Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH Open Access

Convergence theorem of κ-strictly pseudo-contractive mapping and a modification of generalized equilibrium problems

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Abstract

The purpose of this article, we first introduce strong convergence theorem of κ-strictly pseudo-contractive mapping without assumption of the mapping $S = \kappa I + (1 - \kappa)T$. Then, we prove strong convergence of proposed iterative scheme for finding a common element of the set of fixed points of κ-strictly pseudo-contractive mapping and the set of solution of a modification of generalized equilibrium problem. Moreover, by using our main result and a new lemma in the last section we obtain strong convergence theorem for finding a common element of the set of fixed points of κ-strictly pseudo-contractive mapping and two sets of solutions of variational inequalities.

Keywords: nonexpansive mappinga, strictly pseudo-contractive mapping, generalized equilibrium problem, inverse-strongly monotone, variational inequality problem

1 Introduction

Throughout this article, we assume that H is a real Hilbert space and C is a nonempty subset of H. A mapping T of C into itself is nonlinear mapping. A point x is called a fixed point of T if $Tx = x$. We use $F(T)$ to denote the set of fixed point of T. Recalled the following definitions;

Definition 1.1. *The mapping T is said to be nonexpansive if*

$$||Tx - Ty|| \le ||x - y||, \quad \forall x, y \in H$$

Definition 1.2. *The mapping T is said to be strictly pseudo-contractive* [1]*with the coefficient $\kappa \in [0, 1)$ if*

$$||Tx - Ty||^2 \le ||x - y||^2 + \kappa||(I - T)x - (I - T)y||^2 \quad \forall x, y \in H. \tag{1.1}$$

For such case, T is also said to be a *κ-strictly pseudo contractive mapping.*

The class of κ-strictly pseudo-contractive mapping strictly includes the class of non-expansive mapping.

Let $A : C \to H$. The *variational inequality problem* is to find a point $u \in C$ such that

$$\langle Au, v - u\rangle \ge 0 \tag{1.2}$$

for all $v \in C$.

The variational inequality has emerged as a fascinating and interesting branch of mathematical and engineering sciences with a wide range of applications in industry, finance, economics, social, ecology, regional, pure and applied sciences (see, e.g. [2-5]).

A mapping A of C into H is called *α-inverse strongly monotone*; see [6], if there exists a positive real number α such that

$$\langle x - y, Ax - Ay\rangle \geq \alpha||Ax - Ay||^2$$

for all $x, y \in C$.

Let F:$C\times C\to\mathbb{R}$ be a bifunction. The equilibrium problem for F is to determine its equilibrium points, i.e. the set

$$EP(F) = \{x \in C : F(x, y) \geq 0, \quad \forall y \in C\}. \tag{1.3}$$

From (1.2) and (1.3), we have the following generalized equilibrium problem, i.e.

$$\text{Find } z \in C \text{ such that } F(z, y) + \langle Az, y - z\rangle \geq 0, \quad \forall y \in C. \tag{1.4}$$

The set of such $z \in C$ is denoted by EP (F, A), i.e.,

$$EP(F, A) = \{z \in C : F(z, y) + \langle Az,\ y - z\rangle \geq 0, \quad \forall y \in C\}$$

In the case of $A \equiv 0$, EP (F, A) is denoted by $EP(F)$. In the case of $F \equiv 0$, $EP(F, A)$ is also denoted by $VI(C, A)$.

Numerous problems in physics, optimization and economics reduce to find a solution of $EP(F)$ (see, for example [7-9]). Recently, many authors considered the iterative scheme for finding a common element of the set of solution of equilibrium problem and the set of solutions of fixed point problem (see, for example [10-14]). In 2005, Combettes and Hirstoaga [8] introduced an iterative scheme for finding the best approximation to the initial data when $EP(F)$ is nonempty and they also proved the strong convergence theorem.

In 2007, Takahashi and Takahashi [11] introduced viscosity approximation method in framework of a real Hilbert space H. They defined the iterative sequence $\{x_n\}$ and $\{u_n\}$ as follows:

$$\begin{cases} x_1 \in H, \text{ arbitrarily;} \\ F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Tu_n, \quad \forall n \in \mathbb{N}, \end{cases} \tag{1.5}$$

where $f : H \to H$ is a contraction mapping with constant $\alpha \in (0, 1)$ and $\{\alpha_n\} \subset [0,1]$, $\{r_n\} \subset (0, \infty)$. They proved under some suitable conditions on the sequence $\{\alpha_n\}$, $\{r_n\}$ and bifunction F that $\{x_n\}$, $\{u_n\}$ strongly converge to $z \in F(T) \cap EP(F)$, where $z = P_{F(T) \cap EP(F)}f(z)$.

Recently, in 2008, Takahashia and Takahashi [14] introduced a general iterative method for finding a common element of EP (F, A) and $F(T)$. They defined $\{x_n\}$ in the following way:

$$\begin{cases} u, x_1 \in C, \text{ arbitrarily;} \\ F(z_n, y) + \langle Ax_n, y - z_n\rangle + \frac{1}{\lambda_n}\langle y - z_n, z_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = \beta_n x_n + (1 - \beta_n)T(a_n u + (1 - a_n)z_n), \quad \forall n \in \mathbb{N}, \end{cases} \tag{1.6}$$

where A be an α-inverse strongly monotone mapping of C into H with positive real number α and $\{a_n\} \in [0, 1]$, $\{\beta_n\} \subset [0, 1]$, $\{\lambda_n\} \subset [0, 2\alpha]$, and proved strong convergence of the scheme (1.6) to $z \in \cap_{i=1}^{N} F(T_i) \cap EP(F, A)$, where $z = P_{\cap_{i=1}^{N} F(T_i) \cap EP} u$ in the framework of a Hilbert space, under some suitable conditions on $\{a_n\}$, $\{\beta_n\}$, $\{\lambda_n\}$ and bifunction F.

