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Strong convergence of a hybrid projection iterative algorithm for common solutions of operator equations and of inclusion problems

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Abstract

In this article, zero points of the sum of a maximal monotone operator and an inverse-strongly monotone mapping, solutions of a monotone variational inequality, and fixed points of a strict pseudocontraction are investigated. A hybrid projection iterative algorithm is considered for analyzing the convergence of the iterative sequences. Strong convergence theorems are established in the framework of real Hilbert spaces without any compact assumptions. Some applications of the main results are also provided.

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1. Introduction

The theory of monotone operators has emerged as an effective and powerful tool for studying a wide class of unrelated problems arising in various branches of social, engineering, and pure sciences in unified and general framework. Two notions related to monotone operators have turned out to be very useful in the study of various problems involving such operators. The first one, which is inspired by the notion of subdifferential of a convex function, is the concept of enlargement of a given operator; see [1-3] and the references therein. It allows to make a quantitative analysis in different problems involving monotone operators, like for example variational inequalities, inclusions, etc. The second notion is the one of generalized sum of two monotone operators; see [4,5] and the references therein. In recent years, much attention has been given to develop efficient numerical methods for treating zero point problems of monotone operators and fixed point problems of mappings which are Lipschitz continuous; see [6-28] and the references therein. The gradient-projection method is a powerful tool for solving constrained convex optimization problems and has extensively been studied; see [29-31] and the references therein. It has recently applied to solve split feasibility problems which find applications in image reconstructions and the intensity modulated radiation theory; see [32-35] and the reference therein.

In this article, zero points of the sums of a maximal monotone operator and an inverse-strongly monotone mapping, solutions of a monotone variational inequality, and fixed points of a strict pseudocontraction are investigated based on a hybrid iterative method.

The organization of this article is as follows. In Section 2, we provide some necessary preliminaries. In Section 3, a hybrid iterative method is proposed and analyzed. Strong convergence theorems for common elements in the zero point set of the sums of a maximal monotone operator and an inverse-strongly monotone mapping, the solution set of a monotone variational inequality, and the fixed point set of a strict pseudocontraction are established in the framework of real Hilbert spaces without any compact assumptions. In Section 4, applications of the main results are discussed.

2. Preliminaries

In what follows, we always assume that H is a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. Let C be a nonempty, closed, and convex subset of H. Let $S : C \to C$ be a nonlinear mapping. $F(S)$ stands for the fixed point set of S; that is, $F(S) := \{x \in C : x = Tx\}$.

Recall that S is said to be *nonexpansive* iff

$$\|Sx - Sy\| \leq \|x - y\|, \quad \forall x, \ y \in C.$$

If C is a bounded, closed, and convex subset of H, then $F(S)$ is not empty, closed, and convex; see [36].

S is said to be *κ-strictly pseudocontractive* iff there exists a constant $\kappa \in [0, 1)$ such that

$$\|Sx - Sy\|^2 \leq \|x - y\|^2 + \kappa\|x - Sx - y + Sy\|^2, \quad \forall x, \ y \in C.$$

It is clear that the class of κ-strictly pseudocontractive mappings includes the class of non-
expansive mappings.

Let $A : C \to H$ be a mapping. A is said to be *monotone* iff

$$\langle Ax - Ay, \ x - y \rangle \geq 0, \quad \forall x, \ y \in C.$$

A is said to be *inverse-strongly monotone* iff there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, \ x - y \rangle \geq \alpha\|Ax - Ay\|^2, \quad \forall x, \ y \in C.$$

For such a case, A is also said to be *α-inverse-strongly monotone*.

A is said to be *Lipschitz continuous* iff there exists a positive constant L such that

$$\|Ax - Ay\| \leq L\|x - y\|, \quad \forall x, \ y \in C.$$

Recall that the classical variational inequality is to find an $x \in C$ such that

$$\langle Ax, y - x \rangle \geq 0, \quad \forall y \in C. \tag{2.1}$$

It is known that $x \in C$ is a solution to (2.1) if and only if x is a fixed point of the mapping $\mathrm{Proj}_C(I - rA)$, where $r > 0$ is a constant, I stands for the identity mapping, and Proj_C stands for the metric projection from H onto C. If A is α-inverse-strongly monotone and $r \in (0, 2\alpha]$, then the mapping $\mathrm{Proj}_C(I - rA)$ is nonexpansive; see [37] for more details. It follows that $VI(C, A)$, where $VI(C, A)$ stands for the solution set of (2.1), is closed and convex.

A set-valued mapping $R : H \rightrightarrows H$ is said to be *monotone* iff, for all $x, y \in H, f \in Rx$ and $g \in Ry$ imply $\langle x - y, f - g\rangle > 0$. A monotone mapping $R : H \rightrightarrows H$ is *maximal* iff the graph $G(R)$ of R is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping R is maximal if and only if, for any $(x, f) \in H \times H$, $\langle x - y, f - g\rangle \geq 0$, for all $(y, g) \in G(R)$ implies $f \in Rx$.

