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Approximating fixed points of amenable semigroup and infinite family of nonexpansive mappings and solving systems of variational inequalities and systems of equilibrium problems

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Abstract
We introduce an iterative scheme for finding a common element of the set of solutions for systems of equilibrium problems and systems of variational inequalities and the set of common fixed points for an infinite family and left amenable semigroup of nonexpansive mappings in Hilbert spaces. The results presented in this paper mainly extend and improved some well-known results in the literature.
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1. Introduction

Let H be a real Hilbert space and let C be a nonempty closed convex subset of H.

Let A: $C \to H$ be a nonlinear mapping. The classical variational inequality problem is to fined $x \in C$ such that

$$\langle Ax, y - x\rangle \geq 0, \quad \forall y \in C. \tag{1}$$

The set of solution of (1) is denoted by VI(C, A), i.e.,

$$\mathrm{VI}(C,\ A) = \{x \in C\ :\ \langle Ax,\ y - x\rangle \geq 0, \quad \forall y \in C\}. \tag{2}$$

Recall that the following definitions:

(1) A is called monotone if

$$\langle Ax - Ay,\ x - y\rangle \geq 0, \quad \forall x, y \in C.$$

(2) A is called α-strongly monotone if there exists a positive constant α such that

$$\langle Ax - Ay,\ x - y\rangle \geq \alpha \|x - y\|^2, \quad \forall x, y \in C.$$

(3) A is called μ-Lipschitzian if there exist a positive constant μ such that

$$\|Ax - Ay\| \leq \mu \|x - y\|, \quad \forall x,\ y \in C.$$

(4) A is called α-inverse strongly monotone, if there exists a positive real number $\alpha > 0$

such that

$$\langle Ax - Ay,\ x - y\rangle \geq \alpha \|Ax - Ay\|^2, \quad \forall x,\ y \in C.$$

It is obvious that any α-inverse strongly monotone mapping B is $\frac{1}{\alpha}$-Lipschitzian.

(5) A mapping $T : C \to C$ is called nonexpansive if $\| Tx - Ty \| \leq \| x - y \|$ for all $x, y \in C$. Next, we denote by Fix(T) the set of fixed point of T.
(6) A mapping $f : C \to C$ is said to be contraction if there exists a coefficient $\alpha \in (0, 1)$ such that

$$\|f(x) - f(y)\| \leq \alpha \|x - y\|, \quad \forall x,\ y \in C.$$

(7) A set-valued mapping $U : H \to 2^H$ is called monotone if for all $x, y \in H, f \in Ux$ and $g \in Uy$ imply $\langle x - y, f - g\rangle \geq 0$.
(8) A monotone mapping $U : H \to 2^H$ is maximal if the graph $G(U)$ of U is not properly contained in the graph of any other monotone mapping.

It is known that a monotone mapping U is maximal if and only if for $(x, f) \in H \times H$, $\langle x - y, f - g\rangle \leq 0$ for every $(y, g) \in G(U)$ implies that $f \in Ux$. Let B be a monotone mapping of C into H and let $N_C x$ be the normal cone to C at $x \in C$, that is, $N_C x = \{y \in H : \langle x - z, y\rangle \leq 0, \forall z \in C\}$ and define

$$Ux = \begin{cases} Bx + N_C x, & x \in C, \\ \emptyset & x \notin C. \end{cases}$$

Then U is the maximal monotone and $0 \in Ux$ if and only if $x \in \mathrm{VI}(C, B)$; see [1].

Let F be a bi-function of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The equilibrium problem for $F : C \times C \to \mathbb{R}$ is to determine its equilibrium points, i.e the set

$$\mathrm{EP}(F) = \{x \in C : F(x,\ y) \geq 0,\ \forall y \in C\}.$$

Let $\mathcal{J} = \{F_i\}_{i\in I}$ be a family of bi-functions from $C \times C$ into $\mathbb{R}$. The system of equilibrium problems for $\mathcal{J} = \{F_i\}_{i\in I}$ is to determine common equilibrium points for $\mathcal{J} = \{F_i\}_{i\in I}$, i.e the set

$$\mathrm{EP}(\mathcal{J}) = \{x \in C : F_i(x,\ y) \geq 0,\ \forall y \in C,\ \forall i \in I\}. \tag{3}$$

Numerous problems in physics, optimization, and economics reduce into finding some element of EP(F). Some method have been proposed to solve the equilibrium problem; see, for instance [2-5]. The formulation (3), extend this formalism to systems of such problems, covering in particular various forms of feasibility problems [6,7].

Given any $r > 0$ the operator $J_r^F : H \to C$ defined by

$$J_r^F(x) = \{z \in C \ : \ F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \ \forall y \in C\},$$

is called the resolvent of F, see [3]. It is shown [3] that under suitable hypotheses on F (to be stated precisely in Sect. 2), $J_r^F : H \to C$ is single- valued and firmly nonexpansive and

satisfies

$$\mathrm{Fix}(J_r^F) = \mathrm{EP}(F), \quad \forall r > 0.$$

Using this result, in 2007, Yao et al. [8], proposed the following explicit scheme with respect to W-mappings for an infinite family of nonexpansive mappings:

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n W_n J_{r_n}^F x_n \tag{4}$$

They proved that if the sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{r_n\}$ of parameters satisfy appropriate conditions, then, the sequences $\{x_n\}$ and $\{J_{r_n}^F x_n\}$ both converge strongly to the unique $x^* \in \cap_{i=1}^{\infty} \mathrm{Fix}(T_i) \cap \mathrm{EP}(F)$, where $x^* \in P_{\cap_{i=1}^{\infty}\mathrm{Fix}(T_i)\cap \mathrm{EP}(F)} f(x^*)$. Their results extend and improve the corresponding results announced by Combettes and Hirstoaga [3] and Takahashi and Takahashi [5].

Very recently, Jitpeera et al. [9], introduced the iterative scheme based on viscosity and Cesàro mean

$$\begin{cases} \phi(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, & \forall y \in C, \\ y_n = \delta_n u_n + (1 - \delta_n) P_C(u_n - \lambda_n B u_n), \\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1 - \beta_n) I - \alpha_n A) \frac{1}{n+1} \sum_{i=0}^{n} T^i y_n, & \forall n \geq 0, \end{cases}$$

where $B : C \to H$ is β-inverse strongly monotone, ϕ: $C \to \mathbb{R} \cup \{\infty\}$ is a proper lower semi-continuous and convex function, $T^i : C \to C$ is a nonexpansive mapping for all $i = 1, 2, ..., n$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\delta_n\} \subset (0, 1)$, $\{\lambda_n\} \subset (0, 2\beta)$ and $\{r_n\} \subset (0, \infty)$ satisfy the following conditions

(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $\lim_{n\to\infty} \delta_n = 0$

(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

(iv) $\{\lambda_n\} \subset [a, b] \subset (0, 2\beta)$ and $\liminf_{n\to\infty} | \lambda_{n+1} - \lambda_n |= 0$,

(v) $\liminf_{n\to\infty} r_n > 0$ and $\liminf_{n\to\infty} | r_{n+1} - r_n |= 0$.