In 2009, Inchan [15] proved the following theorem:

Theorem 1.1. *Let H be a Hilbert space, C be a nonempty closed convex subset of H such that $C \pm C \subset C$, and let $T : C \to H$ be a κ-strictly pseudo-contractive mapping with a fixed point for some $0 \le \kappa < 1$. Let A be a strongly positive bounded linear operator on C with coefficient $\bar{\gamma}$ and $f : C \to C$ be a contraction with the contractive constant* $(0 < \alpha < 1)$ *such that* $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$. *Let $\{x_n\}$ be the sequence generated by*

$$\begin{cases} x_1 \in C, \\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + \big((1 - \beta_n) I - \alpha_n A\big) P_C S x_n, \end{cases}$$

where $S : C \to H$ is a mapping defined by

$$Sx = \kappa x + (1 - \kappa) Tx \tag{1.7}$$

If the control sequence $\{\alpha_n\}$, $\{\beta_n\} \subset (0, 1)$ *satisfying*

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\lim_{n\to\infty} \beta_n = 0$,

(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(iii) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \quad \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty.$

Then $\{x_n\}$ converges strongly to a fixed point q of T, which solves the following solution of variational inequality;

$$\langle (A - \gamma f) q, q - x \rangle \le 0, \quad \forall x \in F(T).$$

In 2010, Jung [16] proved the following theorem:

Theorem 1.2. *Let H be a Hilbert space, C be a nonempty closed convex subset of H such that $C \pm C \subset C$, and let $T : C \to H$ be a κ-strictly pseudo-contractive mapping with $F(T) \neq \emptyset$ for some $0 \le \kappa < 1$. Let A be a strongly positive bounded linear operator on C with coefficient $\bar{\gamma}$ and $f : C \to C$ be a contraction with the contractive coefficient $0 < \alpha < 1$ such that* $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$. *Let $\{\alpha_n\}$ and $\{\beta_n\} \subset (0, 1)$ be sequences which satisfy the following conditions:*

(C1) $\lim_{n\to\infty} \alpha_n = 0$,

(C2) $\sum_{n=0}^{\infty} \alpha_n = \infty$,

(B) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < a$ *for some a constant* $a \in (0, 1)$.

Let $\{x_n\}$ *be a sequence in* C *generated by*

$$\begin{cases} x_0 = x \in C, \\ y_n = \beta_n x_n + (1 - \beta_n) P_C S x_n \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A) y_n, \quad n \geq 0, \end{cases}$$

where $S : C \to H$ *is a mapping defined by*

$$Sx = \kappa x + (1 - \kappa) Tx \tag{1.8}$$

Then $\{x_n\}$ *converges strongly to a fixed point* q *of* T*, which solves the following solution of variational inequality;*

$$\langle (A - \gamma f) q, q - x \rangle \leq 0, \quad \forall x \in F(T).$$

Question A. How can we prove strong convergence theorem of κ-strictly pseudo-contractive mapping without assumption of the mapping $S = \kappa I + (1 - \kappa)T$ in Theorems 1.1 and 1.2?

Let A, $B : C \to H$ be two mappings. By modification of (1.2), we have

$$VI\,(C, aA + (1 - a)B) = \left\{ x \in C : \langle y - x,\ (aA + (1 - a)B)x \rangle \geq 0, \right. \\ \left. \forall y \in C, \quad a \in (0, 1) \right\}. \tag{1.9}$$

From (1.4) and (1.9), we have

$$EP\left(F, (aA + (1 - a)B)\right) = \{ z \in C : F(z, y) + \langle (aA + (1 - a)B) z, y - z \rangle \geq 0, \\ \forall y \in C \text{ and } a \in (0, 1) \}.$$

In this article, we prove strong convergence theorem to answer question A and to approximate a common element of the set of fixed points of κ-strictly pseudo-contractive mapping and the set of solution of a modification of generalized equilibrium problem. Moreover, by using our main result and a new lemma in the last section we obtain strong convergence theorem for finding a common element of the set of fixed points of κ-strictly pseudo-contractive mapping and two sets of solutions of variational inequalities.

2 Preliminaries

Let H be a real Hilbert space and let C be a nonempty closed convex subset of H, let P_C be the metric projection of H onto C i.e., for $x \in H$, $P_C x$ satisfies the property

$$||x - P_C x|| = \min_{y \in C} ||x - y||.$$

The following characterizes the projection P_C.

Lemma 2.1. [17]*Given* $x \in H$ *and* $y \in C$*. Then* $P_C x = y$ *if and only if there holds the inequality*

$$\langle x - y, y - z \rangle \geq 0 \quad \forall z \in C.$$

Lemma 2.2. [18]*Let* $\{s_n\}$ *be a sequence of nonnegative real number satisfying*

$$s_{n+1} = (1 - \alpha_n) s_n + \alpha_n \beta_n, \quad \forall n \geq 0$$

where $\{\alpha_n\}$, $\{\beta_n\}$ *satisfy the conditions*

$$(1)\quad \{\alpha_n\} \subset [0,1], \quad \sum_{n=1}^{\infty} \alpha_n = \infty;$$

$$(2)\quad \limsup_{n\to\infty} \beta_n \le 0 \text{ or } \sum_{n=1}^{\infty} |\alpha_n \beta_n| < \infty.$$

Then $\lim_{n\to\infty} s_n = 0$.

Lemma 2.3. [17]*Let H be a Hibert space, let C be a nonempty closed convex subset of H and let A be a mapping of C into H. Let* $u \in C$. *Then for* $\lambda > 0$,

$$u = P_C(I - \lambda A)u \Leftrightarrow u \in VI(C, A),$$

where P_C *is the metric projection of H onto C.*

Lemma 2.4. [19]*Let* $\{x_n\}$ *and* $\{z_n\}$ *be bounded sequences in a Banach space X and let* $\{\beta_n\}$ *be a sequence in [0,1] with* $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$. *Suppose*

$$x_{n+1} = \beta_n x_n + (1 - \beta_n) z_n$$

for all $n \ge 0$ *and*

$$\limsup_{n\to\infty} (||z_{n+1} - z_n|| - ||x_{n+1} - x_n||) \le 0.$$

Then $\lim_{n\to\infty} ||x_n - z_n|| = 0$.

Lemma 2.5. [20]*Let E be a uniformly convex Banach space, C be a nonempty closed convex subset of E and* $S : C \to C$ *be a nonexpansive mapping. Then, I - S is demi-closed at zero.*

For solving the equilibrium problem for a bifunction $F: C \times C \to \mathbb{R}$, let us assume that F satisfies the following conditions:

$(A1)$ $F(x, x) = 0\ \forall x \in C$;

$(A2)$ F is monotone, i.e. $F(x, y) + F(y, x) \le 0$, $\forall x, y \in C$;

$(A3)$ $\forall x, y, z \in C$,

$\lim_{t\to 0+} F(tz + (1 - t)x, y) \le F(x, y)$;

$(A4)$ $\forall x \in C$, $y\ \alpha\ F(x, y)$ is convex and lower semicontinuous.

The following lemma appears implicitly in [7].

Lemma 2.6. [7]*Let C be a nonempty closed convex subset of H, and let F be a bifunction of* $C \times C$ *into* $\mathbb{R}$ *satisfying* $(A1)$-$(A4)$. *Let* $r > 0$ *and* $x \in H$. *Then, there exists* $z \in C$ *such that*

$$F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \ge 0,$$

for all $x \in C$.

