beta \eta K_0 < 1/2$ and

$$\left\| [J_F(x_0)]^{-1} \right\| \leq \beta,$$
$$\left\| [J_F(x_0)]^{-1} F(x_0) \right\| \leq \eta,$$

and $B_r[x_0]$, where $r = \frac{1 - \sqrt{(1-2h)}}{h} \eta = 3 - \sqrt{3}$, is contained in $D = X$. Hence, all the conditions of Corollary 5.1 are satisfied. One can write sequence $\{x_n\}$ generated by (5.1) as

$$x_{n+1} = \begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{3} - \frac{\alpha^2}{72}\right) + \left(1 - \frac{\alpha}{6}\right) x_n - \left(\frac{1}{2} + \frac{\alpha}{2} - \frac{\alpha^2}{12}\right) x_n^2 + \frac{\alpha}{2} x_n^3 - \frac{\alpha^2}{8} x_n^4 \\ \left(\frac{1}{3} - \frac{\alpha^2}{72}\right) + \left(1 - \frac{\alpha}{6}\right) y_n - \left(\frac{1}{2} + \frac{\alpha}{2} - \frac{\alpha^2}{12}\right) y_n^2 + \frac{\alpha}{2} y_n^3 - \frac{\alpha^2}{8} y_n^4 \end{bmatrix}.$$

Table 1 Comparison of the iteration processes defined by (1

n	Newton method	SIP of Newton-like	n	Newton method	SIP of Newton-like
1	0.260000000000000	0.260000000000000	9	0.000000955573754	0.000000000272069
2	0.031051585803566	0.010093382981211	10	0.000000218776743	0.000000000022642
3	0.006733609382298	0.000820742954637	11	0.000000050088570	0.000000000001884
4	0.001524129572619	0.000068174499880	12	0.000000011467698	0.000000000000157
5	0.000348051028303	0.000005672583045	13	0.000000002625511	0.000000000000013
6	0.000079639023162	0.000000472064696	14	0.000000000601107	0.000000000000001
7	0.000018230782847	0.000000039285051	15	0.000000000137622	0.000000000000000
8	0.000004173780688	0.000000003269291	16	0.000000000031508	0.000000000000000

Therefore, in view of Corollary 5.1, the sequence $\{x_n\}$ is in $B_r[x_0]$ and it converges to a unique $x^* = (x^*, y^*) \in B_r[x_0]$. The Figure 2 and Table 2 show that sequence $\{x_n\}$ generated by (5.1) with $\alpha = 0.5$ is faster than the modified Newton method defined by (1.3).

Now, we study the convergence of (1.11) for infinite dimensional cases,

Example 5.4 *Let $X = D = C0[1]$ be the space of real-valued continuous functions defined on the interval $0[1]$ with norm*

$$\|x\| = \max_{0 \leq t \leq 1} |x(t)|.$$

Consider the integral equation $F(x) = 0$, where

$$F(x)(s) = -1 + x(s) + \pi x(s) \int_0^1 \frac{s}{s+t} x(t) dt,$$

with $s \in 0[1]$, $x \in C0[1]$ and $\pi \in (0, 2]$. Integral equations of this kind called Chandrasekhar equations arise in elasticity or neutron transport problems [19]. The norm is taken as sup-norm. Now it is easy to find the Fréchet derivative of F as
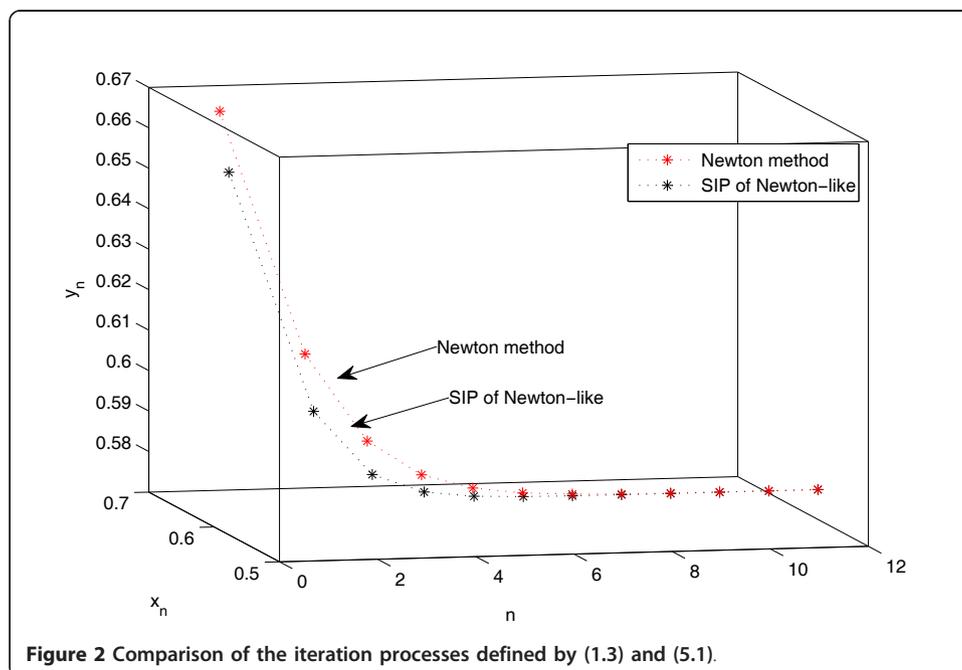

Figure 2 Comparison of the iteration processes defined by (1.3) and (5.1).