They show that if $\theta = \cap_{i=1}^{n} \mathrm{Fix}(T^i) \cap \mathrm{VI}(C, B) \cap \mathrm{MEP}(\phi, \varphi)$ is nonempty, then the sequence $\{x_n\}$ converges strongly to the $z = P_\theta(I - A + \gamma f)z$ which is the unique solution of the variational inequality

$$\langle(\gamma f - A)z, x - z\rangle \leq 0 \quad \forall y \in \theta.$$

In this paper, motivated and inspired by Yao et al. [8,10-15], Lau et al. [16], Jitpeera et al. [9], Kangtunyakarn [17] and Kim [18], Atsushiba and Takahashi [19], Saeidi [20], Piri [21-23] and Piri and Badali [24], we introduce the following iterative scheme for finding a common element of the set of solutions for a system of equilibrium problems

$\mathcal{J} = \{F_k : k = 1, 2, 3, \ldots, M\}$ for a family $\mathcal{J} = \{F_k : k = 1, 2, 3, \ldots, M\}$ of equilibrium bifunctions, systems of variational inequalities, the set of common fixed points for an infinite family $\psi = \{T_i, i = 1, 2, \ldots\}$ of nonexpansive mappings and a left amenable semigroup $\phi = \{T_t : t \in S\}$ of nonexpansive mappings, with respect to W-mappings and a left regular sequence $\{\mu_n\}$ of means defined on an appropriate space of bounded real-valued functions of the semigroup

$$\begin{cases} z_n = J^{F_M}_{r_M,n} \ldots J^{F_2}_{r_2,n} J^{F_1}_{r_1,n} x_n, \\ y_n = \eta_n P_C(z_n - \zeta_n A z_n) + (1 - \eta_n) P_C(z_n - \delta_n B z_n), \\ x_{n+1} = \alpha_n f(T_{\mu_n} W_n y_n) + \beta_n x_n + \gamma_n T_{\mu_n} W_n y_n, \qquad n \geq 1, \end{cases} \tag{5}$$

where $A: C \to H$ be β-inverse monotone map and $B : C \to H$ be δ-inverse monotone map. We prove that under mild assumptions on parameters like that in Yao et al. [8], the sequences $\{x_n\}$ and $\{J^{F_k}_{r_{k,n}} x_n\}_{k=1}^{M}$ converge strongly to $x^* \in \mathcal{F} = \cap_{i=1}^{\infty} \mathrm{Fix}(T_i) \cap \mathrm{Fix}(\varphi) \cap \mathrm{EP}(\mathcal{J}) \cap \mathrm{VI}(C, A) \cap \mathrm{VI}(C, B)$, where $x^* = P_{\mathcal{F}} f(x^*)$.

Compared to the similar works, our results have the merit of studying the solutions of systems of equilibrium problems, systems of variational inequalities and fixed point problems of amenable semigroup of nonexpansive mappings. Consequence for nonnegative integer numbers is also presented.

2. Preliminaries

Let S be a semigroup and let $B(S)$ be the space of all bounded real valued functions defined on S with supremum norm. For $s \in S$ and $f \in B(S)$, we define elements $l_s f$ and $r_s f$ in $B(S)$ by

$$(l_s f)(t) = f(st), \quad (r_s f)(t) = f(ts), \quad \forall t \in S.$$

Let X be a subspace of $B(S)$ containing 1 and let X^* be its topological dual. An element μ of X^* is said to be a mean on X if $\| \mu \| = \mu(1) = 1$. We often write $\mu_t(f(t))$ instead of $\mu(f)$ for $\mu \in X^*$ and $f \in X$. Let X be left invariant (respectively right invariant), i.e., $l_s(X) \subset X$ (respectively $r_s(X) \subset X$) for each $s \in S$. A mean μ on X is said to be left invariant (respectively right invariant) if $\mu(l_s f) = \mu(f)$ (respectively $\mu(r_s f) = \mu(f)$) for each $s \in S$ and $f \in X$. X is said to be left (respectively right) amenable if X has a left (respectively right) invariant mean. X is amenable if X is both left and right amenable. As is well known, $B(S)$ is amenable when S is a commutative semigroup, see [25]. A net $\{\mu_\alpha\}$ of means on X is said to be strongly left regular if

$$\lim_{\alpha} \left\| l_s^* \mu_\alpha - \mu_\alpha \right\| = 0,$$

for each $s \in S$, where l_s^* is the adjoint operator of l_s.

Let S be a semigroup and let C be a nonempty closed and convex subset of a reflexive Banach space E. A family $\phi = \{T_t : t \in S\}$ of mapping from C into itself is said to be a nonexpansive semigroup on C if T_t is nonexpansive and $T_{ts} = T_t T_s$ for each $t, s \in S$. By $\mathrm{Fix}(\phi)$ we denote the set of common fixed points of ϕ, i.e.

$$\mathrm{Fix}(\varphi) = \bigcap_{t \in S} \{x \in C : T_t x = x\}.$$

Lemma 2.1. [25]*Let S be a semigroup and C be a nonempty closed convex subset of a reflexive Banach space E. Let $\phi = \{T_t : t \in S\}$ be a nonexpansive semigroup on H such that $\{T_t x : t \in S\}$ is bounded for some $x \in C$, let X be a subspace of $B(S)$ such that $1 \in X$ and the mapping $t \to \langle T_t x, y^* \rangle$ is an element of X for each $x \in C$ and $y^* \in E^*$, and μ is a mean on X. If we write $T_\mu x$ instead of $\int T_t x d\mu(t)$, then the followings hold.*

(i) *T_μ is nonexpansive mapping from C into C.*

(ii) *$T_\mu x = x$ for each $x \in Fix(\phi)$.*

(iii) *$T_\mu x \in \overline{co}\{T_t x : t \in S\}$ for each $x \in C$.*

Let C be a nonempty subset of a Hilbert space H and $T : C \to H$ a mapping. Then T is said to be demiclosed at $v \in H$ if, for any sequence $\{x_n\}$ in C, the following implication holds:

$$x_n \rightharpoonup u \in C, \quad Tx_n \to v \quad \text{imply} \quad Tu = v,$$

where $\to$ (respectively $\rightharpoonup$) denotes strong (respectively weak) convergence.