The class of monotone operators is one of the most important classes of operators. Within the past several decades, many authors have been devoting to the studies on the existence and convergence of zero points for maximal monotone operators; see [38-45] and the references therein. For a maximal monotone operator M on H and $r > 0$, we may define the single-valued resolvent $J_r : H \to D(M)$, where $D(M)$ denotes the domain of M. It is known that J_r is firmly nonexpansive and $M^{-1}(0) = F(J_r)$, where $F(J_r) := \{x \in D(M) : x = J_r x\}$, and $M^{-1}(0)$: $\{x \in H : 0 \in Mx\}$.

In this article, zero points of the sums of a maximal monotone operator and an inverse-strongly monotone mapping, solutions of a monotone variational inequality, and fixed points of a strict pseudocontraction are investigated. A hybrid iterative algorithm is considered for analyzing the convergence of iterative sequences. Strong convergence theorems are established in the framework of real Hilbert spaces without any compact assumptions.

In order to prove our main results, we also need the following definitions and lemmas.

Lemma 2.1 [46]. *Let C be a nonempty, closed, and convex subset of H, and $S : C \to C$ a κ-strict pseudocontraction. Define a mapping $S_\alpha x = \beta x + (1 - \beta)Sx$ for all $x \in C$. If $\beta \in [\kappa, 1)$, then the mapping S_β is a nonexpansive mapping such that $F(S_\beta) = F(S)$.*

Lemma 2.2 [47]. *Let C be a nonempty, closed, and convex subset of H. Let $S : C \to C$ be a nonexpansive mapping. Then the mapping $I - S$ is demiclosed at zero, that is, if $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup \bar{x}$ and $x_n - Sx_n \to 0$, then $\bar{x} \in F(S)$.*

Lemma 2.3. *Let C be a nonempty, closed, and convex subset of H, $B : C \to H$ a mapping, and $M : H \rightrightarrows H$ a maximal monotone operator. Then $F(J_r(I - sB)) = (B + M)^{-1}(0)$.*

Proof. Notice that

$$\begin{aligned} p \in F(J_r(I - sB)) &\Leftrightarrow p = J_r(I - sB)p \Leftrightarrow p - sBp \in p + sMp \\ &\Leftrightarrow 0 \in (B + M)^{-1}(0) \Leftrightarrow p \in (B + M)^{-1}(0). \end{aligned}$$

This completes the proof.

Lemma 2.4 [48]. *Let C be a nonempty, closed, and convex subset of H, $A : C \to H$ a Lipschitz monotone mapping, and $N_C x$ the normal cone to C at $x \in C$; that is, $N_C x = \{y \in H : \langle x - u, y\rangle, \forall u \in C\}$. Define*

$$Wx = \begin{cases} Ax + N_C x, & x \in C, \\ \emptyset & x \notin C. \end{cases}$$

Then W is maximal monotone and $0 \in Wx$ if and only if $x \in VI(C, A)$.

3. Main results

Now, we are in a position to give our main results.

Theorem 3.1. *Let C be a nonempty, closed, and convex subset of H. Let $S : C \to C$ be a κ-strict pseudocontraction with a nonempty fixed point set, $A : C \to H$ an α-inverse-strongly monotone mapping, and $B : C \to H$ a β-inverse-strongly monotone mapping.*

Let $M : H \rightrightarrows H$ be a maximal monotone operator such that $D(M) \subset C$. Assume that $\mathcal{F} := F(S) \cap (B + M)^{-1}(0) \cap VI(C, A)$ is not empty. Let $\{x_n\}$ be a sequence generated by the following iterative process:

$$\begin{cases} x_1 \in C, \\ C_1 = C, \\ z_n = \operatorname{Proj}_C(J_{s_n}(x_n - s_n Bx_n) - r_n AJ_{s_n}(x_n - s_n Bx_n)), \\ y_n = \alpha_n x_n + (1 - \alpha_n)(\beta_n z_n + (1 - \beta_n) Sz_n), \\ C_{n+1} = \{v \in C_n \ : \ \|y_n - v\| \ \leq \ \|x_n - v\|\}, \\ x_{n+1} = \operatorname{Proj}_{C_{n+1}} x_1, \quad n \geq 0, \end{cases} \tag{3.1}$$

where $J_{s_n} = (I + s_n M)^{-1}$, $\{r_n\}$ is a sequence in $(0, 2\alpha)$, $\{s_n\}$ is a sequence in $(0, 2\beta)$, and $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$. Assume that the following restrictions are satisfied

(a) $0 \leq \alpha_n \leq a < 1$, $\kappa \leq \beta_n \leq b < 1$;
(b) $0 < r \leq r_n \leq r' < 2\alpha$;
(c) $0 < s \leq s_n \leq s' < 2\beta$,

where a, b, r, r', s, and s' are real constants. Then the sequence $\{x_n\}$ converges strongly to $\operatorname{Proj}_{\mathcal{F}} x_1$.