Lemma 2.7. [8]*Assume that* $F: C \times C \to \mathbb{R}$ *satisfies* $(A1)$-$(A4)$. *For* $r > 0$ *and* $x \in H$, *define a mapping* $T_r : H \to C$ *as follows:*

$$T_r(x) = \left\{ z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \ge 0, \quad \forall y \in C \right\}.$$

for all $z \in H$. *Then, the following hold:*

(1) T_r *is single-valued;*

(2) T_r *is firmly nonexpansive i.e.*

$||T_r(x) - T_r(y)||^2 \le \langle T_r(x) - T_r(y), x - y\rangle\ \forall x, y \in H$;

(3) $F(T_r) = EP(F)$;

(4) $EP(F)$ *is closed and convex.*

Remark 2.8. If C is nonempty closed convex subset of H and $T : C \to C$ is κ-strictly pseudocontractive mapping with $F(T) \neq \emptyset$. Then $F(T) = VI(C, (I - T))$. To show this, put $A = I - T$. Let $z \in VI(C, (I - T))$ and $z^* \in F(T)$. Since $z \in VI(C, (I - T))$, $\langle y - z, (I - T)z\rangle \geq 0$, $\forall y \in C$. Since $T : C \to C$ is κ-strictly pseudocontractive mapping, we have

$$\begin{aligned} ||Tz - Tz^*||^2 &= ||(I-A)z - (I-A)z^*||^2 = ||z - z^* - (Az - Az^*)||^2 \\ &= ||z - z^*||^2 - 2\langle z - z^*, Az - Az^*\rangle + ||Az - Az^*||^2 \\ &= ||z - z^*||^2 - 2\langle z - z^*, (I-T)z\rangle + ||(I-T)z||^2 \\ &\leq ||z - z^*||^2 + \kappa||(I-T)z||^2. \end{aligned}$$

It implies that

$$(1-\kappa)||(I-T)z||^2 \leq 2\langle z - z^*, (I-T)z\rangle \leq 0.$$

Then, we have $z = Tz$, therefore $z \in F(T)$. Hence $VI(C, (I - T)) \subseteq F(T)$. It is easy to see that $F(T) \subseteq VI(C, (I - T))$.

Remark 2.9. $A = I - T$ is $\frac{1-\kappa}{2}$- inverse strongly monotone mapping. To show this, let $x, y \in C$, we have

$$\begin{aligned} ||Tx - Ty||^2 &= ||(I-A)x - (I-A)y||^2 = ||x - y - (Ax - Ay)||^2 \\ &= ||x - y||^2 - 2\langle x - y, Ax - Ay\rangle + ||Ax - Ay||^2 \\ &\leq ||x - y||^2 + \kappa||(I-T)x - (I-T)y||^2. \\ &= ||x - y||^2 + \kappa||Ax - Ay||^2. \end{aligned}$$

Then, we have

$$\langle x - y, Ax - Ay\rangle \geq \frac{1-\kappa}{2}||Ax - Ay||^2.$$

3 Main result

Theorem 3.1. *Let C be a closed convex subset of Hilbert space H and let F:$C\times C\to\mathbb{R}$ be a bifunction satisfying (A_1)-(A_4), let $A, B : C \to H$ be α and β-inverse strongly monotone, respectively. Let $T : C \to C$ be κ-*strictly *pseudo contractive mapping with $\mathbb{F} = F(T) \cap EP\left(F, aA + (1-a)B\right) \neq \emptyset$for all $a \in (0, 1)$. Let $\{x_n\}$ and $\{u_n\}$ be the sequences generated by $x_1, u \in C$ and*

$$\begin{cases} F(u_n, y) + \langle (aA + (1-a)B)x_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, & \forall y \in C, \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n P_C(I - \lambda(I-T))u_n, & \forall n \geq 1, \end{cases} \tag{3.1}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset [0, 1]$, $\lambda \in (0, 1 - \kappa)$, $\alpha_n + \beta_n + \gamma_n = 1$, $\forall n\in\mathbb{N}$ and $\{r_n\} \subset [0, 2\gamma]$, $\gamma = \min\{\alpha, \beta\}$ satisfy;

$$(i)\ \sum_{n=1}^{\infty}\alpha_n = \infty, \quad \lim_{n\to\infty}\alpha_n = 0;$$

$$(ii)\ 0 < c \leq \beta_n \leq d < 1, \quad 0 < e \leq r_n \leq f < 2\gamma;$$

$$(iii)\ \lim_{n\to\infty}|r_{n+1} - r_n| = 0.$$

Then $\{x_n\}$ *converges strongly to* $z_0 = P_{\mathbb{F}}u$.

Proof. We divide the proof into seven steps.

Step 1. For every $a \in (0, 1)$, we prove that $aA + (1 - a)B$ is γ-inverse strongly monotone mapping. Put $D = aA + (1 - a)B$. For $x, y \in C$, we have

$$\begin{aligned}\langle Dx - Dy, x - y\rangle &= \langle aAx + (1-a)Bx - aAy - (1-a)By, x - y\rangle \\ &= \langle a(Ax - Ay) + (1-a)(Bx - By), x - y\rangle \\ &= a\langle Ax - Ay, x - y\rangle + (1-a)\langle Bx - By, x - y\rangle \\ &\geq a\alpha||Ax - Ay||^2 + (1-a)\beta||Bx - By||^2 \\ &\geq \gamma(a||Ax - Ay||^2 + (1-a)||Bx - By||^2) \\ &\geq \gamma||a(Ax - Ay) + (1-a)(Bx - By)||^2 \\ &= \gamma||aAx + (1-a)Bx - aAy - (1-a)By||^2 \\ &= \gamma||Dx - Dy||^2\end{aligned} \tag{3.2}$$

Step 2. We show that $I - r_nD$ is a nonexpansive mapping for every $n \in \mathbb{N}$ and so is $P_C(I - \lambda(I - T))$. For every $n \in \mathbb{N}$, let $x, y \in C$. From step 1, we have

$$\begin{aligned}||(I - r_nD)x - (I - r_nD)y||^2 &= ||x - y - r_n(Dx - Dy)||^2 \\ &= ||x - y||^2 - 2r_n\langle x - y, Dx - Dy\rangle + r_n^2||Dx - Dy||^2 \\ &\leq ||x - y||^2 - 2r_n\gamma||Dx - Dy||^2 + r_n^2||Dx - Dy||^2 \\ &= ||x - y||^2 + r_n(r_n - 2\gamma)||Dx - Dy||^2 \\ &\leq ||x - y||^2.\end{aligned} \tag{3.3}$$

Then $I - r_nD$ is a nonexpansive mapping.

Putting $E = I - T$, from Remark 2.9, we have E is η-inverse strong monotone mapping, where $\eta = \frac{1-\kappa}{2}$. By using the same method as (3.3), we have $I - \lambda E$ is nonexpansive mapping. Then, we have $P_C(I - \lambda(I - T))$ is a nonexpansive mapping.

