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The hybrid algorithm for the system of mixed equilibrium problems, the general system of finite variational inequalities and common fixed points for nonexpansive semigroups and strictly pseudo-contractive mappings

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Abstract

In this article, we introduce a new iterative algorithm by the shrinking projection method for finding a common element of the set of solutions of generalized mixed equilibrium problems, the set of common solutions of general system of finite variational inequalities, the set of solutions of fixed points for nonexpansive semigroups and the set of common fixed points for an infinite family of strictly pseudo-contractive mappings in a real Hilbert space. We prove that the sequence converges strongly to a common element of the above four sets under some mind conditions. Our results improve and extend the corresponding recent results in literature work.

1 Introduction

Throughout this article, we assume that C is a closed convex subset of a real Hilbert space H with inner product and norm are denoted by $\langle .,. \rangle$ and $\|.\|$, respectively.

Let A, B: $C \to H$ be two mappings. We consider the following problem of finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \lambda Ay^* + x^* - y^*, x - x^* \rangle \geq 0, & \forall x \in C, \\ \langle \mu Bx^* + y^* - x^*, x - y^* \rangle \geq 0, & \forall x \in C, \end{cases} \tag{1}$$

which is called a *general system of variational inequalities*, where $\lambda \geq 0$ and $\mu \geq 0$ are two constants. The set of solution of (1) is denoted by $SVI(C, A, B)$. In particular, if $A = B$, then problem (1) reduces to finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \lambda Ay^* + x^* - y^*, x - x^* \rangle \geq 0, & \forall x \in C, \\ \langle \mu Ax^* + y^* - x^*, x - y^* \rangle \geq 0, & \forall x \in C, \end{cases} \tag{2}$$

which is defined by Verma [1] (see also Verma [2]), and is called the *new general system of variational inequalities.* Further, if we set $B = 0$, then problem (1) reduces to the classical variational inequality $VI(C, A)$ which was originally introduced and studied by Stampacchia [3] in 1964.

By the system of variational inequality problems above, we extend into the *general system of finite variational inequalities* is to find $(x_1^*, x_2^*, \ldots, x_M^*) \in C \times C \times \cdots \times C$ and is defined by

$$\begin{cases} \langle \lambda_M A_M x_M^* + x_1^* - x_M^*, x - x_1^* \rangle \geq 0, \quad \forall x \in C, \\ \langle \lambda_{M-1} A_{M-1} x_{M-1}^* + x_M^* - x_{M-1}^*, x - x_M^* \rangle \geq 0, \quad \forall x \in C, \\ \vdots \\ \langle \lambda_2 A_2 x_2^* + x_3^* - x_2^*, x - x_3^* \rangle \geq 0, \quad \forall x \in C, \\ \langle \lambda_1 A_1 x_1^* + x_2^* - x_1^*, x - x_2^* \rangle \geq 0, \quad \forall x \in C, \end{cases} \tag{3}$$

where $\{A_l\}_{l=1}^M : C \to H$ is a family of mappings, $\lambda_l \geq 0$, $l \in \{1,2, ..., M\}$. The set of solution of (3) is denoted by $GSVI(C, A_l)$. In particular, if $M = 2, A_1 = B, A_2 = A, \lambda_1 = \mu, \lambda_2 = \lambda, x_1^* = x^*$, and $x_2^* = y^*$, then the problem (3) is reduced to the problem (1).

Recall that a mapping $T : C \to C$ is said to be a *k-strict pseudo-contraction* (see also [4]) if there exists $0 \leq k < 1$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \forall x, y \in C,$$

where I denotes the identity operator on C (see also [5]). If $k = 0$, a mapping $T : C \to C$ is said to be *nonexpansive* [6], that is,

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$$

If $k = 1$, a mapping $T : C \to C$ is said to be *pseudo-contraction,* that is,

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|(I - T)x - (I - T)y\|^2, \forall x, y \in C.$$

Clearly, the class of k-strict pseudo-contraction falls into the one between classes of nonexpansive mappings and pseudo-contraction mappings. We denote the set of fixed points of T by $F(T)$.

Let $\Im = \{F_k\}_{k \in \Gamma}$ be a countable family of bifunctions from $C \times C$ to $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers and Γ is an arbitrary index set. Let $\phi : C \to \mathbb{R} \cup \{+\infty\}$ be a proper extended real-valued function. The *system of mixed equilibrium problems* is to find $x \in C$ such that

$$F_k(x, y) + \varphi(y) \geq \varphi(x), \quad \forall k \in \Gamma, \quad \forall k \in C. \tag{4}$$

The set of solutions of (4) is denoted by $SMEP(F_k, \phi)$, that is

$$SMEP(F_k, \varphi) = \{x \in C : F_k(x, y) + \varphi(y) \geq \varphi(x), \quad \forall k \in \Gamma, \forall y \in C\}. \tag{5}$$

If Γ is a singleton, the problem (4) reduces to find the following *mixed equilibrium problem* (see also Flores-Bazán [7]). For finding $x \in C$ such that

$$F(x, y) + \varphi(y) \geq \varphi(x), \quad \forall y \in C. \tag{6}$$

The set of solutions of (6) is denoted by $MEP(F, \phi)$. Combettes and Hirstoaga [8] introduced the following *system of equilibrium problems.* For finding $x \in C$ such that,

$$F_k(x, y) \geq 0, \forall k \in \Gamma, \forall y \in C. \tag{7}$$

The set of solutions of (7) is denoted by $SEP(\Im)$, that is,

$$SEP(\Im) = \{x \in C : F_k(x, y) \geq 0,\ \forall k \in \Gamma,\ \forall y \in C\}. \tag{8}$$

If Γ is a singleton, the problem (7) becomes the following *equilibrium problem.* For finding $x \in C$ such that

$$F(x, y) \geq 0,\ \forall y \in C. \tag{9}$$

The set of solution of (9) is denoted by $EP(F)$. The mixed equilibrium problems include fixed point problems, variational inequality problems, optimization problems, Nash equilibrium problems, noncooperative games, economics and the equilibrium problem as special cases (see [9-19]). In the last two decades, many articles have appeared in the literature on the existence of solutions of equilibrium problems; see, for example [13] and references therein. Some solution methods have been proposed to solve the mixed equilibrium problems; see, for example, (see [11-14,16-24]) and references therein.

A family $\mathcal{S} = \{S(s) : 0 \leq s < \infty\}$ of mappings of C into itself is called a *nonexpansive semigroup* on C if it satisfies the following conditions:

(i) $S(0)x = x$ for all $x \in C$;

(ii) $S(s + t) = S(s)\ S(t)$ for all $s, t \geq 0$;

(iii) $\|S(s)x - S(s)y\| \leq \|x - y\|$ for all $x, y \in C$ and $s \geq 0$;

(iv) for all $x \in C$, $s \mapsto S(s)x$ is continuous.

We denote by $F(\mathcal{S})$ the set of all common fixed points of $\mathcal{S} = \{S(s) : s \geq 0\}$, i.e., $F(\mathcal{S}) = \cap_{s \geq 0} F(S(s))$. It is well known that $F(\mathcal{S})$ is closed and convex (see also [25,26]).

In 2011, Shehu [21] introduced a new iterative scheme by hybrid method for finding a common element of the set of common fixed points of an infinite family of k-strictly pseudocontractive mappings and the set of common solutions to a system of generalized mixed equilibrium problems and the set of solution of variational inequality problems in Hilbert spaces. Starting with an arbitrary $x_0 \in C, C_{1,i} = C, C_1 = \cap_{i=1}^{\infty} C_{1,i}, x_1 = P_{C_1} x_0$ define sequences $\{x_n\}$, $\{w_n\}$, $\{u_n\}$, $\{z_n\}$, and $\{y_{n,i}\}$ as follows:

$$\begin{cases} z_n = T_{r_n}^{(F_1,\varphi_1)}(x_n - r_n A x_n) \\ y_n = T_{\lambda_n}^{(F_2,\varphi_2)}(z_n - \lambda_n B z_n) \\ w_n = P_C(u_n - s_n D u_n) \\ y_{n,i} = \alpha_{n,i} w_n + (1 - \alpha_{n,i}) T_i w_n, \quad n \geq 1, \\ C_{n+1,i} = \{z \in C_{n,i} : \left\| y_{n,i} - z \right\| \leq \|x_n - z\|\}, \quad n \geq 1, \\ C_{n+1} = \cap_{i=1}^{\infty} C_{n+1,i}, \\ x_{n+1} = P_{C_{n+1}} x_0, \quad n \geq 1, \end{cases} \tag{10}$$

where T_i be a k_i -strictly pseudocontractive mapping and for some $0 \leq k_i < 1$, A, B are α, β-inverse-strongly monotone mappings of C into H, respectively. He proved that if the sequences $\{\alpha_{n,i}\}$, $\{r_n\}$, $\{s_n\}$, and $\{\lambda_n\}$ of parameters satisfies appropriate conditions, then $\{x_n\}$ is generated by (10) converges strongly to $P_\Omega x_0$, where P_Ω is metric projection on H in to $\Omega := MEP(F_1, \varphi_1) \cap MEP(F_2, \varphi_2) \cap VI(C, A) \cap (\cap_{i=1}^{\infty} F(T_i))$. For using the hybrid method, we can see [27-29].