Proof. First, we show that C_n is closed and convex for each $n \geq 1$. From the assumption, we see that $C_1 = C$ is closed and convex. Suppose that C_m is closed and convex for some $m \geq 1$. We show that C_{m+1} is closed and convex for the same m. Let $v_1, v_2 \in C_{m+1}$ and $v = tv_1 + (1 - t)v_2$, where $t \in (0, 1)$. Notice that

$$\|y_m - v\| \ \leq \ \|x_m - v\|$$

is equivalent to

$$\|y_m\|^2 - \|x_m\|^2 - \ 2\langle v, \ y_m - x_m\rangle \geq 0.$$

It is clearly to see that $v \in C_{m+1}$. This shows that C_n is closed and convex for each $n \geq 1$. Put

$$v_n = J_{s_n}(x_n - s_n Bx_n),$$

and

$$u_n = S_n z_n,$$

where S_n is defined by

$$S_n x = \beta_n x + (1 - \beta_n) Sx, \quad \forall x \in C.$$

We see from Lemma 2.1 that S_n is nonexpansive with $F(S_n) = F(S)$. Since A is α-inverse-strongly monotone, and B is β-inverse-strongly monotone, we see from the restriction (b) that

$$\begin{aligned} \|(x - r_m Ax) - (y - r_m Ay)\|^2 &= \|x - y\|^2 - 2r_m\langle x - y, Ax - Ay\rangle + r_m^2 m\|Ax - Ay\|^2 \\ &\leq \|x - y\|^2 - r_m(2\alpha - r_m)\|Ax - Ay\|^2 \\ &\leq \|x - y\|^2, \quad \forall x, \ y \in C, \end{aligned} \tag{3.2}$$

and

$$\begin{aligned}||(I-s_mB)x-(I-s_mB)y||^2 &= ||x-y||^2-2s_m\langle x-y, Bx-By\rangle+s_m^2||Bx-By||^2\\ &\le ||x-y||^2-s_m(2\beta-s_m)||Bx-By||^2\\ &\le ||x-y||^2, \quad \forall x,\ y\in C.\end{aligned} \tag{3.3}$$

Now, we show that $\mathcal{F}\subset C_n$ for each $n\ge 1$. Notice that $\mathcal{F}\subset C=C_1$. *Suppose that* $\mathcal{F}\subset C_m$ for some $m\ge 1$. For any $p\in\mathcal{F}\subset C_m$, we see from (3.2), and (3.3) that

$$\begin{aligned}||y_m-p|| &\le \alpha_m||x_m-p||+(1-\alpha_m)||u_m-p||\\ &\le \alpha_m||x_m-p||+(1-\alpha_m)||z_m-p||\\ &\le \alpha_m||x_m-p||+(1-\alpha_m)||(v_m-r_mAv_m)-(p-r_mAp)||\\ &\le \alpha_m||x_m-p||+(1-\alpha_m)||(x_m-s_mBx_m)-(p-s_mBp)||\\ &\le \alpha_m||x_m-p||+(1-\alpha_m)||x_m-p||\\ &= ||x_m-p||.\end{aligned} \tag{3.4}$$

This shows that $p\in C_{m+1}$. This proves that $\mathcal{F}\subset C_n$. Note that $x_n=\mathrm{Proj}_{C_n}x_1$. For each $p\in\mathcal{F}\subset C_n$, we have $||x_1-x_n||\le||x_1-p||$. Since B is inverse-strongly monotone, we see from Lemma 2.3 that $(B+M)^{-1}(0)$ is closed, and convex. Since A is Lipschitz continuous, we find that $VI(C, A)$ is close, and convex. In view of Lemma 2.2, we obtain $F(S)$ is closed, and convex. This proves that $\mathcal{F}$ is closed and convex. It follows that

$$||x_1-x_n|| \le ||x_1-\mathrm{Proj}_{\mathcal{F}}x_1||. \tag{3.5}$$

This implies that $\{x_n\}$ is bounded. Since $x_n=\mathrm{Proj}_{C_n}x_1$ and $x_{n+1}=\mathrm{Proj}_{C_{n+1}}x_1\in C_{n+1}\subset C_n$, we have

$$\begin{aligned}0 &\le \langle x_1-x_n,\ x_n-x_{n+1}\rangle\\ &= \langle x_1-x_n,\ x_n-x_1+x_1-x_{n+1}\rangle\\ &\le -||x_1-x_n||^2+||x_1-x_n||\ ||x_1-x_{n+1}||.\end{aligned}$$

It follows that

$$||x_n-x_1|| \le ||x_{n+1}-x_1||.$$

This proves that $\lim_{n\to\infty}||x_n-x_1||$ exists. Notice that

$$\begin{aligned}&||x_n-x_{n+1}||^2\\ &= ||x_n-x_1||^2+2\langle x_n-x_1,\ x_1-x_{n+1}\rangle+||x_1-x_{n+1}||^2\\ &= ||x_n-x_1||^2-2||x_n-x_1||^2+2\langle x_n-x_1,\ x_n-x_{n+1}\rangle+||x_1-x_{n+1}||^2\\ &\le ||x_1-x_{n+1}||^2-||x_n-x_1||^2.\end{aligned}$$

It follows that

$$\lim_{n\to\infty}||x_n-x_{n+1}|| = 0. \tag{3.6}$$

In view of $x_{n+1}=\mathrm{Proj}_{C_{n+1}}x_1\in C_{n+1}$, we see that

$$||y_n-x_{n+1}|| \le ||x_n-x_{n+1}||.$$

This implies that

$$||y_n - x_n|| \le ||y_n - x_{n+1}|| + ||x_n - x_{n+1}|| \le 2||x_n - x_{n+1}||.$$