Step 3. We prove that the sequence $\{x_n\}$ is bounded. From $\mathbb{F} \neq \emptyset$ and (3.1), we have $u_n = T_{r_n}(I - r_nD)x_n$, $\forall n \in \mathbb{N}$. Let $z \in \mathbb{F}$. From Remark 2.8 and Lemma 2.3, we have $z = P_C(I - \lambda E)z$, where $E = I - T$. Since $z \in EP(F, D)$, we have $F(z, y) + \langle y - z, Dz\rangle \geq 0$ $\forall y \in C$, so we have

$$F(z, y) + \frac{1}{r_n}\langle y - z, z - z + r_nDz\rangle \geq 0, \quad \forall n \in \mathbb{N} \text{ and } y \in C.$$

From Lemma 2.7, we have $z = T_{r_n}(I - r_nD)z$, $\forall n \in \mathbb{N}$. By nonexpansiveness of $T_{r_n}(I - r_nD)$, we have

$$\begin{aligned}||x_{n+1} - z|| &= ||\alpha_n(u - z) + \beta_n(x_n - z) + \gamma_n(P_C(I - \lambda E)u_n - z)|| \\ &\leq \alpha_n||u - z|| + \beta_n||x_n - z|| + \gamma_n||P_C(I - \lambda E)u_n - z|| \\ &\leq \alpha_n||u - z|| + \beta_n||x_n - z|| + \gamma_n||T_{r_n}(I - r_nD)x_n - z|| \\ &\leq \alpha_n||u - z|| + (1 - \alpha_n)||x_n - z|| \\ &\leq \max\{||x_n - z||, ||u - z||\}.\end{aligned}$$

By induction we can prove that $\{x_n\}$ is bounded and so are $\{u_n\}$, $\{P_C(I - \lambda E)u_n\}$.

Step 4. We will show that

$$\lim_{n\to\infty} ||x_{n+1} - x_n|| = 0. \tag{3.4}$$

Let $p_n = \dfrac{x_{n+1} - \beta_n x_n}{1 - \beta_n}$, we have

$$x_{n+1} = (1 - \beta_n)p_n + \beta_n x_n. \tag{3.5}$$

From (3.5), we have

$$\begin{aligned}
||p_{n+1} - p_n|| &= \left\| \frac{x_{n+2} - \beta_{n+1}x_{n+1}}{1 - \beta_{n+1}} - \frac{x_{n+1} - \beta_n x_n}{1 - \beta_n} \right\| \\
&= \left\| \frac{\alpha_{n+1}u + \gamma_{n+1}P_C(I - \lambda E)u_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n u + \gamma_n P_C(I - \lambda E)u_n}{1 - \beta_n} \right\| \\
&= \left\| \left(\frac{\alpha_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n}{1 - \beta_n} \right) u + \frac{\gamma_{n+1}}{1 - \beta_{n+1}} \left(P_C(I - \lambda E)u_{n+1} - P_C(I - \lambda E)u_n \right) \right. \\
&\quad \left. + \left(\frac{\gamma_{n+1}}{1 - \beta_{n+1}} - \frac{\gamma_n}{1 - \beta_n} \right) P_C(I - \lambda E)u_n \right\| \\
&\leq \left| \frac{\alpha_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n}{1 - \beta_n} \right| ||u|| + \frac{\gamma_{n+1}}{1 - \beta_{n+1}} ||u_{n+1} - u_n|| \\
&\quad + \left| \frac{\gamma_{n+1}}{1 - \beta_{n+1}} - \frac{\gamma_n}{1 - \beta_n} \right| ||P_C(I - \lambda E)u_n|| \\
&= \left| \frac{\alpha_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n}{1 - \beta_n} \right| ||u|| + \frac{\gamma_{n+1}}{1 - \beta_{n+1}} ||u_{n+1} - u_n|| \\
&\quad + \left| \frac{1 - \beta_{n+1} - \alpha_{n+1}}{1 - \beta_{n+1}} - \frac{1 - \beta_n - \alpha_n}{1 - \beta_n} \right| ||P_C(I - \lambda E)u_n|| \\
&= \left| \frac{\alpha_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n}{1 - \beta_n} \right| ||u|| + \frac{\gamma_{n+1}}{1 - \beta_{n+1}} ||u_{n+1} - u_n|| \\
&\quad + \left| \frac{\alpha_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n}{1 - \beta_n} \right| ||P_C(I - \lambda E)u_n|| \\
&= \left| \frac{\alpha_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n}{1 - \beta_n} \right| (||u|| + ||P_C(I - \lambda E)u_n||) \\
&\quad + \frac{\gamma_{n+1}}{1 - \beta_{n+1}} ||u_{n+1} - u_n||.
\end{aligned} \tag{3.6}$$

Putting $v_n = x_n - r_n Dx_n$, we have $u_n = T_{r_n}(x_n - r_n Dx_n) = T_{r_n} v_n$. From definition of u_n, we have

$$F(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - v_n \rangle \geq 0, \quad \forall y \in C, \tag{3.7}$$

and

$$F(u_{n+1}, y) + \frac{1}{r_{n+1}} \langle y - u_{n+1}, u_{n+1} - v_{n+1} \rangle \geq 0, \quad \forall y \in C. \tag{3.8}$$

Putting $y = u_{n+1}$ in (3.7) and $y = u_n$ in (3.8), we have

$$F(u_n, u_{n+1}) + \frac{1}{r_n} \langle u_{n+1} - u_n, u_n - v_n \rangle \geq 0, \tag{3.9}$$

and

$$F(u_{n+1}, u_n) + \frac{1}{r_{n+1}} \langle u_n - u_{n+1}, u_{n+1} - v_{n+1} \rangle \geq 0. \tag{3.10}$$

Summing up (3.9) and (3.10) and using ($A2$), we have

$$\begin{aligned}0 &\le \frac{1}{r_n}\langle u_{n+1}-u_n, u_n-v_n\rangle + \frac{1}{r_{n+1}}\langle u_n-u_{n+1},\ u_{n+1}-v_{n+1}\rangle\\ &= \left\langle u_{n+1}-u_n, \frac{u_n-v_n}{r_n}\right\rangle + \left\langle u_n-u_{n+1}, \frac{u_{n+1}-v_{n+1}}{r_{n+1}}\right\rangle\\ &= \left\langle u_{n+1}-u_n,\ \frac{u_n-v_n}{r_n} - \frac{u_{n+1}-v_{n+1}}{r_{n+1}}\right\rangle.\end{aligned}$$

It implies that

$$\begin{aligned}0 &\le \left\langle u_{n+1}-u_n,\ u_n-v_n-\frac{r_n}{r_{n+1}}(u_{n+1}-v_{n+1})\right\rangle\\ &= \left\langle u_{n+1}-u_n,\ u_n-u_{n+1}+u_{n+1}-v_n-\frac{r_n}{r_{n+1}}(u_{n+1}-v_{n+1})\right\rangle.\end{aligned}$$