In this article, motivated by the above results, we present a new iterative algorithm for finding a common element of the set of solutions for a system of mixed equilibrium problems, the set of common solutions of general system of finite variational inequality problems, the set of solutions of fixed points for nonexpansive semigroup mappings and the set of common fixed points for an infinite family of strictly pseudo-contractive mappings in a real Hilbert space. Then, we prove strong convergence theorem under some mind conditions. The results presented in this article extend and improve the results of Shehu [21] and many authors.

2 Preliminaries

Let H be a real Hilbert space with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$, respectively. Let C be a closed convex subset of H. The sequence $\{x_n\}$ is a sequence in H, $x_n \rightharpoonup x$ means $\{x_n\}$ converges weakly to x and $x_n \to x$ means $\{x_n\}$ converges strongly to x. In a real Hilbert space H, we have

$$\|x-y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x-y, y\rangle \tag{11}$$

and

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2, \quad \forall x, y \in H, \tag{12}$$

and $\lambda \in \mathbb{R}$. For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \le \|x-y\|, \quad \forall y \in C. \tag{13}$$

P_C is called the *metric projection* of H onto C. It is well known that P_C is a nonexpansive mapping of H onto C and satisfies

$$\langle x-y, P_C x - P_C y\rangle \ge \|P_C x - P_C y\|^2, \quad \forall x, y \in H. \tag{14}$$

Moreover, $P_C x$ is characterized by the following properties: $P_C x \in C$ and

$$\langle x - P_C x, y - P_C x\rangle \le 0, \tag{15}$$

$$\|x-y\|^2 \ge \|x - P_C x\|^2 + \|y - P_C x\|^2, \forall x \in H, y \in C. \tag{16}$$

Recall that a mapping A of C into H is called *α-inverse-strongly monotone* if there exists a positive real number α such that

$$\langle Ax - Ay, x-y\rangle \ge \alpha\|Ax - Ay\|^2, \quad \forall x, y \in C. \tag{17}$$

It is obvious that any α-inverse-strongly monotone mappings A is $\left(\frac{1}{\alpha}\right)$-Lipschitz monotone and continuous mappings.

In order to prove our main results, we need the following Lemmas.

Lemma 2.1. [30]*Let $V : C \to H$ be a k-strict pseudo-contraction, then*

(1) the fixed point set $F(V)$ of V is closed convex so that the projection $P_{F(V)}$ is well defined;

(2) define a mapping $T : C \to H$ by

$$Tx = tx + (1-t)Vx, \quad \forall x \in C. \tag{18}$$

If $t \in [k, 1)$, then T is a nonexpansive mapping such that $F(V) = F(T)$.

A family of mappings $\{V_i : C \to H\}_{i=1}^{\infty}$ is called *a family of uniformly k-strict pseudo-contractions,* if there exists a constant $k \in [0, 1)$ such that

$$\|V_i x - V_i y\|^2 \le \|x - y\|^2 + k\|(1 - V_i)x - (I - V_i)y\|^2, \quad \forall x, y \in C, \quad \forall i \ge 1.$$

Let $\{V_i : C \to C\}_{i=1}^{\infty}$ be a countable family of uniformly k-strict pseudo-contractions. Let $\{T_i : C \to C\}_{i=1}^{\infty}$ be the sequence of nonexpansive mappings defined by (18), i.e.,

$$T_i x = tx + (1 - t)V_i x, \forall x \in C, \quad \forall i \ge 1, \quad t \in [k, 1). \tag{19}$$

Let $\{T_i\}$ be a sequence of nonexpansive mappings of C into itself defined by (19) and let $\{\mu_i\}$ be a sequence of nonnegative numbers in [0,1]. For each $n \ge 1$, define a mapping W_n of C into itself as follows:

$$\begin{aligned}
U_{n,n+1} &= I,\\
U_{n,n} &= \mu_n T_n U_{n,n+1} + (1 - \mu_n)I,\\
U_{n,n-1} &= \mu_{n-1} T_{n-1} U_{n,n} + (1 - \mu_{n-1})I,\\
&\vdots\\
U_{n,k} &= \mu_k T_k U_{n,k+1} + (1 - \mu_k)I,\\
U_{n,k-1} &= \mu_{k-1} T_{k-1} U_{n,k} + (1 - \mu_{k-1})I,\\
&\vdots\\
U_{n,2} &= \mu_2 T_2 U_{n,3} + (1 - \mu_2)I,\\
W_n = U_{n,1} &= \mu_1 T_1 U_{n,2} + (1 - \mu_1)I.
\end{aligned} \tag{20}$$

Such a mapping W_n is nonexpansive from C to C and it is called the W-mapping generated by T_1, T_2, ..., T_n and μ_1, μ_2, ..., μ_n.

For each $n, k \in \mathbb{N}$, let the mapping $U_{n,k}$ be defined by (20). Then we can have the following crucial conclusions concerning W_n. You can find them in [31]. Now we only need the following similar version in Hilbert spaces.

Lemma 2.2. [31]*Let C be a nonempty closed convex subset of a real Hilbert space H. Let T_1, T_2, ... be nonexpansive mappings of C into itself such that $\cap_{n=1}^{\infty} F(T_n)$ is nonempty, let μ_1, μ_2, ... be real numbers such that $0 \le \mu_n \le b < 1$ for every $n \ge 1$. Then,*

(1) W_n is nonexpansive and $F(W_n) = \cap_{i=1}^{n} F(T_i), \forall n \ge 1$;

(2) for every $x \in C$ and $k \in \mathbb{N}$, the limit $\lim_{n\to\infty} U_{n,k}\, x$ exists;

(3) a mapping $W : C \to C$ defined by

$$Wx := \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1} x, \forall x \in C \tag{21}$$

is a nonexpansive mapping satisfying $F(W) = \cap_{i=1}^{\infty} F(T_i)$ and it is called the W-mapping generated by T_1,T_2, ... and μ_1, μ_2, ...

Lemma 2.3. [32]*Let C be a nonempty closed convex subset of a Hilbert space H, $\{T_i : C \to C\}$ be a countable family of nonexpansive mappings with $\cap_{i=1}^{\infty} F(T_i) \ne \emptyset$, $\{\mu_i\}$ be a real sequence such that $0 < \mu_i \le b < 1$, $\forall i \ge 1$. If D is any bounded subset of C, then*

$$\lim_{n\to\infty} \sup_{x \in D} \|Wx - W_n x\| = 0.$$

Lemma 2.4. [33]*Each Hilbert space H satisfies Opial's condition, i.e., for any sequence* $\{x_n\} \subset H$ *with* $x_n \rightharpoonup x$, *the inequality*

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|,$$

hold for each $y \in H$ *with* $y \neq x$.

Lemma 2.5. [22]*Let C be a nonempty bounded closed convex subset of a Hilbert space H and let* $\mathcal{S} = \{S(s) : 0 \le s < \infty\}$ *be a nonexpansive semigroup on C, then for any* $h \geq 0$,

$$\lim_{t\to\infty} \sup_{x\in C} \left\| \frac{1}{t}\int_0^t S(s)xds - S(h)\left(\frac{1}{t}\int_0^t S(s)xds\right)\right\| = 0.$$

Lemma 2.6. [34]*Let C be a nonempty bounded closed convex subset of H,* $\{x_n\}$ *be a sequence in C and* $\mathcal{S} = \{S(s) : 0 \le s < \infty\}$ *be a nonexpansive semigroup on C. If the following conditions are satisfied:*

(i) $x_n \rightharpoonup z$;
(ii) $\limsup_{s\to\infty} \limsup_{n\to\infty} \|S(s)x_n - x_n\| = 0$, *then* $z \in F(\mathcal{S})$.

Lemma 2.7. *Let C be a nonempty closed convex subset of Hilbert space H,* $A_l : C \to H$ *be a* β_l*-inverse-strongly monotone and* $\lambda_l \in (0, 2\beta_l)$ *where* $l \in \{1, 2, ..., M\}$. *If* $\mathcal{P} : C \to C$ *is defined by*

$$\mathcal{P}(x) = P_C(I - \lambda_M A_M)P_C(I - \lambda_{M-1}A_{M-1})\dots P_C(I - \lambda_2 A_2)P_C(I - \lambda_1 A_1)x, \forall x \in C,$$

then $\mathcal{P}$ *is nonexpansive.*

Proof. Taking $\mathcal{P}_C^l = P_C(I - \lambda_l A_l)P_C(I - \lambda_{l-1}A_{l-1})\dots P_C(I - \lambda_2 A_2)P_C(I - \lambda_1 A_1), l \in \{1, 2, 3, \dots, M\}$ and $\mathcal{P}_C^0 = I$, where I is the identity mapping on H. Then we have $\mathcal{P} = \mathcal{P}_C^M$.

For any $x, y \in C$, we have

$$\begin{aligned}
\|\mathcal{P}(x) - \mathcal{P}(y)\| &= \|\mathcal{P}_C^M x - \mathcal{P}_C^M y\| \\
&= \|P_C(I - \lambda_M A_M)\mathcal{P}_C^{M-1}x - P_C(I - \lambda_M A_M)\mathcal{P}_C^{M-1}y\| \\
&\le \|(I - \lambda_M A_M)\mathcal{P}_C^{M-1}x - (I - \lambda_M A_M)\mathcal{P}_C^{M-1}y\| \\
&\le \|\mathcal{P}_C^{M-1}x - \mathcal{P}_C^{M-1}y\| \\
&\ \ \vdots \\
&\le \|\mathcal{P}_C^0 x - \mathcal{P}_C^0 y\| \\
&= \|x - y\|.
\end{aligned}$$

This show that $\mathcal{P}$ is nonexpansive on C.