We, therefore, obtain from (3.6) that

$$\lim_{n\to\infty} ||x_n - y_n|| = 0. \tag{3.7}$$

On the other hand, we see from (3.3) that

$$\begin{aligned} ||y_n - p||^2 &\le \alpha_n||x_n - p||^2 + (1-\alpha_n)||u_n - p||^2 \\ &\le \alpha_n||x_n - p||^2 + (1-\alpha_n)||z_n - p||^2 \\ &\le \alpha_n||x_n - p||^2 + (1-\alpha_n)||v_n - p||^2 \\ &= \alpha_n||x_n - p||^2 + (1-\alpha_n)||J_{s_n}(x_n - s_nBx_n) - J_{s_n}(p - s_nBp)||^2 \\ &\le ||x_n - p||^2 - (1-\alpha_n)s_n(2\beta - s_n)||Bx_n - Bp||^2. \end{aligned}$$

It follows that

$$\begin{aligned} (1 - \alpha_n)s_n(2\beta - s_n)||Bx_n - Bp||^2 &\le ||x_n - p||^2 - ||y_n - p||^2 \\ &\le ||x_n - y_n||(||x_n - p|| + ||y_n - p||). \end{aligned}$$

In view of the restrictions (a), and (c), we find from (3.7) that

$$\lim_{n\to\infty} ||Bx_n - Bp|| = 0. \tag{3.8}$$

Since J_{s_n} is firmly nonexpansive, we find that

$$\begin{aligned} ||v_n - p||^2 &= ||J_{s_n}(x_n - s_nBx_n) - J_{s_n}(p - s_nBp)||^2 \\ &\le \langle v_n - p,\ (x_n - s_nBx_n) - (p - s_nBp)\rangle \\ &= \frac{1}{2}(||v_n - p||^2 + ||(x_n - s_nBx_n) - (p - s_nBp)||^2 \\ &\quad -||(v_n - p) - ((x_n - s_nBx_n) - (p - s_nBp))||^2) \\ &\le \frac{1}{2}(||v_n - p||^2 + ||x_n - p||^2 - ||v_n - x_n + s_n(Bx_n - Bp)||^2) \\ &= \frac{1}{2}(||v_n - p||^2 + ||x_n - p||^2 - ||v_n - x_n||^2 - s_n^2||Bx_n - Bp||^2 \\ &\quad -2s_n\langle v_n - x_n,\ Bx_n - Bp\rangle) \\ &\le \frac{1}{2}(||v_n - p||^2 + ||x_n - p||^2 - ||v_n - x_n||^2 + 2s_n||v_n - x_n||\ ||Bx_n - Bp||). \end{aligned}$$

This finds that

$$||v_n - p||^2 \le ||x_n - p||^2 - ||v_n - x_n||^2 + 2s_n||v_n - x_n||\ ||Bx_n - Bp||. \tag{3.9}$$

It follows from (3.1) that

$$\begin{aligned} ||y_n - p||^2 &\le \alpha_n||x_n - p||^2 + (1-\alpha_n)||u_n - p||^2 \\ &\le ||x_n - p||^2 - (1-\alpha_n)||v_n - x_n||^2 + 2s_n||v_n - x_n||\ ||Bx_n - Bp||, \end{aligned}$$

which in turn implies that

$$\begin{aligned} (1 - \alpha_n)||v_n - x_n||^2 &\le ||x_n - p||^2 - ||y_n - p||^2 + 2s_n||v_n - x_n||\ ||Bx_n - Bp|| \\ &\le ||x_n - y_n||(||x_n - p|| + ||y_n - p||) + 2s_n||v_n - x_n||\ ||Bx_n - Bp||. \end{aligned}$$

In view of the restriction (a), we see from (3.7), and (3.8) that

$$\lim_{n\to\infty} ||v_n - x_n|| = 0. \tag{3.10}$$

On the other hand, we see from (3.2) that

$$\begin{aligned}||y_n - p||^2 &\le \alpha_n ||x_n - p||^2 + (1-\alpha_n)||u_n - p||^2\\ &\le \alpha_n ||x_n - p||^2 + (1-\alpha_n)||(v_n - r_nAv_n) - (p - r_nAp)||^2\\ &\le ||x_n - p||^2 - (1-\alpha_n)r_n(2\alpha - r_n)||Av_n - Ap||^2.\end{aligned}$$

It follows that

$$\begin{aligned}(1-\alpha_n)r_n(2\alpha - r_n)||Av_n - Ap||^2 &\le ||x_n - p||^2 - ||y_n - p||^2\\ &\le ||x_n - y_n||(||x_n - p|| + ||y_n - p||).\end{aligned}$$

In view of the restrictions (a), and (b), we find from (3.7) that

$$\lim_{n\to\infty} ||Av_n - Ap|| = 0. \tag{3.11}$$

Since Proj_C is firmly nonexpansive, we arrive at

$$\begin{aligned}||z_n - p||^2 &= ||\mathrm{Proj}_C(v_n - r_nAv_n) - \mathrm{Proj}_C(p - r_nAp)||^2\\ &\le \langle z_n - p,\ (v_n - r_nAv_n) - (p - r_nAp)\rangle\\ &= \frac{1}{2}(||z_n - p||^2 + ||(v_n - r_nAv_n) - (p - r_nAp)||^2\\ &\quad - ||(z_n - p) - ((v_n - r_nAv_n) - (p - r_nAp))||^2)\\ &\le \frac{1}{2}(||z_n - p||^2 + ||x_n - p||^2 - ||z_n - v_n + r_n(Av_n - Ap)||^2)\\ &= \frac{1}{2}(||z_n - p||^2 + ||x_n - p||^2 - ||z_n - v_n||^2 - r_n^2||Av_n - Ap||^2\\ &\quad - 2r_n\langle z_n - v_n,\ Av_n - rp\rangle)\\ &\le \frac{1}{2}(||z_n - p||^2 + ||x_n - p||^2 - ||z_n - v_n||^2 + 2r_n||z_n - v_n||\ ||Av_n - Ap||),\end{aligned}$$