It implies that

$$\begin{aligned}||u_{n+1}-u_n||^2 &\le \left\langle u_{n+1}-u_n,\ u_{n+1}-v_n-\frac{r_n}{r_{n+1}}(u_{n+1}-v_{n+1})\right\rangle\\ &= \left\langle u_{n+1}-u_n,\ u_{n+1}-v_{n+1}+v_{n+1}-v_n-\frac{r_n}{r_{n+1}}(u_{n+1}-v_{n+1})\right\rangle\\ &= \left\langle u_{n+1}-u_n,\ v_{n+1}-v_n+\left(1-\frac{r_n}{r_{n+1}}\right)(u_{n+1}-v_{n+1})\right\rangle\\ &\le ||u_{n+1}-u_n||\left(||v_{n+1}-v_n||+\frac{1}{r_{n+1}}|r_{n+1}-r_n|\,||u_{n+1}-v_{n+1}||\right).\end{aligned}$$

It follows that

$$||u_{n+1}-u_n|| \le ||v_{n+1}-v_n||+\frac{1}{e}|r_{n+1}-r_n|\,||u_{n+1}-v_{n+1}||. \tag{3.11}$$

Since $v_n = x_n - r_nDx_n$, we have

$$\begin{aligned}||v_{n+1}-v_n|| &= ||x_{n+1}-r_{n+1}Dx_{n+1}-x_n+r_nDx_n||\\ &= ||(I-r_{n+1}D)x_{n+1}-(I-r_{n+1}D)x_n\\ &\quad +(I-r_{n+1}D)x_n-(I-r_nD)x_n||\\ &\le ||(I-r_{n+1}D)x_{n+1}-(I-r_{n+1}D)x_n||\\ &\quad +||(r_n-r_{n+1})Dx_n||\\ &\le ||x_{n+1}-x_n||+|r_n-r_{n+1}|\,||Dx_n||.\end{aligned} \tag{3.12}$$

Substitute (3.12) into (3.11), we have

$$\begin{aligned}||u_{n+1}-u_n|| &\le ||v_{n+1}-v_n||+\frac{1}{e}|r_{n+1}-r_n|\,||u_{n+1}-v_{n+1}||\\ &\le ||x_{n+1}-x_n||+|r_n-r_{n+1}|\,||Dx_n||\\ &\quad +\frac{1}{e}|r_{n+1}-r_n|\,||u_{n+1}-v_{n+1}||\\ &\le ||x_{n+1}-x_n||+|r_n-r_{n+1}|L+\frac{1}{e}|r_{n+1}-r_n|L,\end{aligned} \tag{3.13}$$

where $L=\max_{n\in\mathbb{N}}\{||Dx_n||, ||u_n\text{-}v_n||\}$. Substitute (3.13) into (3.6), we have

$$\begin{aligned}||p_{n+1}-p_n|| &\leq \left|\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n}\right|(||u||+||P_C(I-\lambda E)u_n||)\\&+\frac{\gamma_{n+1}}{1-\beta_{n+1}}||u_{n+1}-u_n||\\&\leq \left|\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n}\right|(||u||+||P_C(I-\lambda E)u_n||)\\&+||x_{n+1}-x_n||+|r_n-r_{n+1}|L+\frac{1}{e}|r_{n+1}-r_n|L,\end{aligned} \tag{3.14}$$

From conditions (*i*), (*iii*) and (3.14), we have

$$\limsup_{n\to\infty}\left(||p_{n+1}-p_n||-||x_{n+1}-x_n||\right)\leq 0. \tag{3.15}$$

From Lemma 2.4, (3.15) and (3.5), we have

$$\lim_{n\to\infty}||p_n-x_n|| = 0. \tag{3.16}$$

From (3.5), we have

$$x_{n+1}-x_n=(1-\beta_n)(p_n-x_n). \tag{3.17}$$

From (3.16), (3.17) and condition (*ii*), we have

$$\lim_{n\to\infty}||x_{n+1}-x_n|| = 0. \tag{3.18}$$

Since

$$x_{n+1}-x_n=\alpha_n(u-x_n)+\gamma_n(P_C(I-\lambda(I-T))u_n-x_n),$$

from conditions (*i*), (*ii*) and (3.18), we have

$$\lim_{n\to\infty}||P_C(I-\lambda E)u_n-x_n|| = 0, \tag{3.19}$$

where $E = I - T$.

Step 5. We will show that

$$\lim_{n\to\infty}||u_n-x_n|| = 0. \tag{3.20}$$

Since $u_n=T_{r_n}(x_n-r_nDx_n)$, we have

$$\begin{aligned}\|u_n-z\|^2 &= \|T_{r_n}(x_n-r_nDx_n)-T_{r_n}(I-r_nD)z\|^2\\&\leq \langle (I-r_nD)x_n-(I-r_nD)z,\ u_n-z\rangle\\&= \frac{1}{2}\left(\|(I-r_nD)x_n-(I-r_nD)z\|^2+\|u_n-z\|^2\right.\\&\left.-\|(I-r_nD)x_n-(I-r_nD)z-u_n+z\|^2\right)\\&\leq \frac{1}{2}\left(\|x_n-z\|^2+\|u_n-z\|^2-\|(x_n-u_n)-r_n(Dx_n-Dz)\|^2\right)\\&\leq \frac{1}{2}\left(\|x_n-z\|^2+\|u_n-z\|^2-\|x_n-u_n\|^2-r_n^2\|Dx_n-Dz\|^2\right.\\&\left.+2r_n\langle x_n-u_n,\ Dx_n-Dz\rangle\right),\end{aligned}$$

it implies that

$$||u_n-z||^2\leq ||x_n-z||^2-||x_n-u_n||^2-r_n^2||Dx_n-Dz||^2+2r_n\langle x_n-u_n,Dx_n-Dz\rangle. \tag{3.21}$$

By nonexpansiveness of T_{r_n} and using the same method as (3.3), we have

$$\begin{aligned} ||u_n - z||^2 &= ||T_{r_n}(I - r_nD)x_n - T_{r_n}(I - r_nD)z||^2 \\ &\leq ||(I - r_nD)x_n - (I - r_nD)z||^2 \\ &\leq ||x_n - z||^2 + r_n(r_n - 2\gamma)||Dx_n - Dz||^2 \\ &= ||x_n - z||^2 - r_n(2\gamma - r_n)||Dx_n - Dz||^2. \end{aligned} \tag{3.22}$$