Lemma 2.2. [26]*Let C be a nonempty closed convex subset of a Hilbert space H and suppose that $T : C \to H$ is nonexpansive. then, the mapping $I - T$ is demiclosed at zero.*

Lemma 2.3. [27]*For a given $x \in H$, $y \in C$,*

$$y = P_C x \Leftrightarrow \langle y - x,\ z - y \rangle \geq 0, \quad \forall z \in C.$$

It is well known that P_C is a firmly nonexpansive mapping of H onto C and satisfies

$$\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y \rangle, \quad \forall x, y \in H. \tag{6}$$

Moreover, P_C is characterized by the following properties: $P_C x \in C$ and for all $x \in H$, $y \in C$,

$$\langle x - P_C x, y - P_C x \rangle \leq 0. \tag{7}$$

It is easy to see that (7) is equivalent to the following inequality

$$\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2. \tag{8}$$

Using Lemma 2.3, one can see that the variational inequality (1) is equivalent to a fixed point problem. It is easy to see that the following is true:

$$u \in \mathrm{VI}(C, A) \Leftrightarrow u = P_C(u - \lambda Au), \quad \lambda > 0. \tag{9}$$

Lemma 2.4. [28]*Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space E and let $\{\alpha_n\}$ be a sequence in [0, 1] with $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$. Suppose $x_{n+1} = \alpha_n x_n$ $+(1-\alpha_n)y_n$ for all integers $n \geq 0$ and*

$$\limsup_{n\to\infty} (\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 1.$$

Then, $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

Let $F : C \times C \to \mathbb{R}$ be a bi-function. Given any $r > 0$, the operator $J_r^F : H \to C$ defined by

$$J_r^F x = \left\{ z \in C \ : \ F(z,\ y) + \frac{1}{r} \langle y - z,\ z - x \rangle \geq 0,\ \forall y \in C \right\}$$

is called the resolvent of F, see [3]. The equilibrium problem for F is to determine its equilibrium points, i.e., the set

$$\mathrm{EP}(F) = \{x \in C : F(x, y) \geq 0, \forall y \in C\}.$$

Let $\mathcal{J} = \{F_i\}_{i \in I}$ be a family of bi-functions from $C \times C$ into $\mathbb{R}$. The system of equilibrium problems for $\mathcal{J}$ is to determine common equilibrium points for $\mathcal{J} = \{F_i\}_{i \in I}$. i.e, the set

$$\mathrm{EP}(\mathcal{J}) = \{x \in C : F_i(x, y) \geq 0, \forall y \in C, \forall i \in I\}.$$

Lemma 2.5. [3]*Let C be a nonempty closed convex subset of H and $F : C \times C \to \mathbb{R}$ satisfy*

(A_1) *$F(x, x) = 0$ for all $x \in C$,*

(A_2) *F is monotone, i.e, $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$,*

(A_3) *for all $x, y, z \in C$, $\lim_{t \to 0} F(tz + (1 - t)x, y) \leq F(x, y)$,*

(A_4) *for all $x \in C$, $y \to F(x, y)$ is convex and lower semi-continuous.*

Given $r > 0$, define the operator $J_r^F : H \to C$, the resolvent of F, by

$$J_r^F(x) = \{z \in C \ : \ F(z, y) + \frac{1}{r}\langle y - z, \ z - x\rangle \geq 0, \ \forall y \in C\}.$$

Then,

(1) *J_r^F is single valued,*

(2) *J_r^F is firmly nonexpansive, i.e, $\|J_r^F x - J_r^F y\|^2 \leq \langle J_r^F x - J_r^F y, x - y\rangle$ for all $x, y \in H$,*

(3) $\mathrm{Fix}(J_r^F) = \mathrm{EP}(F)$,

(4) *$\mathrm{EP}(F)$ is closed and convex.*

Let $T_1, T_2, \ldots$ be an infinite family of mappings of C into itself and let $\lambda_1, \lambda_2, \ldots$ be a real numbers such that $0 \leq \lambda_i < 1$ for every $i \in \mathbb{N}$. For any $n \in \mathbb{N}$, define a mapping W_n of C into C as follows:

$$\begin{aligned}
&U_{n,n+1} = I,\\
&U_{n,n} = \lambda_n T_n U_{n,n+1} + (1 - \lambda_n)I,\\
&U_{n,n-1} = \lambda_{n-1} T_{n-1} U_{n,n} + (1 - \lambda_{n-1})I,\\
&\vdots\\
&U_{n,k} = \lambda_k T_k U_{n,k+1} + (1 - \lambda_k)I,\\
&U_{n,k-1} = \lambda_{k-1} T_{k-1} U_{n,k} + (1 - \lambda_{k-1})I,\\
&\vdots\\
&U_{n,2} = \lambda_2 T_2 U_{n,3} + (1 - \lambda_2)I,\\
&W_n = U_{n,1} = \lambda_1 T_1 U_{n,2} + (1 - \lambda_1)I.
\end{aligned} \tag{10}$$

Such a mapping W_n is called the W-mapping generated by $T_1, T_2, \ldots, T_n$ and $\lambda_1, \lambda_2, \ldots, \lambda_n$.

Lemma 2.6. [29]*Let C be a nonempty closed convex subset of a Hilbert space H, $\{T_i : C \to C\}$ be an infinite family of nonexpansive mappings with $\cap_{i=1}^{\infty} \mathrm{Fix}(T_i) \neq \emptyset$, $\{\lambda_i\}$ be a real sequence such that $0 < \lambda_i \leq b < 1$, $\forall i \geq 1$. Then*

(1) *W_n is nonexpansive and $\mathrm{Fix}(W_n) = \cap_{i=1}^{n} \mathrm{Fix}(T_i)$ for each $n \geq 1$,*

(2) *for each* $x \in C$ *and for each positive integer j, the limit* $\lim_{n\to\infty} U_{n,j}$ *exists.*
(3) *The mapping* $W : C \to C$ *defined by*

$$Wx := \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1} x, \quad \forall x \in C,$$

is a nonexpansive mapping satisfying $Fix(W) = \cap_{i=1}^{\infty} Fix(T_i)$ *and it is called the W-mapping generated by* $T_1, T_2, \ldots$ *and* $\lambda_1, \lambda_2, \ldots$.