which finds that

$$||z_n - p||^2 \le ||x_n - p||^2 - ||z_n - v_n||^2 + 2r_n||z_n - v_n||\ ||Av_n - Ap||. \tag{3.12}$$

This implies that

$$\begin{aligned}||y_n - p||^2 &\le \alpha_n||x_n - p||^2 + (1-\alpha_n)||u_n - p||^2\\ &\le ||x_n - p||^2 - (1-\alpha_n)||z_n - v_n||^2 + 2r_n||z_n - v_n||\ ||Av_n - Ap||,\end{aligned}$$

It follows that

$$\begin{aligned}(1-\alpha_n)||z_n - v_n||^2 &\le ||x_n - p||^2 - ||y_n - p||^2 + 2r_n||z_n - v_n||\ ||Ax_n - Ap||\\ &\le ||x_n - y_n||(||x_n - p|| + ||y_n - p||) + 2r_n||z_n - v_n||\ ||Ax_n - Ap||.\end{aligned}$$

In view of the restriction (a), we see from (3.7), and (3.11) that

$$\lim_{n\to\infty} ||z_n - v_n|| = 0. \tag{3.13}$$

On the other hand, we have

$$||x_n - y_n|| = ||x_n - \alpha_n x_n - (1-\alpha_n)S_n z_n|| = (1-\alpha_n)||x_n - S_n z_n||.$$

In view of (3.7), we see from the restriction (a) that

$$\lim_{n\to\infty} ||x_n - S_n z_n|| = 0. \tag{3.14}$$

Note that

$$||z_n - x_n|| \le ||z_n - v_n|| + ||v_n - x_n||.$$

It follows from (3.10) and (3.13) that

$$\lim_{n\to\infty} ||z_n - x_n|| = 0. \tag{3.15}$$

In view of

$$\begin{aligned} ||x_n - S_n x_n|| &\le ||S_n x_n - S_n z_n|| + ||S_n z_n - x_n|| \\ &\le ||x_n - z_n|| + ||S_n z_n - x_n||, \end{aligned}$$

we see from (3.14) and (3.15) that

$$\lim_{n\to\infty} ||x_n - S_n x_n|| = 0. \tag{3.16}$$

Note that

$$\begin{aligned} ||Sx_n - x_n|| &\le ||Sx_n - S_n x_n|| + ||S_n x_n - x_n|| \\ &\le \beta_n ||Sx_n - x_n|| + ||S_n x_n - x_n||, \end{aligned}$$

which yields that

$$(1-\beta_n)||Sx_n - x_n|| \le ||S_n x_n - x_n||.$$

In view of the restriction (b), we conclude from (3.16) that

$$\lim_{n\to\infty} ||Sx_n - x_n|| = 0. \tag{3.17}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup q$. In view of Lemma 2.2, we obtain from (3.17) that $q \in F(S)$. In view of (3.10), and (3.15), we see that $u_{n_i} \rightharpoonup q$, and $z_{n_i} \rightharpoonup q$, respectively. Now, we are in a position to show that $q \in VI(C, A)$.

Define

$$Wx = \begin{cases} Ax + N_C x, & x \in C, \\ \emptyset, & x \notin C. \end{cases}$$

Then W is maximal monotone. Let $(x, y) \in G(W)$. Since $y - Ax \in N_C x$ and $z_n \in C$, we have

$$\langle x - z_n, y - Ax\rangle \ge 0.$$

On the other hand, we have from $z_n = \mathrm{Proj}_C(I - r_n A_1)v_n$ that

$$\langle x - z_n, z_n - (I - r_n A)v_n\rangle \ge 0$$

and hence

$$\left\langle x - z_n, \frac{z_n - v_n}{r_n} + Av_n \right\rangle \geq 0.$$

It follows that

$$\begin{aligned} \langle x - z_{n_i}, y\rangle &\geq \langle x - z_{n_i}, Ax\rangle \\ &\geq \langle x - z_{n_i}, Ax\rangle - \left\langle x - z_{n_i}, \frac{z_{n_i} - v_{n_i}}{r_{n_i}} + Av_{n_i}\right\rangle \\ &\geq \langle x - z_{n_i}, Ax - Az_{n_i}\rangle + \langle x - z_{n_i}, Az_{n_i} - Av_{n_i}\rangle - \left\langle x - z_{n_i}, \frac{z_{n_i} - v_{n_i}}{r_{n_i}}\right\rangle \\ &\geq \langle x - z_{n_i}, Az_{n_i} - Av_{n_i}\rangle - \left\langle x - z_{n_i}, \frac{z_{n_i} - v_{n_i}}{r_{n_i}}\right\rangle. \end{aligned}$$

In view of the restriction (b), we obtain from (3.13) that $\langle x - q, y\rangle \geq 0$. We have $q \in A^{-1}0$ and hence $q \in VI(C, A)$.