Lemma 2.8. *Let C be a nonempty closed and convex subset of a real Hilbert space H,* $A_l : C \to H$ *be nonlinear mappings, where* $l \in \{1, 2, ..., M\}$. *For* $x_l^* \in C, l \in \{1, 2, \dots, M\}$, *then* $(x_1^*, x_2^*, \dots, x_M^*)$ *is a solution of problem* (3) *if and only if*

$$\begin{cases}
x_1^* = P_C(I - \lambda_M A_M)x_M^* \\
x_2^* = P_C(I - \lambda_1 A_1)x_1^* \\
x_3^* = P_C(I - \lambda_2 A_2)x_2^* \\
\ \vdots \\
x_M^* = P_C(I - \lambda_{M-1}A_{M-1})x_{M-1}^*,
\end{cases} \tag{22}$$

that is

$$x_1^* = P_C(I - \lambda_M A_M)P_C(I - \lambda_{M-1}A_{M-1})\ldots P_C(I - \lambda_2 A_2)P_C(I - \lambda_1 A_1)x_1^*.$$

Proof. From the problem (3), we can rewrite as

$$\begin{cases} \langle x_1^* - (x_M^* - \lambda_M A_M x_M^*), x - x_1^* \rangle \geq 0, \forall x \in C, \\ \langle x_M^* - (x_{M-1}^* - \lambda_{M-1} A_{M-1} x_{M-1}^*), x - x_M^* \rangle \geq 0, \forall x \in C, \\ \vdots \\ \langle x_3^* - (x_2^* - \lambda_2 A_2 x_2^*), x - x_3^* \rangle \geq 0, \forall x \in C, \\ \langle x_2^* - (x_1^* - \lambda_1 A_1 x_1^*), x - x_2^* \rangle \geq 0, \forall x \in C. \end{cases} \tag{23}$$

From (15), we conclude that (23) is equivalent to (22).

Lemma 2.9. *(Demi-closedness Principle* [6]*) Assume that S is a nonexpansive self-mapping of a nonempty closed convex subset C of a real Hilbert space H. If S has a fixed point, the I - S is demi-closed: that is, whenever $\{x_n\}$ is a sequence in C converging weakly to some $x \in C$ (for short, $x_n \rightharpoonup x$), and the sequence $\{(I - S)x_n\}$ converges strongly to some y (for short, $(I - S)x_n \to y$), it follows that $(I - S)x = y$. Here I is the identity operator of H.*

For solving the system of mixed equilibrium problems, let us assume that bifunction $F_k : C \times C \to \mathbb{R}$, $k = 1,2, \ldots, N$ satisfies the following conditions:

(H1) F_k is monotone, i.e., $F_k(x, y) + F_k(y, x) \leq 0$, $\forall x, y \in C$;

(H2) for each fixed $y \in C$, $x \mapsto F_k(x,y)$ is convex and upper semicontinuous;

(H3) for each fixed $x \in C$, $y \mapsto F_k(x, y)$ is convex.

Lemma 2.10. [35]*Let C be a nonempty closed convex subset of a real Hilbert space H and let ϕ be a lower semicontinuous and convex functional from C to $\mathbb{R}$. Let F be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (H1)-(H3). Assume that*

(i) $\eta : C \times C \to H$ is k Lipschitz continuous with constant $k > 0$ such that;

(a) $\eta(x, y) + \eta(y, x) = 0$, $\forall x, y \in C$,

(b) $\eta(\cdot,\cdot)$ is affine in the first variable,

(c) for each fixed $x \in C$, $y \mapsto \eta(x, y)$ is sequentially continuous from the weak topology to the weak topology,

(ii) $\mathcal{K} : C \to \mathcal{R}$ is η-strongly convex with constant $\sigma > 0$ and its derivative $\mathcal{K}'$ is sequentially continuous from the weak topology to the strong topology;

(iii) for each $x \in C$, there exist a bounded subset $D_x \subset C$ and $z_x \in C$ such that for any $y \in C \setminus D_x$,

$$F(y, z_x) + \varphi(z_x) - \varphi(y) + \frac{1}{r}\langle \mathcal{K}'(y) - \mathcal{K}'(x), \eta(z_x, y) \rangle < 0.$$

For given $r > 0$, Let $K_r^F : C \to C$ be the mapping defined by:

$$K_r^F(x) = \left\{ y \in C : F(y, z) + \varphi(z) - \varphi(y) + \frac{1}{r}\langle \mathcal{K}'(y) - \mathcal{K}'(x), \eta(z, y) \rangle \geq 0, \forall z \in C \right\} \tag{24}$$

for all $x \in C$. Then the following hold

(1) K_r^F is single-valued;

(2) K_r^F *is nonexpansive if* $\mathcal{K}'$ *is Lipschitz continuous with constant* $v > 0$ *such that* $\sigma \geq kv$;

(3) $F(K_r^F) = MEP(F, \varphi)$;

(4) $MEP(F, \phi)$ *is closed and convex.*

3 Main result

In this section, we prove a strong convergence theorem in a real Hilbert space.

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert Space H. Let* $\{F_k : C \times C \to \mathbb{R}, k = 1, 2, ..., N\}$ *be a finite family of bifunctions satisfying conditions (H1)-(H3). Let* A_l *be* β_l*-inverse-strongly monotone mappings of C into H, where* $l \in \{1, 2, ..., M\}$. *Let* $\mathcal{S} = \{S(s) : 0 \leq s < \infty\}$ *be a nonexpansive semigroup on C and let* $\{t_n\}$ *be a positive real divergent sequence. Let* $\{V_i : C \to C\}_{i=1}^{\infty}$ *be a countable family of uniformly k-strict pseudo-contractions,* $\{T_i : C \to C\}_{i=1}^{\infty}$ *be a countable family of nonexpansive mappings defined by* $T_i x = tx + (1 - t)V_i x$, $\forall x \in C$, $\forall i \geq 1$, $t \in [k, 1)$. *For* $l \in \{1,2, ..., M\}$, *suppose* $\Theta := F(\mathcal{S}) \cap (\cap_{i=1}^{\infty} F(T_i)) \cap (\cap_{k=1}^{N} SMEP(F_k)) \cap GSVI(C, A_l) \neq \emptyset$. *Let* $\{x_n\}$ *be a sequence generated by* $x_0 \in C, C_{1,i} = C, C_1 = \cap_{i=1}^{\infty} C_{1,i}, x_1 = P_{C_1} x_0$ *and*

$$\begin{cases} u_n = K_{r_{N,n}}^{F_N} K_{r_{N-1,n}}^{F_{N-1}} \dots K_{r_{2,n}}^{F_2} K_{r_{1,n}}^{F_1} x_n, \\ w_n = P_C(I - \lambda_M A_M) P_C(I - \lambda_{M-1} A_{M-1}) \dots P_C(I - \lambda_2 A_2) P_C(I - \lambda_1 A_1) u_n, \\ y_{n,i} = \alpha_{n,i} x_0 + (1 - \alpha_{n,i}) \dfrac{1}{t_n} \int_0^{t_n} S(s) W_n w_n ds, \\ C_{n+1,i} = \{z \in C_{n,i} : \|y_{n,i} - z\|^2 \leq \|x_n - z\|^2 + \alpha_{n,i}(\|x_0\|^2 + 2\langle x_n - x_0, z\rangle)\}, \\ C_{n+1} = \cap_{i=1}^{\infty} C_{n+1,i}, \\ x_{n+1} = P_{C_{n+1}} x_0 \end{cases} \tag{25}$$

for every $n \geq 0$, *where* $K_{r_k}^{F_k} : C \to C$, *is the mapping defined by* (24), $r_k > 0$, $k = 1, 2, ..., N$ *are constants and* $\{\alpha_{n,i}\}_{n=1}^{\infty} \subset (0, 1)$ *satisfy the following conditions:*

(i) $\eta_k : C \times C \to H$ *is* L_k*-Lipschitz continuous with constant* $k = 1,2, ..., N$ *such that*

(1) $\eta_k(x, y) + \eta_k(y, x) = 0$, $\forall x, y \in C$,

(2) $x \mapsto \eta_k(x, y)$ *is affine,*

(3) for each fixed $y \in C$, $y \mapsto \eta_k(x, y)$ *is sequentially continuous from the weak topology to the weak topology;*

(ii) $\mathcal{K}_k : C \to \mathcal{R}$ *is* η_k*-strongly convex with constant* $\sigma_k > 0$ *and its derivative* $\mathcal{K}_k'$ *is not only sequentially continuous from the weak topology to the strong topology but also Lipschitz continuous with a Lipschitz constant* $v_k > 0$ *such that* $\sigma_k > L_k v_k$;

(iii) For each $k \in \{1, 2, ..., N\}$ *and for all* $x \in C$, *there exist a bounded subset* $D_x \subset C$ *and* $z_x \in C$ *such that for any* $y \in C \backslash D_x$,

$$F_k(y, z_x) + \varphi(z_x) - \varphi(y) + \frac{1}{r_k} \langle \mathcal{K}'(y) - \mathcal{K}'(x), \eta(z_x, y) \rangle < 0;$$

(iv) $\lim_{n\to\infty} \alpha_{n,i} = 0, \forall i \geq 1$;

(v) $\{\lambda_l\} \subset (0, 2\beta_l)$, $\forall l = 1, 2, ..., M$;

(vi) $\liminf_{n\to\infty} r_{k,n} > 0$, $\forall k = 1, 2, 3, ..., N$.