By nonexpansiveness of $P_C(I - \lambda E)$ and (3.22), we have

$$\begin{aligned} ||x_{n+1} - z||^2 &= ||\alpha_n(u - z) + \beta_n(x_n - z) + \gamma_n(P_C(I - \lambda E)u_n - z)||^2 \\ &\leq \alpha_n||u - z||^2 + \beta_n||x_n - z||^2 + \gamma_n||u_n - z||^2 \\ &\leq \alpha_n||u - z||^2 + \beta_n||x_n - z||^2 + \gamma_n(||x_n - z||^2 \\ &\quad - r_n(2\gamma - r_n)||Dx_n - Dz||^2) \\ &\leq \alpha_n||u - z||^2 + ||x_n - z||^2 - r_n\gamma_n(2\gamma - r_n)||Dx_n - Dz||^2, \end{aligned} \tag{3.23}$$

it implies that

$$\begin{aligned} r_n\gamma_n(2\gamma - r_n)||Dx_n - Dz||^2 &\leq \alpha_n||u - z||^2 + ||x_n - z||^2 - ||x_{n+1} - z||^2 \\ &\leq \alpha_n||u - z||^2 + (||x_n - z|| \\ &\quad + ||x_{n+1} - z||)||x_{n+1} - x_n||. \end{aligned} \tag{3.24}$$

From (3.18), (3.24), conditions (*i*) and (*ii*), we have

$$\lim_{n\to\infty} ||Dx_n - Dz|| = 0 \tag{3.25}$$

From (3.23) and (3.21). we have

$$\begin{aligned} ||x_{n+1} - z||^2 &\leq ||\alpha_n(u - z) + \beta_n(x_n - z) + \gamma_n(P_C(I - \lambda E)u_n - z)||^2 \\ &\leq \alpha_n||u - z||^2 + \beta_n||x_n - z||^2 + \gamma_n||u_n - z||^2 \\ &\leq \alpha_n||u - z||^2 + \beta_n||x_n - z||^2 + \gamma_n\left(||x_n - z||^2 - ||x_n - u_n||^2\right. \\ &\quad \left. -r_n^2||Dx_n - Dz||^2 + 2r_n\langle x_n - u_n, Dx_n - Dz\rangle\right) \\ &\leq \alpha_n||u - z||^2 + \beta_n||x_n - z||^2 + \gamma_n||x_n - z||^2 - \gamma_n||x_n - u_n||^2 \\ &\quad + 2r_n\gamma_n||x_n - u_n||\,||Dx_n - Dz|| \\ &\leq \alpha_n||u - z||^2 + ||x_n - z||^2 - \gamma_n||x_n - u_n||^2 + 2r_n\gamma_n||x_n - u_n||\,||Dx_n - Dz||, \end{aligned}$$

which implies that

$$\begin{aligned} \gamma_n||x_n - u_n||^2 &\leq \alpha_n||u - z||^2 + ||x_n - z||^2 - ||x_{n+1} - z||^2 + 2r_n\gamma_n||x_n - u_n||\,||Dx_n - Dz|| \\ &\leq \alpha_n||u - z||^2 + (||x_n - z|| + ||x_{n+1} - z||)||x_{n+1} - x_n|| \\ &\quad + 2r_n\gamma_n||x_n - u_n||\,||Dx_n - Dz||, \end{aligned}$$

from condition (*i*), (3.25) and (3.18), we have

$$\lim_{n\to\infty} ||x_n - u_n|| = 0.$$

Step 6. We prove that

$$\limsup_{n\to\infty}\langle u - z_0, x_n - z_0\rangle \leq 0, \tag{3.26}$$

where $z_0 = P_{\mathbb{F}}u$. To show this equality, take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle u - z_0, x_n - z_0\rangle = \lim_{k\to\infty}\langle u - z_0, x_{n_k} - z_0\rangle, \tag{3.27}$$

Without loss of generality, we may assume that $x_{n_k} \rightharpoonup \omega$ as $k \to \infty$ where $\omega \in C$. We first show $\omega \in EP(F, D)$, where $D = aA + (1 - a)B$, $\forall a \in [0,1]$. From (3.20), we have $u_{n_k} \rightharpoonup \omega$ as $k \to \infty$. Since $u_n = T_{r_n}(x_n - r_n D x_n)$, we obtain

$$F(u_n, y) + \langle Dx_n,\ y - u_n\rangle + \frac{1}{r_n}\langle y - u_n,\ u_n - x_n\rangle \geq 0, \quad \forall y \in C.$$

From $(A2)$, we have $\langle Dx_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq F(y, u_n)$. Then

$$\langle Dx_{n_k}, y - u_{n_k}\rangle + \frac{1}{r_{n_k}}\langle y - u_{n_k}, u_{n_k} - x_{n_k}\rangle \geq F(y,\ u_{n_k}), \quad \forall y \in C. \tag{3.28}$$

Put $z_t = ty + (1 - t)\omega$ for all $t \in (0, 1]$ and $y \in C$. Then, we have $z_t \in C$. So, from (3.28) we have

$$\begin{aligned}\langle z_t - u_{n_k}, Dz_t\rangle &\geq \langle z_t - u_{n_k}, Dz_t\rangle - \langle z_t - u_{n_k}, Dx_{n_k}\rangle - \left\langle z_t - u_{n_k}, \frac{u_{n_k} - x_{n_k}}{r_{n_k}}\right\rangle + F(z_t,\ u_{n_k}) \\ &= \langle z_t - u_{n_k},\ Dz_t - Du_{n_k}\rangle + \langle z_t - u_{n_k},\ Du_{n_k} - Dx_{n_k}\rangle \\ &\quad - \left\langle z_t - u_{n_k},\ \frac{u_{n_k} - x_{n_k}}{r_{n_k}}\right\rangle + F(z_t,\ u_{n_k}).\end{aligned}$$

Since $||u_{n_k} - x_{n_k}|| \to 0$, we have $||Du_{n_k} - Dx_{n_k}|| \to 0$. Further, from monotonicity of D, we have $\langle z_t - u_{n_k}, Dz_t - Du_{n_k}\rangle \geq 0$. So, from (A4) we have

$$\langle z_t - \omega, Dz_t\rangle \geq F(z_t,\ \omega) \text{ as } k \to \infty. \tag{3.29}$$

From $(A1)$, $(A4)$ and (3.29), we also have

$$\begin{aligned}0 = \ & F(z_t,\ z_t) \leq tF(z_t,\ y) + (1 - t)F(z_t,\ \omega) \\ \leq \ & tF(z_t,\ y) + (1 - t)\langle z_t - \omega,\ Dz_t\rangle \\ = \ & tF(z_t,\ y) + (1 - t)t\langle y - \omega,\ Dz_t\rangle,\end{aligned}$$

hence

$$0 \leq F(z_t,\ y) + (1 - t)\langle y - \omega,\ Dz_t\rangle.$$

Letting $t \to 0$, we have

$$0 \leq F(\omega, y) + \langle y - \omega, D\omega\rangle \quad \forall y \in C. \tag{3.30}$$