Next, we prove that $q \in (B + M)^{-1}(0)$. Notice that

$$x_n - s_n Bx_n \in v_n + s_n Mv_n;$$

that is,

$$\frac{x_n - v_n}{s_n} - Bx_n \in Mv_n. \tag{3.18}$$

Let $\mu \in v$. Since M is monotone, we find from (3.18) that

$$\left\langle \frac{x_n - v_n}{s_n} - Bx_n - \mu, v_n - v \right\rangle \geq 0.$$

In view of the restriction (c), we see from (3.10) that

$$\langle -Bq - \mu, q - v\rangle \geq 0.$$

This implies that $-Bq \in Mq$, that is, $q \in (B + M)^{-1}(0)$. This completes $q \in \mathcal{F}$. Assume that there exists another subsequence $\{x_{n_i}\}$ of $\{x_n\}$ weak converges weakly to $q' \in \mathcal{F}$. We can easily conclude from Opial's condition (see [49]) that $q = q'$.

Finally, we show that $q = \mathrm{Proj}_{\mathcal{F}} x_1$ and $\{x_n\}$ converges strongly to q. This completes the proof of Theorem 3.1. In view of the weak lower semicontinuity of the norm, we obtain from (3.5) that

$$\begin{aligned} ||x_1 - \mathrm{Proj}_{\mathcal{F}} x_1|| \ \leq \ ||x_1 - q|| \ &\leq \liminf_{n\to\infty} ||x_1 - x_n|| \\ &\leq \limsup_{n\to\infty} ||x_1 - x_n|| \ \leq \ ||x_1 - \mathrm{Proj}_{\mathcal{F}} x_1||, \end{aligned}$$

which yields that $\lim_{n\to\infty} ||x_1 - x_n|| = ||x_1 - \mathrm{Proj}_{\mathcal{F}} x_1|| = ||x_1 - q||$. It follows that $\{x_n\}$ converges strongly to $\mathrm{Proj}_{\mathcal{F}} x_1$. This completes the proof.

We conclude from Theorem 3.1 the following results on nonexpansive mappings.

Corollary 3.2. *Let C be a nonempty, closed, and convex subset of H. Let $S : C \to C$ be a nonexpansive mapping with a nonempty fixed point set, $A : C \to H$ be an α-inverse-*

strongly monotone mapping, and $B : C \to H$ *be a* β*-inverse-strongly monotone mapping. Let* $M : H \rightrightarrows H$ *be a maximal monotone operator such that* $D(M) \subset C$*. Assume that* $\mathcal{F} := F(S) \cap (B+M)^{-1}(0) \cap VI(C, A)$ *is not empty. Let* $\{x_n\}$ *be a sequence generated by the following iterative process:*

$$\begin{cases} x_1 \in C, \\ C_1 = C, \\ z_n = \operatorname{Proj}_C(J_{s_n}x_n - s_nBx_n) - r_nAJ_{s_n}(x_n - s_nBx_n)), \\ y_n = \alpha_n x_n + (1-\alpha_n)Sz_n, \\ C_{n+1} = \{v \in C_n \ : \ \|y_n - v\| \ \le \ \|x_n - v\|\}, \\ x_{n+1} = \operatorname{Proj}_{C_{n+1}} x_1, \quad n \ge 0, \end{cases}$$

where $J_{s_n} = (I + s_nM)^{-1}$, $\{r_n\}$ *is a sequence in* $(0, 2\alpha)$, $\{s_n\}$ *is a sequence in* $(0, 2\beta)$*, and* $\{\alpha_n\}$ *is a sequence in* $(0, 1)$*. Assume that the following restrictions are satisfied*

(a) $0 \le \alpha_n \le a < 1$;
(b) $0 < r \le r_n \le r' < 2\alpha$;
(c) $0 < s \le s_n \le s' < 2\beta$,

where a, r, r', s, *and* s' *are real constants. Then the sequence* $\{x_n\}$ *converges strongly to* $\operatorname{Proj}_{\mathcal{F}} x_1$.

If $A = 0$, then Corollary 3.2 is reduced to the following.

Corollary 3.3. *Let* C *be a nonempty, closed, and convex subset of* H*. Let* $S : C \to C$ *be a nonexpansive mapping with a nonempty fixed point set, and* $B : C \to H$ *be a* β*-inverse-strongly monotone mapping. Let* $M : H \rightrightarrows H$ *be a maximal monotone operator such that* $D(M) \subset C$*. Assume that* $\mathcal{F} := F(S) \cap (B+M)^{-1}(0)$ *is not empty. Let* $\{x_n\}$ *be a sequence generated by the following iterative process:*

$$\begin{cases} x_1 \in C, \\ C_1 = C, \\ y_n = \alpha_n x_n + (1-\alpha_n)SJ_{s_n}(x_n - s_nBx_n), \\ C_{n+1} = \{v \in C_n \ : \ \|y_n - v\| \ \le \ \|x_n - v\|\}, \\ x_{n+1} = \operatorname{Proj}_{C_{n+1}} x_1, \quad n \ge 0, \end{cases}$$

where $J_{s_n} = (I + s_nM)^{-1}$ $\{s_n\}$ *is a sequence in* $(0, 2\beta)$*, and* $\{\alpha_n\}$ *is a sequence in* $(0, 1)$*. Assume that the following restrictions are satisfied*

(a) $0 \le \alpha_n \le a < 1$;
(b) $0 < s \le s_n \le s' < 2\beta$,

where a, s, *and* s' *are real constants. Then the sequence* $\{x_n\}$ *converges strongly to* $\operatorname{Proj}_{\mathcal{F}} x_1$.