Lemma 2.7. [30]*Let C be a nonempty closed convex subset of a Hilbert space H,* $\{T_i : C \to C\}$ *be a countable family of nonexpansive mappings with* $\cap_{i=1}^{\infty} Fix(T_i) \neq \emptyset$,$\{\lambda_i\}$ *be a real sequence such that* $0 < \lambda_i \le b < 1$, $\forall i \ge 1$. *If D is any bounded subset of C, then*

$$\lim_{n\to\infty} \sup_{x\in D} \|Wx - W_n x\| = 0.$$

Lemma 2.8. [31]*Let* $\{a_n\}$ *be a sequence of nonnegative real numbers such that*

$$a_{n+1} \le (1 - b_n)a_n + b_n c_n, \quad n \ge 0,$$

where $\{b_n\}$ *and* $\{c_n\}$ *are sequences of real numbers satisfying the following conditions:*

(i) $\{b_n\} \subset [0, 1]$, $\sum_{n=0}^{\infty} b_n = \infty$,

(ii) *either* $\limsup_{n\to\infty} c_n \le 0$ *or* $\sum_{n=0}^{\infty} |b_n c_n| < \infty$.

Then, $\lim_{n\to\infty} a_n = 0$.

Lemma 2.9. [32]*Let* $(E, \langle ., .\rangle)$ *be an inner product space. Then for all* $x, y, z \in E$ *and* $\alpha, \beta, \gamma, \in [0, 1]$ *such that* $\alpha + \beta + \gamma = 1$, *we have*

$$\|\alpha x + \beta y + \gamma z\|^2 = \alpha\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 - \alpha\beta\|x - y\|^2 - \alpha\gamma\|x - z\|^2 - \beta\gamma\|y - z\|^2.$$

Notation Throughout the rest of this paper the open ball of radius r centered at 0 is denoted by B_r. For a subset A of H we denote by $\overline{co}A$ the closed convex hull of A. For $\varepsilon > 0$ and a mapping $T : D \to H$, we let $F_\epsilon(T; D)$ be the set of ϵ-approximate fixed points of T, i.e., $F_\epsilon(T; D) = \{x \in D : \| x - Tx \| \le \epsilon\}$. Weak convergence is denoted by $\rightharpoonup$ and strong convergence is denoted by $\to$.

3. Strong convergence

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H, A:* $C \to H$ *a β-inverse strongly monotone,* $B : C \to H$ *a γ-inverse strongly monotone, S a semigroup and* $\phi = \{T_t : t \in S\}$ *be a nonexpansive semigroup from C into C such that* $\text{Fix}(\varphi) = \cap_{t\in S}\text{Fix}(T_t) \neq \emptyset$. *Let X be a left invariant subspace of B(S) such that* $1 \in X$, *and the function* $t \to \langle T_t x, y\rangle$ *is an element of X for each* $x \in C$ *and* $y \in H$, $\{\mu_n\}$ *a left regular sequence of means on X such that* $\lim_{n\to\infty} \|\mu_{n+1} - \mu_n\| = 0$. *Let* $\mathcal{J} = \{F_k : k = 1, 2, \ldots, M\}$ *be a finite family of bi-functions from* $C \times C$ *into* $\mathbb{R}$ *which satisfy* (A_1)-(A_4) *and* $\{T_i\}_{i=1}^{\infty}$ *an infinite family of nonexpansive mappings of C into C such that* $T_i(Fix(\varphi) \cap \text{EP}(\mathcal{J})) \subset \text{Fix}(\varphi)$ *for each* $i \in \mathbb{N}$ *and* $\mathcal{F} = \cap_{i=1}^{\infty}\text{Fix}(T_i) \cap \text{Fix}(\varphi) \cap \text{EP}(\mathcal{J}) \cap \text{VI}(C, A) \cap \text{VI}(C, B) \neq \emptyset$. *Let* $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ *and* $\{\eta_n\}$ *be a sequences in* (0, 1). *Let* $\{\zeta_n\}$ *a sequence in* $(0, 2\beta)$, $\{\delta_n\}$ *a sequence in* $(0, 2\gamma)$, $\{r_{k,n}\}_{k=1}^{M}$ *be sequences in* $(0, \infty)$ *and* $\{\lambda_n\}$ *a sequence of real numbers such that* $0 < \lambda_n \le b < 1$. *Assume that,*

(B_1) $\lim_{n\to\infty} \eta_n = \eta \in (0, 1)$, $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(B_2) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$,

(B_3) $\alpha_n + \beta_n + \gamma_n = 1$,

(B_4) $\lim_{n\to\infty} | \zeta_{n+1} - \zeta_n | = \lim_{n\to\infty} | \delta_{n+1} - \delta_n | = 0$,

(B_5) $\liminf_{n\to\infty} r_{k,n} > 0$ *and* $\lim_{n\to\infty} (r_{k,n+1} - r_{k,n}) = 0$ *for* $k \in \{1, 2, \cdots, M\}$.

Let f be a contraction of C into itself with coefficient $\alpha \in (0, 1)$ *and given* $x_1 \in C$ *arbitrarily. If the sequences* $\{x_n\}$, $\{y_n\}$ *and* $\{z_n\}$ *are generated iteratively by* $x_1 \in C$ *and*

$$\begin{cases} z_n = J^{F_M}_{r_M,n} \dots J^{F_2}_{r_2,n} J^{F_1}_{r_1,n} x_n, \\ y_n = \eta_n P_C(z_n - \zeta_n A z_n) + (1-\eta_n) P_C(z_n - \delta_n B z_n), \\ x_{n+1} = \alpha_n f(T_{\mu_n} W_n y_n) + \beta_n x_n + \gamma_n T_{\mu_n} W_n y_n, \qquad n \ge 1, \end{cases} \tag{11}$$

then, the sequences $\{x_n\}$, $\{y_n\}$ *and* $\{J^{F_k}_{r_{k,n}} x_n\}_{k=1}^{M}$ *converge strongly to* $x^* \in \mathcal{F}$, *which is the unique solution of the system of variational inequalities:*

$$\begin{cases} \langle f(x^*) - x^*,\ x^* - y\rangle \ge 0, \quad \forall y \in \mathcal{F}, \\ \langle Bx^*,\ y - x^*\rangle \ge 0 \quad \forall y \in C, \\ \langle Ax^*,\ y - x^*\rangle \ge 0 \quad \forall y \in C. \end{cases}$$

Proof. Since A is a β-inverse strongly monotone map, for any $x, y \in C$, we have

$$\begin{aligned} &\|(I-\zeta_n A)x - (I-\zeta_n A)y\|^2 \\ &\quad = \|(x-y) - \zeta_n(Ax - Ay)\|^2 \\ &\quad = \||x-y\|^2 - 2\zeta_n\langle x-y,\ Ax-Ay\rangle + \zeta_n^2\|Ax-Ay\|^2 \\ &\quad \le \|x-y\|^2 - 2\zeta_n\beta\|Ax-Ay\|^2 + \zeta_n^2\|Ax-Ay\|^2 \\ &\quad = \|x-y\|^2 + \zeta_n(\zeta_n - 2\beta)\|Ax-Ay\|^2 \\ &\quad \le \|x-y\|^2 \end{aligned}$$