Table 2 Comparison of the iteration processes defined by (1

N	Newton method	SIP of Newton-like	n	Newton method	SIP of Newton-like
1	1	1	7	0.578358062815874	0.577513680188369
	1	1		0.578358062815874	0.577513680188369
2	0.666666666666667	0.652777777777778	8	0.577775705070475	0.577399387702240
	0.666666666666667	0.652777777777778		0.577775705070475	0.577399387702240
3	0.611111111111111	0.598061698654261	9	0.575529989052300	0.577365035376011
	0.611111111111111	0.598061698654261		0.575529989052300	0.577365035376011
4	0.591049382716049	0.583424704288997	10	0.577426211591592	0.577354708436348
	0.591049382716049	0.583424704288997		0.577426211591592	0.577354708436348
5	0.583046362978205	0.579163300958378	11	0.577382363341750	0.577351603803299
	0.583046362978205	0.579163300958378		0.577382363341750	0.577351603803299
6	0.579741498953815	0.577894163483913	12	0.577363833259364	0.577350670428927
	0.579741498953815	0.577894163483913		0.577363833259364	0.577350670428927

Now one can easily compute

$$
\begin{aligned}
\|F(x_0)\| &= \left\| -1 + x_0(s) + \pi x_0(s) \int_0^1 \frac{s}{s+t} x_0(t)\,dt \right\| \\
&\leq \|x_0 - 1\| + |\pi| \max_{s \in [0,1]} \left| \int_0^1 \frac{s}{s+t}\,dt \right| \|x_0\|^2 \\
&\leq \|x_0 - 1\| + \pi \log 2 \|x_0\|^2.
\end{aligned}
$$

Also notice that

$$
\begin{aligned}
\|I - F'_{x_0}\| &= \left\| \pi \int_0^1 \frac{s}{s+t} x_0(t)\,dt + \pi x_0(s) \int_0^1 \frac{s}{s+t}\,dt \right\| \\
&\leq 2\pi \max_{s \in [0,1]} \left| \int_0^1 \frac{s}{s+t}\,dt \right| \|x_0\| \\
&\leq 2\pi \log 2 \|x_0\|
\end{aligned}
$$

Now, we have

$$
\begin{aligned}
\|F'_x - F'_{x_0}\| &= \left\| \pi \int_0^1 \frac{s}{s+t} x(t)\,dt + \pi x(s) \int_0^1 \frac{s}{s+t}\,dt - \pi \int_0^1 \frac{s}{s+t} x_0(t)\,dt - \pi x_0(s) \int_0^1 \frac{s}{s+t}\,dt \right\| \\
&= |\pi| \left\| \int_0^1 \frac{s}{s+t} (x(t) - x_0(t)\,dt + (x(s) - x_0(s)) \int_0^1 \frac{s}{s+t}\,dt \right\| \\
&\leq 2\pi \max_{s \in [0,1]} \left| \int_0^1 \frac{s}{s+t}\,dt \right| \|x - x_0\| \\
&\leq 2\pi \log 2 \|x - x_0\|
\end{aligned}
$$

and, if $2|\pi \mid \log 2|| x_0|| < 1$, then by Lemma 2.3, we obtain

$$\left\|F_{x_0}'^{-1}\right\| \leq \frac{1}{1 - 2\pi \log 2 ||x_0||}.$$

Hence, we have

$$\left\|F_{x_0}'^{-1}F(x_0)\right\| \leq \frac{\|x_0 - 1\| + \pi |\log 2| \|x_0\|^2}{1 - 2\pi \log 2 \|x_0\|}.$$

Now, for $\pi = \frac{1}{4}$ and the initial point $x_0 = x_0(s) = 1$, we obtain

$$|| \left\|F_{x_0}'^{-1}\right\| \leq \beta = 1.17718382, \left\|F_{x_0}'^{-1}F(x_0)\right\| \leq \eta = 0.08859191, K_0 = 0.346573590279973.$$

Hence, $h = \beta\eta K_0 = 0.036143800345579 < \frac{1}{2}$. So the hypotheses of Theorem 3:5 are satisfied. Therefore, the sequence $\{x_n\}$ generated by Algorithm 1.3 is in $B_r[x_0]$ and it converges to a unique solution $x^* \in B_r[x_0]$ of the integral equation.

Authors' contributions

All authors contribute equally and significantly in this research work. All authors read and approved the final manuscript.

Competing interests

The authors declare that they have no competing interests.

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