If $B = 0$, then Corollary 3.2 is reduced to the following.

Corollary 3.4. *Let* C *be a nonempty, closed, and convex subset of* H*. Let* $S : C \to C$ *be a nonexpansive mapping with a nonempty fixed point set,* $A : C \to H$ *a* α*-inverse-*

strongly monotone mapping. Let $M : H \rightrightarrows H$ *be a maximal monotone operator such that* $D(M) \subset C$. *Assume that* $\mathcal{F} := F(S) \cap M^{-1}(0) \cap VI(C, A)$ *is not empty. Let* $\{x_n\}$ *be a sequence generated by the following iterative process:*

$$\begin{cases} x_1 \in C, \\ C_1 = C, \\ z_n = \operatorname{Proj}_C(J_{s_n}x_n - r_nAJ_{s_n}x_n)), \\ y_n = \alpha_n x_n + (1-\alpha_n)Sz_n, \\ C_{n+1} = \{v \in C_n : ||y_n - v|| \le ||x_n - v||\}, \\ x_{n+1} = \operatorname{Proj}_{C_{n+1}}x_1, \quad n \ge 0, \end{cases}$$

where $J_{s_n} = (I + s_nM)^{-1}$,$\{r_n\}$ *is a sequence in* $(0, 2\alpha)$, $\{s_n\}$ *is a sequence in* $(0, +\infty)$, *and* $\{\alpha_n\}$ *is a sequence in* $(0, 1)$. *Assume that the following restrictions are satisfied*

(a) $0 \le \alpha_n \le a < 1$;
(b) $0 < r \le r_n \le r' < 2\alpha$;
(c) $0 < s \le s_n < \infty$,

where a, r, r', *and* s *are real constants. Then the sequence* $\{x_n\}$ *converges strongly to* $\operatorname{Proj}_{\mathcal{F}}x_1$.

Let $f : H \to (-\infty, +\infty]$ be a proper convex lower semicontinuous function. Then the subdifferential ∂ of f is defined as follows

$$\partial f(x) = \{y \in H : f(z) \ge f(x) + \langle z - x, y\rangle, \quad z \in H\}, \quad \forall x \in H.$$

From Rockafellar [50], we know that ∂f is maximal monotone. It is not hard to verify that $0 \in \partial f(x)$ if and only if $f(x) = \min_{y \in H} f(y)$.

Let I_C be the indicator function of C, i.e.,

$$I_C(x) = \begin{cases} 0, & x \in C, \\ +\infty, & x \notin C. \end{cases}$$

Since I_C is a proper lower semicontinuous convex function on H, we see that the subdifferential ∂I_C of I_C is a maximal monotone operator. It is clearly that $J_s x = \operatorname{Proj}_C x$, $\forall x \in H$. Notice that $(B + \partial I_C)^{-1}(0) = VI(C, B)$. Indeed,

$$\begin{aligned} x \in (B + \partial I_C)^{-1}(0) &\Leftrightarrow 0 \in Bx + \partial I_C x \\ &\Leftrightarrow -B_X \in \partial I_C x \\ &\Leftrightarrow \langle Bx, y - x\rangle \ge 0 \\ &\Leftrightarrow x \in VI(C, B). \end{aligned}$$

In view of Theorem 3.1, we have the following.

Corollary 3.5. *Let* C *be a nonempty, closed, and convex subset of* H. *Let* $S : C \to C$ *be a*α κ *-strict pseudocontraction with a nonempty fixed point set,* $A : C \to H$ *be an* α*-inverse-strongly monotone mapping, and* $B : C \to H$ *be a* β*-inverse-strongly monotone mapping. Assume hat* $\mathcal{F} := F(S) \cap VI(C, B) \cap VI(C, A)$ *is not empty. Let* $\{x_n\}$ *be a sequence generated by he following iterative process:*

$$\begin{cases} x_1 \in C, \\ C_1 = C, \\ z_n = \mathrm{Proj}_C(\mathrm{Proj}_C(x_n - s_n B x_n) - r_n A \mathrm{Proj}_C(x_n - s_n B x_n)), \\ y_n = \alpha_n x_n + (1-\alpha_n)(\beta_n z_n + (1-\beta_n) S z_n), \\ C_{n+1} = \{v \in C_n : \| y_n - v \| \le \| x_n - v \|\}, \\ x_{n+1} = \mathrm{Proj}_{C_{n+1}} x_1, \quad n \ge 0, \end{cases}$$

where $\{r_n\}$ *is a sequence in* $(0, 2\alpha)$, $\{s_n\}$ *is a sequence in* $(0, 2\beta)$, *and* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are sequences in* $(0, 1)$. *Assume that the following restrictions are satisfied*

(a) $0 \le \alpha_n \le a < 1$, $\kappa \le \beta_n \le b < 1$;
(b) $0 < r \le r_n \le r' < 2\alpha$;
(c) $0 < s \le s_n \le s' < 2\beta$,

where a, b, r, r', s, *and* s' *are real constants. Then the sequence* $\{x_n\}$ *converges strongly to* $\mathrm{Proj}_{\mathcal{F}} x_1$.