Therefore $\omega \in EP(F, D)$, where $D = aA + (1 - a)B$, $\forall a \in [0,1]$. Since

$$||P_C(I - \lambda E)u_n - u_n|| \ \leq \ ||\ ||P_C(I - \lambda E)u_n - x_n||\ || + ||x_n - u_n||,$$

where $E = I - T$ from (3.19) and (3.20), we have

$$\lim_{n \to 0} ||P_C(I - \lambda E)u_n - u_n|| \ = 0. \tag{3.31}$$

Since $u_{n_k} \to \omega$ as $k \to \omega$, (3.31) and Lemma 2.5, we have $\omega \in F(P_C(I - \lambda E))$. From Lemma 2.3 and Remark 2.8, we have $\omega \in F(T)$. Therefore $\omega \in \mathbb{F}$. Since $x_{n_k} \to \omega$ as $k \to \infty$ and $\omega \in \mathbb{F}$, we have

$$\limsup_{n \to \infty}\langle u - z_0, x_n - z_0\rangle = \lim_{n \to \infty}\langle u - z_0, x_{n_k} - z_0\rangle = \langle u - z_0, \omega - z_0\rangle \leq 0.$$

Step 7. Finally, we show that $\{x_n\}$ converses strongly to $z_0 = P_{\mathbb{F}}u$. From definition of x_n, we have

$$\begin{aligned} ||x_{n+1} - z_0||^2 &= ||\alpha_n(u - z_0) + \beta_n(x_n - z_0) + \gamma_n(P_C(I - \lambda(I - T))u_n - z_0)||^2 \\ &\leq ||\beta_n(x_n - z_0) + \gamma_n(P_C(I - \lambda(I - T))u_n - z_0)||^2 + 2\alpha_n\langle u - z_0,\ x_{n+1} - z_0\rangle \\ &\leq \beta_n||x_n - z_0||^2 + \gamma_n||P_C(I - \lambda(I - T))u_n - z_0||^2 + 2\alpha_n\langle u - z_0,\ x_{n+1} - z_0\rangle \\ &\leq \beta_n||x_n - z_0||^2 + \gamma_n||T_{r_n}(I - r_nD)x_n - z_0||^2 + 2\alpha_n\langle u - z_0,\ x_{n+1} - z_0\rangle \\ &\leq (1 - \alpha_n)||x_n - z_0||^2 + 2\alpha_n\langle u - z_0,\ x_{n+1} - z_0\rangle \end{aligned}$$

From (3.26) and Lemma 2.2, we have $\{x_n\}$ converses strongly to $z_0 = P_{\mathbb{F}}u$. This completes the prove. □

4 Applications

To prove strong convergence theorem in this section, we needed the following lemma.

Lemma 4.1. *Let C be a nonempty closed convex subset of a real Hilbert space H and let A, $B : C \to H$ be α and β-inverse strongly monotone mappings, respectively, with α, $\beta > 0$ and $VI(C, A) \cap VI(C, B) \neq \emptyset$. Then*

$$VI(C,\ aA + (1 - a)B) = VI(C,\ A)\bigcap VI(C,\ B),\ \forall a \in (0, 1). \tag{4.1}$$

Furthermore if $0 < \gamma < 2\eta$, where $\eta = \min\{\alpha, \beta\}$, we have $I - \gamma(aA + (1 - a)B)$ is a nonexpansive mapping.

Proof. It is easy to see that $VI(C, A) \cap VI(C, B) \subseteq VI(C, aA + (1 - a)B)$. Next, we will show that $VI(C, aA + (1 - a)B) \subseteq VI(C, A) \cap VI(C, B)$. Let $x_0 \in VI(C, aA + (1 - a)B)$ and $x^* \in VI(C, A) \cap VI(C, B)$. Then, we have

$$\langle y - x^*,\ Ax^*\rangle \geq 0, \quad \forall y \in C,$$

and

$$\langle y - x^*,\ Bx^*\rangle \geq 0, \quad \forall y \in C.$$

For every $a \in (0, 1)$, we have

$$\langle y - x^*, aAx^*\rangle \geq 0, \quad \forall y \in C, \tag{4.2}$$

and

$$\langle y - x^*, (1 - a)Bx^*\rangle \geq 0, \quad \forall y \in C. \tag{4.3}$$

By monotonicity of A, B and x^*, $x_0 \in C$, we have

$$\begin{aligned} \langle x^* - x_0, aAx_0\rangle &= \langle x^* - x_0, aAx_0 + (1 - a)Bx_0 - (1 - a)Bx_0\rangle \\ &= \langle x^* - x_0,\ aAx_0 + (1 - a)Bx_0\rangle - \langle x^* - x_0,\ (1 - a)Bx_0\rangle \\ &\geq (1 - a)\langle x_0 - x^*,\ Bx_0\rangle \\ &= (1 - a)\left(\langle x_0 - x^*,\ Bx_0 - Bx^*\rangle + \langle x_0 - x^*,\ Bx^*\rangle\right) \\ &\geq 0. \end{aligned} \tag{4.4}$$

It implies that

$$\langle x^* - x_0, Ax_0\rangle \geq 0. \tag{4.5}$$

By monotonicity of A, $x^* \in VI(C, A)$ and (4.5), we have

$$\begin{aligned} 0 &\le \langle x^* - x_0,\ Ax_0\rangle \\ &= \langle x^* - x_0,\ Ax_0 - Ax^* + Ax^*\rangle \\ &= \langle x^* - x_0,\ Ax_0 - Ax^*\rangle + \langle x^* - x_0,\ Ax^*\rangle \\ &\le -\alpha||Ax^* - Ax_0||^2 + \langle x^* - x_0,\ Ax^*\rangle \\ &\le -\alpha||Ax^* - Ax_0||^2, \end{aligned}$$

it implies that

$$Ax^* = Ax_0. \tag{4.6}$$

For every $y \in C$, from (4.5), (4.6) and $x^* \in VI(C, A)$, we have

$$\begin{aligned} \langle y - x_0,\ Ax_0\rangle &= \langle y - x^*,\ Ax_0\rangle + \langle x^* - x_0,\ Ax_0\rangle \\ &\ge \langle y - x^*,\ Ax^*\rangle \ge 0. \end{aligned}$$

Then, we have

$$x_0 \in VI(C,\ A). \tag{4.7}$$

From (4.4), we have

$$\begin{aligned} (1-a)\langle x^* - x_0,\ Bx_0\rangle &\ge a\langle x_0 - x^*,\ Ax_0\rangle \\ &= a\left(\langle x_0 - x^*,\ Ax_0 - Ax^*\rangle + \langle x_0 - x^*,\ Ax^*\rangle\right) \\ &\ge 0. \end{aligned} \tag{4.8}$$