Then, $\{x_n\}$ *converges strongly to* $P_\Theta x_0$.

Proof. First, we show that $I - \lambda_l A_l$ for all $l \in \{1, 2, ..., M\}$ is nonexpansive mappings. Indeed, for all $x, y \in C$ and $\lambda_l \in (0, 2\beta_l)$, we observe that

$$\begin{aligned}\|(I-\lambda_1 A_l)x-(I-\lambda_1 A_l)y\|^2 &= \|x-y-\lambda_l(A_lx-A_ly)\|^2\\ &= \|x-y\|^2-2\lambda_l\langle x-y,A_lx-A_ly\rangle+\lambda_l^2\|A_lx-A_ly\|^2\\ &\le \|x-y\|^2-2\lambda_1\beta_l\|A_lx-A_ly\|^2+\lambda_l^2\|A_lx-A_ly\|^2\\ &\le \|x-y\|^2+\lambda_l(\lambda_l-2\beta_l)\|A_lx-A_ly\|^2\\ &\le \|x-y\|^2,\end{aligned} \tag{26}$$

which implies that the mapping $I - \lambda_l A_l$ is nonexpansive for all $l \in \{1, 2, ..., M\}$.

Let $p \in \Theta$. Taking

$$\Im_n^k = K_{r_{k,n}}^{F_k} K_{r_{k-1,n}}^{F_{k-1}} \dots K_{r_{2,n}}^{F_2} K_{r_{1,n}}^{F_1}, k \in \{1,2,3,\dots,N\}, n \in \mathbb{N}$$

and

$$\mathcal{P}_C^l = P_C(I-\lambda_l A_l)P_C(I-\lambda_{l-1}A_{l-1})\dots P_C(I-\lambda_2 A_2)P_C(I-\lambda_1 A_1) \text{ for all } l \in \{1,2,3,\dots,M\},$$

$\Im_n^0 = \mathcal{P}_C^0 = I$, where I is the identity mapping on H. From the definition of $K_{r_{k,n}}^{F_k}$ and P_C are nonexpansive then $\Im_n^k, k \in \{1,2,3,\dots,N\}$ and $\mathcal{P}_C^l, l \in \{1,2,3,\dots,M\}$ also. We note that $u_n = \Im_n^N x_n$ and $p = \Im_{r_{k,n}}^{F_k} p$, we have

$$\|u_n-p\| = \|\Im_n^N x_n - \Im_n^N p\| \le \|x_n-p\|. \tag{27}$$

It follows that

$$\|w_n-p\| = \left\|\mathcal{P}_C^l u_n - \mathcal{P}_C^l p\right\| \le \|u_n-p\| \le \|x_n-p\|, \forall l \in \{1,2,3,\dots,M\}. \tag{28}$$

Next, we will divide the proof into five steps.

Step 1. We show that $\{x_n\}$ is well defined. Let $n = 1$, then $C_{1,i} = C$ is closed and convex for each $i \geq 1$. Suppose that $C_{n,i}$ is closed convex for some $n > 1$. Then, from definition of $C_{n+1,i}$, we know that $C_{n+1,i}$ is closed convex for the same $n \geq 1$. Hence, $C_{n,i}$ is closed convex for $n \geq 1$ and for each $i \geq 1$. This implies that C_n is closed convex for $n \geq 1$. Furthermore, we show that $\Theta \subset C_n$. For $n = 1$, $\Theta \subset C = C_{1,i}$. For $n \geq 2$, let $p \in \Theta$. Then,

$$\begin{aligned}\|y_{n,i}-p\|^2 &= \left\|\alpha_{n,i}(x_0-p)+(1-\alpha_{n,i})(\frac{1}{t_n}\int_0^{t_n} S(s)W_nw_n ds - p)\right\|^2\\ &\le \alpha_{n,i}\|x_0-p\|^2+(1-\alpha_{n,i})\left\|\frac{1}{t_n}\int_0^{t_n} S(s)W_nw_n ds - p\right\|^2\\ &\le \alpha_{n,i}\|x_0-p\|^2+(1-\alpha_{n,i})\|w_n-p\|^2\\ &= \|w_n-p\|^2+\alpha_{n,i}(\|x_0-p\|^2-\|w_n-p\|^2)\\ &\le \|x_n-p\|^2+\alpha_{n,i}(\|x_0\|^2+2\langle x_n-x_0,p\rangle),\end{aligned} \tag{29}$$

which shows that $p \in C_{n,i}$, $\forall n \geq 2$, $\forall i \geq 1$. Thus, $\Theta \subset C_{n,i}$, $\forall n \geq 1$, $\forall i \geq 1$. Hence, it follows that $\emptyset \neq \Theta \subset C_n, \forall n \geq 1$. This implies that $\{x_n\}$ is well-defined.

Step 2. We claim that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$ and $\lim_{n\to\infty} \|y_{n,i} - x_n\| = 0$, for $i \geq 1$. Since $x_n = P_{C_n}x_0$ and $x_{n+1} = P_{C_{n+1}}x_0 \in C_{n+1} \subset C_n, \forall n \geq 1$, we have

$$\|x_n - x_0\| \leq \|x_{n+1} - x_0\| . \tag{30}$$

Also, as $\Theta \subset C_n$ by (13), it follows that

$$\|x_n - x_0\| \leq \|z - x_0\| , z \in \Theta, \forall n \geq 1. \tag{31}$$

Form (30) and (31), we have that $\lim_{n\to\infty} \|x_n - x_0\|$ exists. Hence $\{x_n\}$ is bounded and so are $\{y_{n,i}\}$, $\forall i \geq 1$, $\{w_n\}$, $\{u_n\}$, $\{W_n w_n\}$, and $\left\{\frac{1}{t_n}\int_0^{t_n} S(s)W_n w_n ds\right\}$. For $m > n \geq 1$, we have that $x_m = P_{C_m}x_0 \in C_m \subset C_n$. By (16), we obtain

$$\|x_m - x_n\|^2 \leq \|x_n - x_0\|^2 - \|x_m - x_0\|^2. \tag{32}$$

Letting m, $n \to \infty$ and taking the limit in (32), we have $\|x_m - x_n\| \to 0$, which shows that $\{x_n\}$ is a Cauchy sequence. In particular,

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{33}$$

Since, $\{x_n\}$ is a Cauchy sequence, we assume that $x_n \to z \in C$. Since $x_{n+1} = P_{C_{n+1}}x_0 \in C_{n+1}$, then for each $i \geq 1$,

$$\left\|y_{n,i} - x_{n+1}\right\|^2 \leq \|x_n - x_{n+1}\|^2 + \alpha_{n,i}(\|x_0\|^2 + 2\langle x_n - x_0, x_{n+1}\rangle) \to 0, n \to \infty.$$

It follows that

$$\left\|y_{n,i} - x_n\right\| \leq \left\|y_{n,i} - x_{n+1}\right\| + \|x_{n+1} - x_n\| .$$

Therefore

$$\lim_{n\to\infty} \left\|y_{n,i} - x_n\right\| = 0, \forall i \geq 1. \tag{34}$$

Step 3. We claim that the following statements hold:

(1) $\lim_{n\to\infty} \left\|\Im_n^k x_n - \Im_n^{k-1} x_n\right\| = 0, \forall k = 1, 2, \ldots, N$;
(2) $\lim_{n\to\infty}\|u_n - x_n\| = 0$;
(3) $\lim_{n\to\infty}\|u_n - w_n\| = 0$.