4. Applications

Let F be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ denotes the set of real numbers. Recall the following equilibrium problem in the terminology of Blum and Oettli [51] (see also Fan [52]).

$$\text{Find } x \in C \text{ such that } F(x, y) \ge 0, \quad \forall y \in C. \tag{4.1}$$

To study the equilibrium problem (4.1), we may assume that F satisfies the following conditions:

(A1) $F(x, x) = 0$ for all $x \in C$;
(A2) F is monotone, i.e., $F(x, y) + F(y, x) = 0$ for all $x, y \in C$;
(A3) for each $x, y, z \in C$,

$$\limsup_{t \downarrow 0} F(tz + (1-t)x, y) \le F(x, y);$$

(A4) for each $x \in C, y \mapsto F(x,y)$ is convex and lower semi-continuous.

Putting $F(x, y) = \langle Ax, y - x \rangle$ for every $x, y \in C$, we see that the equilibrium problem (4.1) is reduced to the variational inequality (2.1).

The following lemma can be found in [51,53].

Lemma 4.1. *Let* C *be a nonempty, closed, and convex subset of* H *and* $F: C \times C \to \mathbb{R}$ *a bifunction satisfying* (A1)-(A4). *Then, for any* $s > 0$ *and* $x \in H$, *there exists* $z \in C$ *such that*

$$F(z, y) + \frac{1}{s}\langle y - z, z - x \rangle \ge 0, \quad \forall y \in C.$$

Further, define

$$T_s x = \left\{ z \in C : F(z, y) + \frac{1}{s}\langle y - z, z - x \rangle \ge 0, \quad \forall y \in C \right\} \tag{4.2}$$

for all $s > 0$ *and* $x \in H$. *Then, the following hold:*

(a) T_s *is single-valued;*

(b) T_s *is firmly nonexpansive; that is,*

$$\left\|T_s x - T_s y\right\|^2 \leq \langle T_s x - T_s y,\ x - y\rangle, \quad \forall x,\ y \in H;$$

(c) $F(T_s) = EP(F)$;
(d) $EP(F)$ *is closed and convex.*

Lemma 4.2 [8]. *Let C be a nonempty, closed, and convex subset of H, F a bifunction from C×C to* $\mathbb{R}$ *which satisfies* (A1)-(A4), *and* A_F *a multivalued mapping of H into itself defined by*

$$A_F x = \begin{cases} \{z \in H\ :\ F(x,\ y) \geq \langle y - x,\ z\rangle,\ \forall y \in C\}, & x \in C, \\ \emptyset, & x \notin C. \end{cases} \tag{4.3}$$

Then A_F *is a maximal monotone operator with the domain* $D(A_F) \subset C$, $EP(F) = A_F^{-1}(0)$, *where FP(F) stands for the solution set of (4.1), and*

$$T_s x = (I + sA_F)^{-1} x, \quad \forall x \in H,\ r > 0,$$

where T_s *is defined as in (4.2).*

In this section, we consider the problem of approximating a solution of the equilibrium problem.

Theorem 4.3. *Let C be a nonempty, closed, and convex subset of H. Let* $S : C \to C$ *be a* κ*-strict pseudocontraction with a nonempty fixed point set, and* $F{:}C{\times}C{\to}\mathbb{R}$ *a bifunction satisfying* (A1)-(A4). *Assume that* $\mathcal{F} := F(S) \cap EP(F)$ *is not empty. Let* $\{x_n\}$ *be a sequence generated by the following iterative process:*

$$\begin{cases} x_1 \in C, \\ C_1 = C, \\ y_n = \alpha_n x_n + (1 - \alpha_n)\left(\beta_n (I + s_n A_F)^{-1} x_n + (1 - \beta_n) S (I + s_n A_F)^{-1} x_n\right), \\ C_{n+1} = \{v \in C_n\ :\ \|\ y_n - v\ \| \leq \|\ x_n - v\ \|\}, \\ x_{n+1} = \ \mathrm{Proj}_{C_{n+1}} x_1, \quad n \geq 0, \end{cases}$$

where A_F *is defined by (4.3),* $\{s_n\}$ *is a positive sequence, and* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are sequences in* (0, 1). *Assume that the following restrictions are satisfied*

(a) $0 \leq \alpha_n \leq a < 1$, $\kappa \leq \beta_n \leq b < 1$;
(b) $0 < s \leq s_n \leq s' < \infty$,

where a, b, s, and s' are real constants. Then the sequence $\{x_n\}$ *converges strongly to* $\mathrm{Proj}_{\mathcal{F}} x_1$.

Proof. Putting $A = B = 0$, we immediately conclude from Lemmas 4.1 and 4.2 the desired conclusion.

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Authors' contributions

CW designed and performed all the steps of proof in this research and also wrote the paper. AL participated in the design of the study. All authors read and approved the final manuscript.

Competing interests

The authors declare that they have no competing interests.

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