It follows that

$$\|(I-\zeta_n A)x - (I-\zeta_n A)y\| \le \|x-y\|. \tag{12}$$

Since B is a β-inverse strongly monotone map, repeating the same argument as above, we can deduce that

$$\|(I-\delta_n B)x - (I-\delta_n B)y\| \le \|x-y\|. \tag{13}$$

Let $p \in \mathcal{F}$, in the context of the variational inequality problem the characterization of projection (9) implies that $p = P_C(p - \zeta_n Ap)$ and $p = P_C(p - \delta_n Bp)$. Using (12) and (13), we get

$$\begin{aligned} \|y_n - p\| &= \|\eta_n P_C(z_n - \zeta_n A z_n) + (1-\eta_n)P_C(z_n - \delta_n B z_n) - p\| \\ &= \|\eta_n[P_C(z_n - \zeta_n A z_n) - P_C(p - \zeta_n Ap)] \\ &\quad +(1-\eta_n)[P_C(z_n - \delta_n B z_n) - P_C(p - \delta_n Bp)]\| \\ &\le \eta_n \|P_C(z_n - \zeta_n A z_n) - P_C(p - \zeta_n Ap)\| \\ &\quad + (1-\eta_n)\|P_C(z_n - \delta_n B z_n) - P_C(p - \delta_n Bp)\| \\ &\le \eta_n\|z_n - p\| + (1-\eta_n)\|z_n - p\| = \|z_n - p\|. \end{aligned} \tag{14}$$

By taking $v_n = P_C(z_n - \zeta_n Az_n)$, $w_n = P_C(z_n - \delta_n Bz_n)$ and $\mathcal{J}_n^k = J_{r_{k,n}}^{F_k} \cdots J_{r_{2,n}}^{F_2} J_{r_{1,n}}^{F_1}$ for $k \in \{1, 2, ..., M\}$ and $\mathcal{J}_n^0 = I$ for all $n \in \mathbb{N}$, we shall equivalently write scheme (11) as follows:

$$\begin{aligned} z_n &= \mathcal{J}_n^M x_n, \\ y_n &= \eta_n v_n + (1 - \eta_n) w_n, \\ x_{n+1} &= \alpha_n f(T_{\mu_n} W_n y_n) + \beta_n x_n + \gamma_n T_{\mu_n} W_n y_n, \quad n \geq 1. \end{aligned}$$

We shall divide the proof into several steps.

Step 1. The sequence $\{x_n\}$ is bounded.

Proof of Step 1. Let $p \in \mathcal{F}$. Since for each $k \in \{1, 2, ..., M\}$, $J_{r_{k,n}}^{F_k}$ is nonexpansive we have

$$\left\| \mathcal{J}_n^k x_n - p \right\| = \left\| \mathcal{J}_n^k x_n - \mathcal{J}_n^k p \right\| \leq \left\| x_n - p \right\|, \quad \forall k \in \{1, 2, \ldots, M\}. \tag{15}$$

Thus, by Lemmas 2.1, 2.5 and (14), we have

$$\begin{aligned} &\left\| x_{n+1} - p \right\| \\ &\leq \alpha_n \left\| f(T_{\mu_n} W_n y_n) - p \right\| + \beta_n \left\| x_n - p \right\| + \gamma_n \left\| T_{\mu_n} W_n \mathcal{J}_n^M y_n - p \right\| \\ &\leq \alpha_n [\left\| f(T_{\mu_n} W_n y_n) - f(p) \right\| + \left\| f(p) - p \right\|] + \beta_n \left\| x_n - p \right\| + \gamma_n \left\| y_n - p \right\| \\ &\leq \alpha_n \alpha \left\| x_n - p \right\| + \alpha_n \left\| f(p) - p \right\| + (\beta_n + \gamma_n) \left\| x_n - p \right\| \\ &= [1 - \alpha_n (1 - \alpha)] \left\| x_n - p \right\| + \alpha_n \left\| f(p) - p \right\| \\ &\leq \max \left\{ \left\| x_n - p \right\|, \frac{1}{1 - \alpha} \left\| f(p) - p \right\| \right\}. \end{aligned}$$

By induction,

$$\left\| x_n - p \right\| \leq \max \left\{ \left\| x_1 - p \right\|, \frac{1}{1 - \alpha} \left\| f(p) - p \right\| \right\}, \quad n \geq 1.$$

Step 2. Let $\{u_n\}$ be a bounded sequence in H. Then

$$\lim_{n \to \infty} \left\| \mathcal{J}_{n+1}^k u_n - \mathcal{J}_n^k u_n \right\| = 0, \tag{16}$$

for every $k \in \{1, 2, ..., M\}$.

Proof of Step 2. This assertion is proved in [27, Step 2].

Step 3. Let $\{u_n\}$ be a bounded sequence in H. Then

$$\lim_{n \to \infty} \left\| W_{n+1} u_n - W_n u_n \right\| = 0 \text{ and } \lim_{n \to \infty} \left\| T_{\mu_{n+1}} u_n - T_{\mu_n} u_n \right\| = 0.$$

This assertion is proved in [21, Step 3].

Step 4. $\lim_{n \to \infty} \| x_{n+1} - x_n \| = 0$.

Proof of Step 4. Setting $x_{n+1} = \beta_n x_n + (1 - \beta_n) t_n$ for all $n \geq 1$, we have

$$\begin{aligned} &t_{n+1} - t_n \\ &= \frac{1}{1 - \beta_{n+1}} [x_{n+2} - \beta_{n+1} x_{n+1}] - \frac{1}{1 - \beta_n} [x_{n+1} - \beta_n x_n] \\ &= \frac{\alpha_{n+1}}{1 - \beta_{n+1}} [f(T_{\mu_{n+1}} W_{n+1} y_{n+1}) - f(T_{\mu_n} W_n y_n)] + \left(\frac{\alpha_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n}{1 - \beta_n} \right) f(T_{\mu_n} W_n y_n) \\ &\quad + \frac{\gamma_{n+1}}{1 - \beta_{n+1}} [T_{\mu_{n+1}} W_{n+1} y_{n+1} - T_{\mu_n} W_n y_n] \\ &\quad + \left[\frac{\gamma_{n+1}}{1 - \beta_{n+1}} - \frac{\gamma_n}{1 - \beta_n} \right] T_{\mu_n} W_n y_n. \end{aligned}$$