Indeed, for $p \in \Theta$, note that $K_{r_{k,n}}^{F_k}, k = 1, 2, \ldots, N$ is the firmly nonexpansive, so we have

$$\begin{aligned}\left\|\Im_n^k x_n - \Im_n^k p\right\|^2 &= \left\|K_{r_{k,n}}^{F_k}\Im_n^{k-1}x_n - K_{r_{k,n}}^{F_k}p\right\|^2 \\ &\leq \left\langle \Im_n^k x_n - p, \Im_n^{k-1}x_n - p\right\rangle \\ &= \frac{1}{2}\left\{\left\|\Im_n^k x_n - p\right\|^2 + \left\|\Im_n^{k-1}x_n - p\right\|^2 - \left\|\Im_n^k x_n - \Im_n^{k-1}x_n\right\|^2\right\}.\end{aligned}$$

Thus, we get

$$\left\|\Im_n^k x_n - \Im_n^k p\right\|^2 \leq \left\|\Im_n^{k-1}x_n - p\right\|^2 - \left\|\Im_n^k x_n - \Im_n^{k-1}x_n\right\|^2.$$

It follows that

$$\begin{aligned}\|u_n - p\|^2 &\le \left\|\Im_n^k x_n - \Im_n^k p\right\|^2 \\ &\le \left\|\Im_n^{k-1} x_n - p\right\|^2 - \left\|\Im_n^k x_n - \Im_n^{k-1} x_n\right\|^2 \\ &\le \|x_n - p\|^2 - \left\|\Im_n^k x_n - \Im_n^{k-1} x_n\right\|^2.\end{aligned} \tag{35}$$

From (29) and (35), we have for $i \ge 1$,

$$\begin{aligned}\|y_{n,i} - p\|^2 &\le \alpha_{n,i}\|x_0 - p\|^2 + (1-\alpha_{n,i})\|w_n - p\|^2 \\ &\le \alpha_{n,i}\|x_0 - p\|^2 + (1-\alpha_{n,i})\|u_n - p\|^2 \\ &\le \alpha_{n,i}\|x_0 - p\|^2 + (1-\alpha_{n,i})[\|x_n - p\|^2 - \left\|\Im_n^k x_n - \Im_n^{k-1} x_n\right\|^2],\end{aligned}$$

it follows that

$$\begin{aligned}(1-\alpha_{n,i})\left\|\Im_n^k x_n - \Im_n^{k-1} x_n\right\|^2 &\le \alpha_{n,i}\|x_0 - p\|^2 + \|x_n - p\|^2 - \|y_{n,i} - p\|^2 \\ &\le \alpha_{n,i}\|x_0 - p\|^2 + \|x_n - y_{n,i}\|\,(\|x_n - p\| + \|y_{n,i} - p\|).\end{aligned}$$

By the condition (iv) and (34), we have

$$\lim_{n\to\infty}\left\|\Im_n^k x_n - \Im_n^{k-1} x_n\right\| = 0. \tag{36}$$

For $p \in \Theta$ and again since $K_{r_{k,n}}^{F_k}, k = 1, 2, \dots, N$ is the firmly nonexpansive, we obtain

$$\begin{aligned}\|u_n - p\|^2 &= \left\|\Im_n^k x_n - \Im_n^k p\right\|^2 \\ &\le \left\langle \Im_n^k x_n - \Im_n^k p, x_n - p\right\rangle \\ &= \frac{1}{2}\left\{\left\|\Im_n^k x_n - \Im_n^k p\right\|^2 + \|x_n - p\|^2 - \left\|\Im_n^k x_n - x_n\right\|^2\right\} \\ &= \frac{1}{2}\left\{\|u_n - p\|^2 + \|x_n - p\|^2 - \|u_n - x_n\|^2\right\}\end{aligned}$$

and hence

$$\|u_n - p\|^2 \le \|x_n - p\|^2 - \|u_n - x_n\|^2. \tag{37}$$

From (29) and (37), for $i \ge 1$, we have

$$\begin{aligned}\|y_{n,i} - p\|^2 &\le \alpha_{n,i}\|x_0 - p\|^2 + (1-\alpha_{n,i})\|w_n - p\|^2 \\ &\le \alpha_{n,i}\|x_0 - p\|^2 + (1-\alpha_{n,i})\|u_n - p\|^2 \\ &\le \alpha_{n,i}\|x_0 - p\|^2 + (1-\alpha_{n,i})[\|x_n - p\|^2 - \|u_n - x_n\|^2],\end{aligned}$$

it follows that

$$\begin{aligned}(1-\alpha_{n,i})\|u_n - x_n\|^2 &\le \alpha_{n,i}\|x_0 - p\|^2 + \|x_n - p\|^2 - \|y_{n,i} - p\|^2 \\ &\le \alpha_{n,i}\|x_0 - p\|^2 + \|x_n - y_{n,i}\|\,(\|x_n - p\| + \|y_{n,i} - p\|).\end{aligned}$$

By the condition (iv) and (34), we have

$$\lim_{n\to\infty} \|u_n - x_n\| = 0. \tag{38}$$

From (26), we note that

$$\begin{aligned}
\|w_n - p\|^2 &= \|\mathcal{P}_C^M u_n - \mathcal{P}_C^M p\|^2 \\
&= \|P_C(I - \lambda_M A_M)\mathcal{P}_C^{M-1} u_n - P_C(I - \lambda_M A_M)\mathcal{P}_C^{M-1} p\|^2 \\
&\le \|(I - \lambda_M A_M)\mathcal{P}_C^{M-1} u_n - (I - \lambda_M A_M)\mathcal{P}_C^{M-1} p\|^2 \\
&\le \|\mathcal{P}_C^{M-1} u_n - \mathcal{P}_C^{M-1} p\|^2 + \lambda_M(\lambda_M - 2\beta_M)\|A_M\mathcal{P}_C^{M-1} u_n - A_M\mathcal{P}_C^{M-1} p\|^2 \\
&\vdots \\
&\le \|u_n - p\|^2 + \sum_{l=1}^{M} \lambda_l(\lambda_l - 2\beta_i)\left\|A_l\mathcal{P}_C^{l-1} u_n - A_l\mathcal{P}_C^{l-1} p\right\|^2 \\
&\le \|x_n - p\|^2 + \sum_{l=1}^{M} \lambda_l(\lambda_l - 2\beta_l)\left\|A_l\mathcal{P}_C^{l-1} u_n - A_l\mathcal{P}_C^{l-1} p\right\|^2.
\end{aligned} \tag{39}$$

It follows that, for $i \ge 1$

$$\begin{aligned}
\|y_{n,i} - p\|^2 &\le \alpha_{n,i}\|x_0 - p\|^2 + (1 - \alpha_{n,i})\|w_n - p\|^2 \\
&= \alpha_{n,i}\|x_0 - p\|^2 + (1 - \alpha_{n,i})\left[\|x_n - p\|^2 + \sum_{l=1}^{M} \lambda_l(\lambda_l - 2\beta_l)\left\|A_l\mathcal{P}_C^{l-1} u_n - A_l\mathcal{P}_C^{l-1} p\right\|^2\right] \\
&\le \alpha_{n,i}\|x_0 - p\|^2 + \|x_n - p\|^2 + (1 - \alpha_{n,i})\sum_{l=1}^{M} \lambda_l(\lambda_l - 2\beta_l)\left\|A_l\mathcal{P}_C^{l-1} u_n - A_l\mathcal{P}_C^{l-1} p\right\|^2,
\end{aligned} \tag{40}$$

which implies that

$$\begin{aligned}
(1 - \alpha_{n,i})\sum\nolimits_{l=1}^{M} \lambda_l(2\beta_l - \lambda_l)\left\|A_l\mathcal{P}_C^{l-1} u_n - A_l\mathcal{P}_C^{l-1} p\right\|^2 \\
= \alpha_{n,i}\|x_0 - p\|^2 + \|x_n - p\|^2 - \|y_{n,i} - p\|^2 \\
\le \alpha_{n,i}\|x_0 - p\|^2 + \|x_n - y_{n,i}\|\left(\|x_n - p\| + \|y_{n,i} - p\|\right).
\end{aligned} \tag{41}$$

By the conditions (iv), (v) and (34), we obtain

$$\lim_{n\to\infty} \left\|A_l\mathcal{P}_C^{l-1} u_n - A_l\mathcal{P}_C^{l-1} p\right\| = 0. \tag{42}$$

On the other hand, we note that

$$\begin{aligned}
\|\mathcal{P}_C^M u_n - \mathcal{P}_C^M p\|^2 &= \|P_C(I - \lambda_M A_M)\mathcal{P}_C^{M-1} u_n - P_C(I - \lambda_M A_M)\mathcal{P}_C^{M-1} p\|^2 \\
&\le \langle (I - \lambda_M A_M)\mathcal{P}_C^{M-1} u_n - (I - \lambda_M A_M)\mathcal{P}_C^{M-1} p, \mathcal{P}_C^M u_n - \mathcal{P}_C^M p\rangle \\
&= \frac{1}{2}\Big\{\|(I - \lambda_M A_M)\mathcal{P}_C^{M-1} u_n - (I - \lambda_M A_M)\mathcal{P}_C^{M-1} p\|^2 + \|\mathcal{P}_C^M u_n - \mathcal{P}_C^M p\|^2 \\
&\quad - \|(I - \lambda_M A_M)\mathcal{P}_C^{M-1} u_n - (I - \lambda_M A_M)\mathcal{P}_C^{M-1} p - (\mathcal{P}_C^M u_n - \mathcal{P}_C^M p)\|^2\Big\} \\
&\le \frac{1}{2}\Big\{\|\mathcal{P}_C^{M-1} u_n - \mathcal{P}_C^{M-1} p\|^2 + \|\mathcal{P}_C^M u_n - \mathcal{P}_C^M p\|^2 \\
&\quad - \|(\mathcal{P}_C^{M-1} u_n - \mathcal{P}_C^{M-1} p - \mathcal{P}_C^M u_n + \mathcal{P}_C^M p) - \lambda_M(A_M\mathcal{P}_C^{M-1} u_n - A_M\mathcal{P}_C^{M-1} p)\|^2\Big\} \\
&= \frac{1}{2}\Big\{\|\mathcal{P}_C^{M-1} u_n - \mathcal{P}_C^{M-1} p\|^2 + \|\mathcal{P}_C^M u_n - \mathcal{P}_C^M p\|^2 - \|\mathcal{P}_C^{M-1} u_n - \mathcal{P}_C^{M-1} p - \mathcal{P}_C^M u_n \\
&\quad + \mathcal{P}_C^M p\|^2 + 2\lambda_M\langle \mathcal{P}_C^{M-1} u_n - \mathcal{P}_C^{M-1} p - \mathcal{P}_C^M u_n + \mathcal{P}_C^M p, A_M\mathcal{P}_C^{M-1} u_n - A_M\mathcal{P}_C^{M-1} p\rangle \\
&\quad - \lambda_M^2\|A_M\mathcal{P}_C^{M-1} u_n - A_M\mathcal{P}_C^{M-1} p\|^2\Big\} \\
&\le \frac{1}{2}\Big\{\|\mathcal{P}_C^{M-1} u_n - \mathcal{P}_C^{M-1} p\|^2 + \|\mathcal{P}_C^M u_n - \mathcal{P}_C^M p\|^2 - \|\mathcal{P}_C^{M-1} u_n - \mathcal{P}_C^{M-1} p - \mathcal{P}_C^M u_n \\
&\quad + \mathcal{P}_C^M p\|^2 + 2\lambda_M\|\mathcal{P}_C^{M-1} u_n - \mathcal{P}_C^{M-1} p - \mathcal{P}_C^M u_n + \mathcal{P}_C^M p\|\,\|A_M\mathcal{P}_C^{M-1} u_n - A_M\mathcal{P}_C^{M-1} p\|\Big\},
\end{aligned}$$