It implies that

$$\langle x^* - x_0, Bx_0\rangle \ge 0. \tag{4.9}$$

By monotonicity of B, $x^* \in VI(C, B)$ and (4.9), we have

$$\begin{aligned} 0 &\le \langle x^* - x_0,\ Bx_0\rangle \\ &= \langle x^* - x_0,\ Bx_0 - Bx^* + Bx^*\rangle \\ &= \langle x^* - x_0,\ Bx_0 - Bx^*\rangle + \langle x^* - x_0,\ Bx^*\rangle \\ &\le -\beta||Bx^* - Bx_0||^2 + \langle x^* - x_0,\ Bx^*\rangle \\ &\le -\beta||Bx^* - Bx_0||^2, \end{aligned}$$

it implies that

$$Bx^* = Bx_0. \tag{4.10}$$

For every $y \in C$, from (4.9), (4.10) and $x^* \in VI(C, B)$, we have

$$\begin{aligned} \langle y - x_0,\ Bx_0\rangle &= \langle y - x^*,\ Bx_0\rangle + \langle x^* - x_0,\ Bx_0\rangle \\ &\ge \langle y - x^*,\ Bx^*\rangle \ge 0. \end{aligned}$$

Then, we have

$$x_0 \in VI(C,\ B). \tag{4.11}$$

By (4.7) and (4.11), we have $x_0 \in VI(C, A) \cap VI(C, B)$. Hence, we have

$$VI\left(C,\ aA + (1-a)B\right) \subseteq VI(C,\ A) \bigcap VI(C,\ B).$$

Next, we will show that $I - \gamma(aA + (1 - a)B)$ is a nonexpansive mapping. To show this let $x, y \in C$, then we have

$$\begin{aligned}
&\left\| \left(I - \gamma\left(aA + (1-a)B\right)\right)x - \left(I - \gamma\left(aA + (1-a)B\right)\right)y \right\|^2 \\
&= \left\| x - y - \gamma\left(\left(aA + (1-a)B\right)x - \left(aA + (1-a)B\right)y\right) \right\|^2 \\
&= \left\| x - y - \gamma\left(a(Ax - Ay) + (1-a)(Bx - By)\right) \right\|^2 \\
&= ||x - y||^2 - 2\gamma\langle a\,(Ax - Ay) + (1-a)\,(Bx - By),\ x - y\rangle \\
&\quad + \gamma^2||a(Ax - Ay) + (1-a)\,(Bx - By)||^2 \\
&\le ||x - y||^2 - 2\gamma a\langle Ax - Ay,\ x - y\rangle - 2\gamma(1-a)\,\langle Bx - By,\ x - y\rangle \\
&\quad + a\gamma^2||Ax - Ay||^2 + (1-a)\gamma^2||\,Bx - By||^2 \\
&\le ||x - y||^2 - 2\gamma a\alpha||Ax - Ay||^2 - 2\gamma(1-a)\beta||Bx - By||^2 \\
&\quad + a\gamma^2||Ax - Ay||^2 + (1-a)\gamma^2||Bx - By||^2 \\
&= ||x - y||^2 + a\gamma(\gamma - 2\alpha)||Ax - Ay||^2 + (1-a)\gamma(\gamma - 2\beta)||Bx - By||^2 \\
&\le ||x - y||^2.
\end{aligned} \tag{4.12}$$

□

Theorem 4.2. *Let C be a closed convex subset of Hilbert space H and let $A, B : C \to H$ be α and β-inverse strongly monotone, respectively. Let T be κ-*strictly *pseudo contractive mapping with $\mathbb{F} = F(T) \cap VI(C, A) \cap VI(C, B) \neq \emptyset$. Let $\{x_n\}$ be the sequence generated by $x_1, u \in C$ and*

$$x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n P_C(I - \lambda(I - T))P_C(I - r_n(aA + (1-a)B))x_n, \quad \forall n \ge 1 \tag{4.13}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset [0, 1]$, $a \in (0, 1)$, $\lambda \in (0, 1 - \kappa)$, $\alpha_n + \beta_n + \gamma_n = 1$, $\forall n \in \mathbb{N}$ and $\{r_n\} \subset [0, 2\gamma]$, $\gamma = \min\{\alpha, \beta\}$ satisfy;

$$(i)\ \sum_{n=1}^{\infty} \alpha_n = \infty, \quad \lim_{n\to\infty} \alpha_n = 0;$$

$$(ii)\ 0 < c \le \beta_n \le d < 1,\ 0 < e \le r_n \le f < 2\gamma;$$

$$(iii)\ \lim_{n\to\infty} |r_{n+1} - r_n| = 0.$$

Then $\{x_n\}$ converges strongly to $z_0 = P_{\mathbb{F}}u$.

Proof. From 3.1 putting $F \equiv 0$ in Theorem 3.1, we have

$$\langle y - u_n,\ u_n - (I - r_n D)x_n\rangle \ge 0, \quad \forall y \in C,$$

where $D = aA + (1 - a)B$, $\forall a \in [0,1]$ It implies that

$$u_n = P_C(I - r_n D)x_n.$$

Then, we have (4.13). From Theorem 3.1 and Lemma 4.1, we can conclude the desired conclusion. □

Theorem 4.3. *Let C be a closed convex subset of Hilbert space H and let $F: C\times C \to \mathbb{R}$ be a bifunction satisfying (A_1)-(A_4), let $A : C \to H$ be α-inverse strongly monotone. Let $T : C \to C$ be κ-*strictly *pseudo contractive mapping with $\mathbb{F} = F(T) \cap EP(F, A) \neq \emptyset$. Let $\{x_n\}$ and $\{u_n\}$ be the sequences generated by $x_1, u \in C$ and*

$$\begin{cases} F(u_n, y) + \langle Ax_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, & \forall y \in C, \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n P_C(I - \lambda(I - T))u_n, & \forall n \geq 1, \end{cases} \tag{4.14}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset [0, 1]$, $\lambda \in (0, 1 - \kappa)$, $\alpha_n + \beta_n + \gamma_n = 1$, $\forall n \in \mathbb{N}$ *and* $\{r_n\} \subset [0, 2\gamma]$, $\gamma = \min\{\alpha, \beta\}$ *satisfy;*

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty, \quad \lim_{n\to\infty} \alpha_n = 0;$

(ii) $0 < c \leq \beta_n \leq d < 1,\ 0 < e \leq r_n \leq f < 2\gamma;$

(iii) $\lim_{n\to\infty} |r_{n+1} - r_n| = 0.$

Then $\{x_n\}$ *converges strongly to* $z_0 = P_{\mathbb{F}}u$.

Proof. From Theorem 3.1, putting $A \equiv B$, we can conclude the desired conclusion. □

Acknowledgements
This research was supported by the Research Administration Division of King Mongkut's Institute of Technology Ladkrabang.

Competing interests
The authors declare that they have no competing interests.

Received: 15 January 2012 Accepted: 23 May 2012 Published: 23 May 2012

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doi:10.1186/1687-1812-2012-89
Cite this article as: Kangtunyakarn: Convergence theorem of κ-strictly pseudo-contractive mapping and a modification of generalized equilibrium problems. *Fixed Point Theory and Applications* 2012 **2012**:89.