Therefore, we have

$$\begin{aligned}&\|t_{n+1}-t_n\|\\&\leq \frac{\alpha_{n+1}}{1-\beta_{n+1}}[\alpha\left\|T_{\mu_{n+1}}W_{n+1}y_{n+1}-T_{\mu_n}W_ny_n\right\|+\left\|T_{\mu_{n+1}}W_{n+1}y_{n+1}-T_{\mu_n}W_ny_n\right\|]\\&\quad+\left|\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n}\right|[\left\|f(T_{\mu_n}W_ny_n)\right\|+\left\|T_{\mu_n}W_ny_n\right\|]\\&\quad+\left\|T_{\mu_{n+1}}W_{n+1}y_{n+1}-T_{\mu_n}W_ny_n\right\|.\end{aligned}$$

On the other hand

$$\begin{aligned}&\left\|T_{\mu_{n+1}}W_{n+1}y_{n+1}-T_{\mu_n}W_ny_n\right\|\\&\leq \left\|T_{\mu_{n+1}}W_{n+1}y_{n+1}-T_{\mu_{n+1}}W_{n+1}y_n\right\|\\&\quad+\left\|T_{\mu_{n+1}}W_{n+1}y_n-T_{\mu_{n+1}}W_ny_n\right\|\\&\quad+\left\|T_{\mu_{n+1}}W_ny_n-T_{\mu_n}W_ny_n\right\|\\&\leq \left\|y_{n+1}-y_n\right\|+\left\|W_{n+1}y_n-W_ny_n\right\|+\left\|T_{\mu_{n+1}}W_ny_n-T_{\mu_n}W_ny_n\right\|\end{aligned}$$

Observing that $z_n = \mathcal{J}_n^M x_n$, $z_{n+1} = \mathcal{J}_{n+1}^M x_{n+1}$ and $\mathcal{J}_n^M x_n = J_{r_M,n}^{F_M}\mathcal{J}_n^{M-1}x_n$ we get

$$\frac{1}{r_{M,n}}\langle y-z_n,z_n-\mathcal{J}_n^{M-1}x_n\rangle+F_M(z_n,\ y)\geq 0,\quad \forall y\in C, \tag{17}$$

and

$$\frac{1}{r_{M,n+1}}\langle y-z_{n+1},z_{n+1}-\mathcal{J}_{n+1}^{M-1}x_{n+1}\rangle+F_M(z_{n+1},\ y)\geq 0,\quad \forall y\in C, \tag{18}$$

Take $y = z_{n+1}$ in (17) and $y = z_n$ in (18), by using (A_2), it follows that

$$\left\langle z_{n+1}-z_n,\ z_n-\mathcal{J}_n^{M-1}x_n-\frac{r_{M,n}}{r_{M,n+1}}(z_{n+1}-\mathcal{J}_{n+1}^{M-1}x_{n+1})\right\rangle\geq 0,$$

and hence

$$\begin{aligned}&\langle z_{n+1}-z_n,\ z_n-\mathcal{J}_n^{M-1}x_n-z_{n+1}+\mathcal{J}_{n+1}^{M-1}x_{n+1}\\&\qquad\qquad+\left(1-\frac{r_{M,n}}{r_{M,n+1}}\right)(z_{n+1}-\mathcal{J}_{n+1}^{M-1}x_{n+1})\Big\rangle\geq 0,\end{aligned}$$

Thus, we have

$$\begin{aligned}&\|z_{n+1}-z_n\|\\&\leq \left\|\mathcal{J}_{n+1}^{M-1}x_{n+1}-\mathcal{J}_n^{M-1}x_n\right\|+\left|1-\frac{r_{M,n}}{r_{M,n+1}}\right|\left\|z_{n+1}-\mathcal{J}_{n+1}^{M-1}x_{n+1}\right\|\\&\leq \left\|\mathcal{J}_{n+1}^{M-1}x_{n+1}-\mathcal{J}_{n+1}^{M-1}x_n\right\|+\left\|\mathcal{J}_{n+1}^{M-1}x_n-\mathcal{J}_n^{M-1}x_n\right\|\\&\quad+\left|1-\frac{r_{M,n}}{r_{M,n+1}}\right|\left\|z_{n+1}-\mathcal{J}_{n+1}^{M-1}x_{n+1}\right\|\\&\leq \|x_{n+1}-x_n\|+\left\|\mathcal{J}_{n+1}^{M-1}x_n-\mathcal{J}_n^{M-1}x_n\right\|\\&\quad+\left|1-\frac{r_{M,n}}{r_{M,n+1}}\right|\left\|z_{n+1}-\mathcal{J}_{n+1}^{M-1}x_{n+1}\right\|.\end{aligned}$$

Since $v_n = P_C(z_n - \zeta_n A z_n)$ and $w_n = P_C(z_n - \delta_n B z_n)$, it follows from the definition of $\{y_n\}$ that