which implies that

$$\begin{aligned}\left\|\mathcal{P}_C^M u_n - \mathcal{P}_C^M p\right\|^2 &\le \left\|\mathcal{P}_C^{M-1}u_n - \mathcal{P}_C^{M-1}p\right\|^2 - \left\|\mathcal{P}_C^{M-1}u_n - \mathcal{P}_C^{M-1}p - \mathcal{P}_C^M u_n + \mathcal{P}_C^M p\right\|^2 \\ &\quad + 2\lambda_M \left\|\mathcal{P}_C^{M-1}u_n - \mathcal{P}_C^{M-1}p - \mathcal{P}_C^M u_n + \mathcal{P}_C^M p\right\| \left\|A_M \mathcal{P}_C^{M-1}u_n - A_M \mathcal{P}_C^{M-1}p\right\| \\ &\quad \vdots \\ &\le \left\|u_n - p\right\|^2 - \sum_{l=1}^{M} \left\|\mathcal{P}_C^{l-1}u_n - \mathcal{P}_C^{l-1}p - \mathcal{P}_C^l u_n + \mathcal{P}_C^l p\right\|^2 \\ &\quad + \sum_{l=1}^{M} 2\lambda_l \left\|\mathcal{P}_C^{l-1}u_n - \mathcal{P}_C^{l-1}p - \mathcal{P}_C^l u_n + \mathcal{P}_C^l p\right\| \left\|A_l \mathcal{P}_C^{l-1}u_n - A_l \mathcal{P}_C^{l-1}p\right\| \\ &\le \left\|x_n - p\right\|^2 - \sum_{l=1}^{M} \left\|\mathcal{P}_C^{l-1}u_n - \mathcal{P}_C^{l-1}p - \mathcal{P}_C^l u_n + \mathcal{P}_C^l p\right\|^2 \\ &\quad + \sum_{l=1}^{M} 2\lambda_l \left\|\mathcal{P}_C^{l-1}u_n - \mathcal{P}_C^{l-1}p - \mathcal{P}_C^l u_n + \mathcal{P}_C^l p\right\| \left\|A_l \mathcal{P}_C^{l-1}u_n - A_l \mathcal{P}_C^{l-1}p\right\|.\end{aligned} \tag{43}$$

From (29) and (43), for $i \ge 1$, we note that

$$\begin{aligned}\left\|y_{n,i} - p\right\|^2 &\le \alpha_{n,i}\left\|x_0 - p\right\|^2 + (1 - \alpha_{n,i})\left\|w_n - p\right\|^2 \\ &= \alpha_{n,i}\left\|x_0 - p\right\|^2 + (1 - \alpha_{n,i})\left\|\mathcal{P}_C^M u_n - \mathcal{P}_C^M p\right\|^2 \\ &\le \alpha_{n,i}\left\|x_0 - p\right\|^2 + (1 - \alpha_{n,i})\Bigg\{\left\|x_n - p\right\|^2 - \sum_{l=1}^{M}\left\|\mathcal{P}_C^{l-1}u_n - \mathcal{P}_C^{l-1}p - \mathcal{P}_C^l u_n + \mathcal{P}_C^l p\right\|^2 \\ &\quad + \sum_{l=1}^{M} 2\lambda_l \left\|\mathcal{P}_C^{l-1}u_n - \mathcal{P}_C^{l-1}p - \mathcal{P}_C^l u_n + \mathcal{P}_C^l p\right\| \left\|A_l \mathcal{P}_C^{l-1}u_n - A_l \mathcal{P}_C^{l-1}p\right\|\Bigg\} \\ &\le \alpha_{n,i}\left\|x_0 - p\right\|^2 + \left\|x_n - p\right\|^2 - (1 - \alpha_{n,i})\sum_{l=1}^{M}\left\|\mathcal{P}_C^{l-1}u_n - \mathcal{P}_C^{l-1}p - \mathcal{P}_C^l u_n + \mathcal{P}_C^l p\right\|^2 \\ &\quad + \sum_{l=1}^{M} 2\lambda_l \left\|\mathcal{P}_C^{l-1}u_n - \mathcal{P}_C^{l-1}p - \mathcal{P}_C^l u_n + \mathcal{P}_C^l p\right\| \left\|A_l \mathcal{P}_C^{l-1}u_n - A_l \mathcal{P}_C^{l-1}p\right\|.\end{aligned} \tag{44}$$

This implies that

$$\begin{aligned}&(1 - \alpha_{n,i})\textstyle\sum_{l=1}^{M}\left\|\mathcal{P}_C^{l-1}u_n - \mathcal{P}_C^{l-1}p - \mathcal{P}_C^l u_n + \mathcal{P}_C^l p\right\|^2 \\ &\quad\le \alpha_{n,i}\left\|x_0 - p\right\|^2 + \left\|x_n - p\right\|^2 - \left\|y_{n,i} - p\right\|^2 \\ &\qquad + \sum_{l=1}^{M} 2\lambda_l \left\|\mathcal{P}_C^{l-1}u_n - \mathcal{P}_C^{l-1}p - \mathcal{P}_C^l u_n + \mathcal{P}_C^l p\right\| \left\|A_l \mathcal{P}_C^{l-1}u_n - A_l \mathcal{P}_C^{l-1}p\right\| \\ &\quad\le \alpha_{n,i}\left\|x_0 - p\right\|^2 + \left\|x_n - y_{n,i}\right\|\left(\left\|x_n - p\right\| + \left\|y_{n,i} - p\right\|\right) \\ &\qquad + \sum_{l=1}^{M} 2\lambda_l \left\|\mathcal{P}_C^{l-1}u_n - \mathcal{P}_C^{l-1}p - \mathcal{P}_C^l u_n + \mathcal{P}_C^l p\right\| \left\|A_l \mathcal{P}_C^{l-1}u_n - A_l \mathcal{P}_C^{l-1}p\right\|.\end{aligned} \tag{45}$$

Then, by condition (iv), (34) and (42), we obtain that

$$\lim_{n\to\infty}\left\|\mathcal{P}_C^{l-1}u_n - \mathcal{P}_C^{l-1}p - \mathcal{P}_C^l u_n + \mathcal{P}_C^l p\right\| = 0. \tag{46}$$

Therefore, we have

$$\left\|u_n - w_n\right\| = \left\|\mathcal{P}_C^0 u_n - \mathcal{P}_C^l u_n\right\| \le \sum_{l=1}^{M}\left\|\mathcal{P}_C^{l-1}u_n - \mathcal{P}_C^{l-1}p - \mathcal{P}_C^l u_n + \mathcal{P}_C^l p\right\|^2 \to 0 \text{ as } n \to \infty. \tag{47}$$

On the other hand, by condition (iv) implies that

$$\left\|y_{n,i} - \frac{1}{t_n}\int_0^{t_n} S(s)W_n w_n ds\right\|^2 = \alpha_{n,i}\left\|x_0 - \frac{1}{t_n}\int_0^{t_n} S(s)W_n w_n ds\right\| \to 0, \forall i \ge 1 \tag{48}$$

it follows that

$$\left\| x_n - \frac{1}{t_n}\int_0^{t_n} S(s)W_n w_n ds \right\| \leq \|x_n - y_{n,i}\| + \left\| y_{n,i} + \frac{1}{t_n}\int_0^{t_n} S(s)W_n w_n ds \right\| \to 0. \quad (49)$$

Step 4. We show that $z \in \Theta := F(\mathcal{S}) \cap (\cap_{i=1}^{\infty} F(T_i)) \cap (\cap_{k=1}^{N} SMEP(F_k)) \cap GSVI(C, A_l), \forall l \in \{1, 2, \ldots, M\}$. Since $\{w_{n_i}\}$ is bounded, there exists a subsequence $\{w_{n_{i_j}}\}$ of $\{w_{n_i}\}$ which converges weakly to $z \in C$. Without loss of generality, we can assume that $w_{n_i} \rightharpoonup z$.