$$
\begin{aligned}
&\left\|y_{n+1} - y_n\right\| \\
&= \left\|\eta_{n+1} v_{n+1} + (1-\eta_{n+1}) w_{n+1} - \eta_n v_n - (1-\eta_n) w_n\right\| \\
&= \left\|\eta_{n+1}(v_{n+1} - v_n) + (\eta_{n+1} - \eta_n) v_n + (1-\eta_{n+1}) w_{n+1}\right. \\
&\quad \left. -(1-\eta_{n+1}) w_n + (\eta_n - \eta_{n+1}) w_n\right\| \\
&\le \eta_{n+1}\|v_{n+1} - v_n\| + |\eta_{n+1} - \eta_n|\left(\|v_n\| + \|w_n\|\right) \\
&\quad +(1-\eta_{n+1})\|w_{n+1} - w_n\| \\
&= \eta_{n+1}\left\|P_C(z_{n+1} - \zeta_{n+1} A z_{n+1}) - P_C(z_n - \zeta_n A z_n)\right\| \\
&\quad + |\eta_{n+1} - \eta_n|\left(\|v_n\| + \|w_n\|\right) \\
&\quad +(1-\eta_{n+1})\left\|P_C(z_{n+1} - \delta_{n+1} B z_{n+1}) - P_C(z_n - \delta_n B z_n)\right\| \\
&= \eta_{n+1}\left\|P_C(z_{n+1} - \zeta_{n+1} A z_{n+1}) - P_C(z_n - \zeta_{n+1} A z_n)\right. \\
&\quad \left. +P_C(z_n - \zeta_{n+1} A z_n) - P_C(z_n - \zeta_n A z_n)\right\| \\
&\quad + |\eta_{n+1} - \eta_n|\left(\|v_n\| + \|w_n\|\right) \\
&\quad +(1-\eta_{n+1})\left\|P_C(z_{n+1} - \delta_{n+1} B z_{n+1}) - P_C(z_n - \delta_{n+1} B z_n)\right. \\
&\quad \left. +P_C(z_n - \delta_{n+1} B z_n) - P_C(z_n - \delta_n B z_n)\right\| \\
&\le \eta_{n+1}\|z_{n+1} - z_n\| + \eta_{n+1}|\zeta_{n+1} - \zeta_n|\,\|A z_n\| \\
&\quad + |\eta_{n+1} - \eta_n|\left(\|v_n\| + \|w_n\|\right) + (1-\eta_{n+1})\|z_{n+1} - z_n\| \\
&\quad +(1-\eta_{n+1})|\delta_{n+1} - \delta_n|\,\|B z_n\| \\
&\le \|z_{n+1} - z_n\| + \eta_{n+1}|\zeta_{n+1} - \zeta_n|\,\|A z_n\| \\
&\quad + |\eta_{n+1} - \eta_n|\left(\|v_n\| + \|w_n\|\right) + |\delta_{n+1} - \delta_n|\,\|B z_n\|.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\|t_{n+1} - t_n\| - \|x_{n+1} - x_n\| \\
&\le \frac{\alpha_{n+1}}{1-\beta_{n+1}}\left[\left\|y_{n+1} - y_n\right\| + \left\|T_{\mu_{n+1}} W_{n+1} y_{n+1} - T_{\mu_n} W_n y_n\right\|\right] \\
&\quad + \left|\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}\right|\left[\left\|f(T_{\mu_n} W_n y_n)\right\| + \left\|T_{\mu_n} W_n y_n\right\|\right] \\
&\quad + \left\|\mathcal{J}_{n+1}^{M-1} x_n - \mathcal{J}_n^{M-1} x_n\right\| + \left|1 - \frac{r_{M,n}}{r_{M,n+1}}\right|\left\|z_{n+1} - \mathcal{J}_{n+1}^{M-1} x_{n+1}\right\| \\
&\quad +\eta_{n+1}|\zeta_{n+1} - \zeta_n|\,\|A z_n\| + |\eta_{n+1} - \eta_n|\left(\|v_n\| + \|w_n\|\right) \\
&\quad + |\delta_{n+1} - \delta_n|\,\|B z_n\| + \left\|W_{n+1} y_n - W_n y_n\right\| + \left\|T_{\mu_{n+1}} W_n y_n - T_{\mu_n} W_n y_n\right\|.
\end{aligned}
$$

This together with conditions (B_1), (B_4), Steps 2 and 3 imply that

$$\limsup_{n\to\infty}\left(\|t_{n+1} - t_n\| - \|x_{n+1} - x_n\|\right) \le 0.$$

Hence by Lemma 2.4, we obtain $\lim_{n\to\infty} \| t_n - x_n \| = 0$. Consequently,

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = (1-\beta_n)\|t_n - x_n\| = 0.$$

Step 5. $\lim_{n\to\infty}\left\|\mathcal{J}_n^{k+1} x_n - \mathcal{J}_n^{k} x_n\right\| = 0$, $\forall k \in \{0, 1, 2, ..., M - 1\}$.

Proof of Step 5. Let $p \in \mathcal{F}$ and $k \in \{1, 2, ..., M - 1\}$. Since $J_{r_{k+1,n}}^{F_{k+1}}$ is firmly nonexpansive, we obtain

$$
\begin{aligned}
&\left\|\mathcal{J}_n^{k+1} x_n - p\right\|^2 \\
&\quad = \left\|J_{r_{k+1,n}}^{F_{k+1}}\mathcal{J}_n^{k} x_n - J_{r_{k+1,n}}^{F_{k+1}} p\right\|^2 = \left\langle J_{r_{k+1,n}}^{F_{k+1}}\mathcal{J}_n^{k} x_n - p,\ \mathcal{J}_n^{k} x_n - p\right\rangle \\
&\quad = \frac{1}{2}\left[\left\|J_{r_{k+1,n}}^{F_{k+1}}\mathcal{J}_n^{k} x_n - p\right\|^2 + \left\|\mathcal{J}_n^{k} x_n - p\right\|^2 - \left\|J_{r_{k+1,n}}^{F_{k+1}}\mathcal{J}_n^{k} x_n - \mathcal{J}_n^{k} x_n\right\|^2\right].
\end{aligned}
$$

It follows that

$$\left\|\mathcal{J}_n^{k+1}x_n - p\right\|^2 \leq \left\|x_n - p\right\|^2 - \left\|\mathcal{J}_n^{k+1}x_n - \mathcal{J}_n^k x_n\right\|^2. \tag{19}$$

Using Lemma 2.9, (14) and (19), we obtain

$$\begin{aligned}
&\left\|x_{n+1} - p\right\|^2 \\
&\leq \alpha_n \left\|f(T_{\mu_n} W_n y_n) - p\right\|^2 + \beta_n \left\|x_n - p\right\|^2 + \gamma_n \left\|T_{\mu_n} W_n y_n - p\right\|^2 \\
&\leq \alpha_n \left\|f(T_{\mu_n} W_n y_n) - p\right\|^2 + \beta_n \left\|x_n - p\right\|^2 + \gamma_n \left\|y_n - p\right\|^2 \\
&\leq \alpha_n \left\|f(T_{\mu_n} W_n y_n) - p\right\|^2 + \beta_n \left\|x_n - p\right\|^2 + \gamma_n \left\|z_n - p\right\|^2 \\
&= \alpha_n \left\|f(T_{\mu_n} W_n y_n) - p\right\|^2 + \beta_n \left\|x_n - p\right\|^2 + \gamma_n \left\|\mathcal{J}_n^M x_n - p\right\|^2 \\
&= \alpha_n \left\|f(T_{\mu_n} W_n y_n) - p\right\|^2 + \beta_n \left\|x_n - p\right\|^2 \\
&\quad + \gamma_n \left\|J_{r_M,n}^{F_M} \ldots J_{r_{k+2},n}^{F_{k+2}} \mathcal{J}_n^{k+1} y_n - J_{r_M,n}^{F_M} \ldots J_{r_{k+2},n}^{F_{k+2}} p\right\|^2 \\
&\leq \alpha_n \left\|f(T_{\mu_n} W_n y_n) - p\right\|^2 + \beta_n \left\|x_n - p\right\|^2 + \gamma_n \left\|\mathcal{J}_n^{k+1} y_n - p\right\|^2 \\
&\leq \alpha_n \left\|f(T_{\mu_n} W_n y_n) - p\right\|^2 + \beta_n \left\|x_n - p\right\|^2 + \gamma_n \left\|\mathcal{J}_n^{k+1} x_n - p\right\|^2 \\
&\leq \alpha_n \left\|f(T_{\mu_n} W_n y_n) - p\right\|^2 + \beta_n \left\|x_n - p\right\|^2 \\
&\quad + \gamma_n \left[\left\|x_n - p\right\|^2 - \left\|\mathcal{J}_n^{k+1} x_n - \mathcal{J}_n^k x_n\right\|\right].
\end{aligned}$$