(1) First, we prove that $z \in F(\mathcal{S})$. From (38), (47), and (49), we get

$$\lim_{n\to\infty} \left\| w_n - \frac{1}{t_n}\int_0^{t_n} S(s)W_n w_n ds \right\| = 0. \quad (50)$$

Since $\{W_n\ w_n\}$ is a bounded and from Lemma 2.5 for any $h \geq 0$, we have

$$\lim_{n\to\infty} \left\| \frac{1}{t_n}\int_0^{t_n} S(s)W_n w_n ds - S(h)\left(\frac{1}{t_n}\int_0^{t_n} S(s)W_n w_n ds\right) \right\| = 0, \quad (51)$$

and since

$$\begin{aligned} \|w_n - S(h)w_n\| &\leq \left\| w_n - \frac{1}{t_n}\int_0^{t_n} S(s)W_n w_n ds \right\| + \left\| \frac{1}{t_n}\int_0^{t_n} S(s)W_n w_n ds - S(h)\frac{1}{t_n}\int_0^{t_n} S(s)W_n w_n ds \right\| \\ &\quad + \left\| S(h)\frac{1}{t_n}\int_0^{t_n} S(s)W_n w_n ds - S(h)w_n \right\| \\ &\leq 2\left\| w_n - \frac{1}{t_n}\int_0^{t_n} S(s)W_n w_n ds \right\| + \left\| \frac{1}{t_n}\int_0^{t_n} S(s)W_n w_n ds - S(h)\frac{1}{t_n}\int_0^{t_n} S(s)W_n w_n ds \right\| \end{aligned}$$

for all $0 \leq s < \infty$. It follows from (50) and (51), we get

$$\lim_{n\to\infty} \|w_n - S(h)w_n\| = 0. \quad (52)$$

Indeed, from Lemma 2.6 and (52), we get $z \in F(\mathcal{S})$, i.e., $z = S(s)z$, $\forall s \geq 0$.

(2) Next, we show that $z \in F(W) = \cap_{n=1}^{\infty} F(W_n) = \cap_{i=1}^{\infty} F(T_i)$, where $F(W_n) = \cap_{i=1}^{n} F(T_i), \forall n \geq 1$ and $F(W_{n+1}) \subset F(W_n)$. Assume that $z \notin F(W)$, then there exists a positive integer m such that $z \notin F(T_m)$ and so $z \notin \cap_{i=1}^{m} F(T_i)$. Hence for any $n \geq m, z \notin \cap_{i=1}^{n} F(T_i) = F(W_n)$, i.e., $z \neq W_n z$. This together with $z = S(s)z$, $\forall s \geq 0$ shows $z = S(s)z \neq S(s)W_n z$, $\forall s \geq 0$, therefore we have $z \neq \frac{1}{t_n}\int_0^{t_n} S(s)W_n z ds, \forall n \geq m$. It follows from the Opial's condition and (50) that

$$\begin{aligned} \liminf_{i\to\infty} \|w_{n_i} - z\| &< \liminf_{i\to\infty} \left\| w_{n_i} - \frac{1}{t_{n_i}}\int_0^{t_{n_i}} S(s)W_{n_i} z ds \right\| \\ &\leq \liminf_{i\to\infty} \left(\left\| w_{n_i} - \frac{1}{t_{n_i}}\int_0^{t_{n_i}} S(s)W_{n_i} w_{n_i} ds \right\| \right. \\ &\quad \left. + \left\| \frac{1}{t_{n_i}}\int_0^{t_{n_i}} S(s)W_{n_i} w_{n_i} ds - \frac{1}{t_{n_i}}\int_0^{t_{n_i}} S(s)W_{n_i} z ds \right\| \right) \\ &\leq \liminf_{i\to\infty} \|w_{n_i} - z\|, \end{aligned}$$

which is a contradiction. Thus, we get $z \in \cap_{i=1}^{\infty} F(T_i)$.

(3) We prove that $z \in \cap_{k=1}^{N} SMEP(F_k, \varphi)$. Since $\Im_n^k = \mathcal{K}_{r_k}^{F_k}$, $k = 1, 2, \ldots, N$ and $u_n^k = \Im_n^k x_n$, we have

$$F_k(\Im_n^k x_n, x) + \varphi(x) - \varphi(\Im_n^k x_n) + \frac{1}{r_k}\left\langle \mathcal{K}'(\Im_n^k x_n) - \mathcal{K}'(\Im_n^{k-1} x_n), \eta(x, \Im_n^k x_n)\right\rangle \geq 0, \quad \forall x \in C.$$

It follows that

$$\frac{1}{r_k}\left\langle \mathcal{K}'(\Im_{n_i}^k x_{n_i}) - \mathcal{K}'(\Im_{n_i}^{k-1} x_{n_i}), \eta(x, \Im_{n_i}^k x_{n_i})\right\rangle \geq -F_k(\Im_{n_i}^k x_{n_i}, x) - \varphi(x) + \varphi(\Im_{n_i}^k x_{n_i}) \quad (53)$$

for all $x \in C$. From (36) and by conditions (i)(3) and (ii), we get

$$\lim_{n_i \to \infty} \frac{1}{r_k}\left\langle \mathcal{K}'(\Im_{n_i}^k x_{n_i}) - \mathcal{K}'(\Im_{n_i}^{k-1} x_{n_i}), \eta(x, \Im_{n_i}^k x_{n_i})\right\rangle = 0.$$

By the assumption and by the condition (H1), we know that the function ϕ and the mapping $x \mapsto (-F_k(x, y))$ both are convex and lower semicontinuous, hence they are weakly lower semicontinuous. These together with $\frac{\mathcal{K}'(\Im_{n_i}^k x_{n_i}) - \mathcal{K}'(\Im_{n_i}^{k-1} x_{n_i})}{r_k} \to 0$ and $\Im_{n_i}^k x_{n_i} \rightharpoonup z$, we have

$$0 = \liminf_{n_i \to \infty}\left\langle \frac{\mathcal{K}'(\Im_{n_i}^k x_{n_i}) - \mathcal{K}'(\Im_{n_i}^{k-1} x_{n_i})}{r_k}, \eta(x, \Im_{n_i}^k x_{n_i})\right\rangle \geq \liminf_{n_i \to \infty}\{-F_k(\Im_{n_i}^k x_{n_i}, x) - \varphi(x) + \varphi(\Im_{n_i}^k x_{n_i})\}.$$

Then, we obtain

$$F_k(z, x) + \varphi(x) - \varphi(z) \geq 0, \quad \forall x \in C, \quad \forall k =, 1, 2, \ldots, N. \quad (54)$$

Therefore $z \in \cap_{k=1}^{N} SMEP(F_k, \varphi)$.

(4) Last, we show that $z \in GSVI(C, A_l)$, $\forall l \in \{1, 2, ..., M\}$. By the nonexpansivity of $\mathcal{P}$ in Lemma 2.7, we have

$$\begin{aligned} \|w_n - \mathcal{P}(w_n)\| &= \|P_C(I - \lambda_M A_M) P_C(I - \lambda_{M-1} A_{M-1}) \ldots P_C(I - \lambda_2 A_2) P_C(I - \lambda_1 A_1) u_n - \mathcal{P}(w_n)\| \\ &= \|\mathcal{P}(u_n) - \mathcal{P}(w_n)\| \\ &\leq \|u_n - w_n\|. \end{aligned}$$

We note that

$$\|x_n - w_n\| \leq \|x_n - u_n\| + \|u_n - w_n\|.$$

Therefore, we conclude that $\lim_{n \to \infty} \|w_n - \mathcal{P}(w_n)\| = 0$ and $\lim_{n \to \infty} \|x_n - w_n\| = 0$. Since $\mathcal{P}$ is nonexpansive, we get

$$\begin{aligned} \|x_n - \mathcal{P}(x_n)\| &\leq \|x_n - w_n\| + \|w_n - \mathcal{P}(w_n)\| + \|\mathcal{P}(w_n) - \mathcal{P}(x_n)\| \\ &\leq 2\|x_n - w_n\| + \|w_n - \mathcal{P}(w_n)\|. \end{aligned}$$

According to Lemma 2.9, we obtain that $z \in GSVI(C, A_l)$ for all $l \in \{1, 2, ..., M\}$. Hence by (1)-(4), we have $z \in \Theta$.

Step 5. Noting that $x_n = P_{C_n} x_0$. By (15), we have

$$\langle x_0 - x_n, y - x_n\rangle \leq 0, \quad \forall y \in C_n.$$

Since $\Theta \subset C_n$ and by the continuity of inner product, we obtain from the above inequality that

$$\langle x_0 - z, y - z \rangle \le 0, \quad \forall y \in C.$$

By (15) again, we conclude that $z = P_\Theta x_0$. This completes the proof.

Corollary 3.2. *Let C be a nonempty closed convex subset of a real Hilbert Space H. Let* $F : C \times C \to \mathbb{R}$ *be a bifunctions satisfying conditions (H1)-(H3). Let A be a* β*-inverse-strongly monotone mappings of C into H. Let* $\mathcal{S} = \{S(s) : 0 \le s < \infty\}$ *be a nonexpansive semigroup on C and let* $\{t_n\}$ *be a positive real divergent sequence. Suppose that* $\Theta := F(\mathcal{S}) \cap MEP(F, \varphi) \cap VI(C, A) \neq \emptyset$. *Let* $\{x_n\}$ *be a sequence generated by* $x_0 \in C, \ \ C_{1,i} = C, \ \ C_1 = \cap_{i=1}^{\infty} C_{1,i}, \ \ x_1 = P_{C_1} x_0$ *and*

$$\begin{cases} u_n = K_r^F x_n, \\ w_n = P_C(I - \lambda A)u_n, \\ y_{n,i} = \alpha_{n,i} x_0 + (1 - \alpha_{n,i}) \frac{1}{t_n} \int_0^{t_n} S(s) w_n ds, \\ C_{n+1,i} = \{z \in C_{n,i} : \|y_{n,i} - z\|^2 \le \|x_n - z\|^2 + \alpha_{n,i}(\|x_0\|^2 + 2\langle x_n - x_0, z\rangle)\}, \\ C_{n+1} = \cap_{i=1}^{\infty} C_{n+1,i}, \\ x_{n+1} = P_{C_{n+1}} x_0 \end{cases} \tag{55}$$

for every $n \ge 0$, *where* $K_r^F : C \to C$ *is the mapping defined by* (24), $r > 0$ *is a constant and* $\{\alpha_{n,i}\}_{n=1}^{\infty} \subset (0, 1)$ *satisfy the conditions (i)-(vi). Then,* $\{x_n\}$ *converges strongly to* $P_\Theta x_0$.

Proof. Putting $W_n = I$ (Identity mapping), $M = 1$ and $N = 1$ in Theorem 3.1, we can conclude the desired conclusion easily. This completes the proof.

4 Deduced theorems

In this section, we deduce Theorem 3.1 to obtain the following four corollaries.

Corollary 4.1. *Let C be a nonempty closed convex subset of a real Hilbert Space H. Let* $\mathcal{S} = \{S(s) : 0 \le s < \infty\}$ *be a nonexpansive semigroup on C and let* $\{t_n\}$ *be a positive real divergent sequence. Suppose that* $F(\mathcal{S}) \neq \emptyset$. *Let* $\{x_n\}$ *be a sequence generated by* $x_0 \in C, \ \ C_{1,i} = C, \ \ C_1 = \cap_{i=1}^{\infty} C_{1,i}, \ \ x_1 = P_{C_1} x_0$ *and*

$$\begin{cases} y_{n,i} = \alpha_{n,i} x_0 + (1 - \alpha_{n,i}) \frac{1}{t_n} \int_0^{t_n} S(s) x_n ds, \\ C_{n+1,i} = \{z \in C_{n,i} : \|y_{n,i} - z\|^2 \le \|x_n - z\|^2 + \alpha_{n,i}(\|x_0\|^2 + 2\langle x_n - x_0, z\rangle)\}, \\ C_{n+1} = \cap_{i=1}^{\infty} C_{n+1,i}, \\ x_{n+1} = P_{C_{n+1}} x_0 \end{cases} \tag{56}$$

for every $n \ge 0$, *where* $\{\alpha_{n,i}\}_{n=1}^{\infty} \subset (0, 1)$ *satisfy the condition (iv). Then,* $\{x_n\}$ *converges strongly to* $P_{F(\mathcal{S})} x_0$.

Proof. Putting $W_n = K_{r_k}^{F_k} = I, \forall k = 1, 2, \dots, N$ (Identity mappings) and $A_l = 0$, $\forall l \in \{1, 2, \dots, M\}$ in Theorem 3.1, we can conclude the desired conclusion easily. This completes the proof.

Corollary 4.2. *Let C be a nonempty closed convex subset of a real Hilbert Space H. Let* $\{V_i : C \to C\}_{i=1}^{\infty}$ *be a countable family of uniformly k-strict pseudo-contractions,* $\{T_i : C \to C\}_{i=1}^{\infty}$ *be a countable family of nonexpansive mappings defined by* $T_i x = tx + (1 - t)V_i x$, $\forall x \in C$, $\forall i \ge 1$, $t \in [k, 1)$. *Suppose that* $\Theta := \cap_{i=1}^{\infty} F(T_i) \neq \emptyset$. *Let* $\{x_n\}$ *be a sequence generated by* $x_0 \in C, \ \ C_{1,i} = C, \ \ C_1 = \cap_{i=1}^{\infty} C_{1,i}, \ \ x_1 = P_{C_1} x_0$ *and*

$$\begin{cases} y_{n,i} = \alpha_{n,i}x_0 + (1-\alpha_{n,i})W_n x_n, \\ C_{n+1,i} = \{z \in C_{n,i} : \|y_{n,i} - z\|^2 \le \|x_n - z\|^2 + \alpha_{n,i}(\|x_0\|^2 + 2\langle x_n - x_0, z\rangle)\}, \\ C_{n+1} = \cap_{i=1}^{\infty} C_{n+1,i}, \\ x_{n+1} = P_{C_{n+1}}x_0 \end{cases} \tag{57}$$

for every $n \ge 0$, *where* $\{\alpha_{n,i}\}_{n=1}^{\infty} \subset (0,1)$ *satisfy the condition (iv). Then,* $\{x_n\}$ *converges strongly to* $P_\Theta x_0$.

Proof. Taking $K_{r_k}^{F_k} = I, \forall k = 1, 2, \dots, N$ (Identity mapping), $s = 0$ and $A_l = 0$, $\forall l \in \{1, 2, \dots, M\}$ in Theorem 3.1, we can conclude the desired conclusion easily. This completes the proof.

Corollary 4.3. *Let C be a nonempty closed convex subset of a real Hilbert Space H. Let* $\{F_k : C \times C \to \mathbb{R}, k = 1, 2, \dots, N\}$ *be a finite family of bifunctions satisfying conditions (H1)-(H3). Suppose that* $\Theta := \cap_{k=1}^{N} SMEP(F_k) \neq \emptyset$. *Let* $\{x_n\}$ *be a sequence generated by* $x_0 \in C$, $C_{1,i} = C$, $C_1 = \cap_{i=1}^{\infty} C_{1,i}$, $x_1 = P_{C_1}x_0$ *and*

$$\begin{cases} u_n = K_{r_{N,n}}^{F_N} K_{r_{N-1,n}}^{F_{N-1}} \dots K_{r_{2,n}}^{F_2} K_{r_{1,n}}^{F_1} x_n, \\ y_{n,i} = \alpha_{n,i}x_0 + (1-\alpha_{n,i})u_n, \\ C_{n+1,i} = \{z \in C_{n,i} : \|y_{n,i} - z\|^2 \le \|x_n - z\|^2 + \alpha_{n,i}(\|x_0\|^2 + 2\langle x_n - x_0, z\rangle)\}, \\ C_{n+1} = \cap_{i=1}^{\infty} C_{n+1,i}, \\ x_{n+1} = P_{C_{n+1}}x_0 \end{cases} \tag{58}$$

for every $n \ge 0$, *where* $K_r^F : C \to C$ *is the mapping defined by* (24), $r > 0$ *is a constant and* $\{\alpha_{n,i}\}_{n=1}^{\infty} \subset (0,1)$ *satisfy the conditions (i)-(iv) and (vi). Then,* $\{x_n\}$ *converges strongly to* $P_\Theta x_0$.

Proof. Putting $W_n = I$ (identity mapping), $A_l = 0$, $\forall l \in \{1, 2, \dots, M\}$ and $s = 0$ in Theorem 3.1, we can conclude the desired conclusion easily. This completes the proof.

Corollary 4.4. *Let C be a nonempty closed convex subset of a real Hilbert Space H. Let* A_l *be* β_l*-inverse-strongly monotone mappings of C into H, where* $l \in \{1, 2, \dots, M\}$. *For* $l \in \{1, 2, \dots, M\}$, *suppose that* $\Theta := GSVI(C, A_l) \neq \emptyset$. *Let* $\{x_n\}$ *be a sequence generated by* $x_0 \in C$, $C_{1,i} = C$, $C_1 = \cap_{i=1}^{\infty} C_{1,i}$, $x_1 = P_{C_1}x_0$ *and*

$$\begin{cases} w_n = P_C(I - \lambda_M A_M)P_C(I - \lambda_{M-1}A_{M-1}) \dots P_C(I - \lambda_2 A_2)P_C(I - \lambda_1 A_1)x_n, \\ y_{n,i} = \alpha_{n,i}x_0 + (1-\alpha_{n,i})w_n, \\ C_{n+1,i} = \{z \in C_{n,i} : \|y_{n,i} - z\|^2 \le \|x_n - z\|^2 + \alpha_{n,i}(\|x_0\|^2 + 2\langle x_n - x_0, z\rangle)\}, \\ C_{n+1} = \cap_{i=1}^{\infty} C_{n+1,i}, \\ x_{n+1} = P_{C_{n+1}}x_0 \end{cases} \tag{59}$$

for every $n \ge 0$, *where* $\{\alpha_{n,i}\}_{n=1}^{\infty} \subset (0,1)$ *satisfy the conditions (iv) and (v). Then,* $\{x_n\}$ *converges strongly to* $P_{F(\mathcal{S})}x_0$.

Proof. Taking $W_n = K_{r_k}^{F_k} = I, \forall k = 1, 2, \dots, N$ (identity mapping) and $s = 0$ in Theorem 3.1, we can conclude the desired conclusion easily. This completes the proof.

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Authors' contributions

Both authors read and approved the final manuscript.

Competing interests

The authors declare that they have no competing interests.

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