Then, we have

$$\begin{aligned}
&\gamma_n \left\|\mathcal{J}_n^{k+1} x_n - \mathcal{J}_n^k x_n\right\|^2 \\
&\leq \alpha_n \left\|f(T_{\mu_n} W_n y_n) - p\right\|^2 + \beta_n \left\|x_n - p\right\|^2 \\
&\quad + (1 - \alpha_n - \beta_n)\left\|x_n - p\right\|^2 - \left\|x_{n+1} - p\right\|^2 \\
&= \alpha_n \left[\left\|f(T_{\mu_n} W_n y_n) - p\right\|^2 - \left\|x_n - p\right\|^2\right] + \left\|x_n - p\right\|^2 - \left\|x_{n+1} - p\right\|^2 \\
&\leq \alpha_n \left[\left\|f(T_{\mu_n} W_n y_n) - p\right\|^2 - \left\|x_n - p\right\|^2\right] \\
&\quad + \|x_n - x_{n+1}\| \left[\left\|x_n - p\right\| + \left\|x_{n+1} - p\right\|\right].
\end{aligned}$$

It is easily seen that $\liminf_{n\to\infty} \gamma_n > 0$. So we have

$$\lim_{n\to\infty} \left\|\mathcal{J}_n^{k+1} x_n - \mathcal{J}_n^k x_n\right\| = 0.$$

Step 6. $\lim_{n\to\infty} \left\|x_n - T_{\mu_n} W_n \mathcal{J}_n^M y_n\right\| = 0$.

Proof of Step 6. Observe that

$$\begin{aligned}
&\left\|x_n - T_{\mu_n} W_n \mathcal{J}_n^M y_n\right\| \\
&\leq \|x_n - x_{n+1}\| + \left\|x_{n+1} - T_{\mu_n} W_n \mathcal{J}_n^M y_n\right\| \\
&= \|x_n - x_{n+1}\| + \left\|\alpha_n [f(T_{\mu_n} W_n y_n) - T_{\mu_n} W_n \mathcal{J}_n^M y_n] \right. \\
&\quad \left. + \beta_n [x_n - T_{\mu_n} W_n \mathcal{J}_n^M y_n]\right\| \\
&\leq \|x_n - x_{n+1}\| + \alpha_n \left[\left\|f(T_{\mu_n} W_n y_n)\right\| + \left\|T_{\mu_n} W_n \mathcal{J}_n^M y_n\right\|\right] \\
&\quad + \beta_n \left\|x_n - T_{\mu_n} W_n \mathcal{J}_n^M y_n\right\|,
\end{aligned}$$

hence

$$\begin{aligned}
&\left\|x_n - T_{\mu_n} W_n \mathcal{J}_n^M y_n\right\| \\
&\quad \leq \frac{1}{1 - \beta_n} \|x_{n+1} - x_n\| + \frac{\alpha_n}{1 - \beta_n} \left[\left\|f(T_{\mu_n} W_n y_n)\right\| + \left\|T_{\mu_n} W_n \mathcal{J}_n^M y_n\right\|\right].
\end{aligned}$$

It follows from conditions (B_1), (B_2) and Step 4, that

$$\lim_{n\to\infty} \left\| x_n - T_{\mu_n} W_n \mathcal{J}_n^M y_n \right\| = 0.$$

Step 7. $\lim_{n\to\infty} \| x_n - T_t x_n \| = 0$, for all $t \in S$.

Proof of Step 7. Let $p \in \mathcal{F}$ and set $M_0 = \max\{\|x_1 - p\|, \frac{1}{1-\alpha}\|f(p) - p\|\}$ and $D = \{y \in H : \| y - p \| \le M_0\}$, we remark that D is bounded closed convex set, $\{y_n\} \subset D$ and it is invariant under $\{J_{r_{k,n}}^{F_k} : k = 1, 2, \ldots, M, \forall n \in \mathbb{N}\}$, ϕ and W_n for all $n \in \mathbb{N}$. We will show that

$$\limsup_{n\to\infty} \sup_{y\in D} \left\| T_{\mu_n} y - T_t T_{\mu_n} y \right\| = 0, \quad \forall t \in S \tag{20}$$

Let $\epsilon > 0$. By [33, Theorem 1.2], there exists $\delta > 0$ such that

$$\overline{co} F_\delta(T_t; D) + B_\delta \subset F_\varepsilon(T_t; D), \quad \forall t \in S. \tag{21}$$

Also by [33, Corollary 1.1], there exists a natural number N such that

$$\left\| \frac{1}{N+1} \sum_{i=0}^{N} T_{t^i s} y - T_t \left(\frac{1}{N+1} \sum_{i=0}^{N} T_{t^i s} y \right) \right\| \le \delta, \tag{22}$$

for all $t, s \in S$ and $y \in D$. Let $t \in S$. Since $\{\mu_n\}$ is strongly left regular, there exists $N_0 \in \mathbb{N}$ such that $\left\| \mu_n - l_{t^i}^* \mu_n \right\| \le \frac{\delta}{(M_0 + \|p\|)}$ for $n \ge N_0$ and $i = 1, 2, \ldots, N$. Then, we have

$$\begin{aligned}
&\sup_{y\in D} \left\| T_{\mu_n} y - \int \frac{1}{N+1} \sum_{i=0}^{N} T_{t^i s} y \, d\mu_n(s) \right\| \\
&= \sup_{y\in D} \sup_{\|z\|=1} \left| \langle T_{\mu_n} y, z\rangle - \left\langle \int \frac{1}{N+1} \sum_{i=0}^{N} T_{t^i s} y \, d\mu_n(s), z \right\rangle \right| \\
&= \sup_{y\in D} \sup_{\|z\|=1} \left| \frac{1}{N+1} \sum_{i=0}^{N} (\mu_n)_s \langle T_s y, z\rangle - \frac{1}{N+1} \sum_{i=0}^{N} (\mu_n)_s \langle T_{t^i s} y, z\rangle \right| \\
&\le \frac{1}{N+1} \sum_{i=0}^{N} \sup_{y\in D} \sup_{\|z\|=1} |(\mu_n)_s \langle T_s y, z\rangle - (l_{t^i}^* \mu_n)_s \langle T_s y, z\rangle| \\
&\le \max_{i=1,2,\ldots,N} \left\| \mu_n - l_{t^i}^* \mu_n \right\| (M_0 + \|p\|) \le \delta, \quad \forall n \ge N_0.
\end{aligned} \tag{23}$$

By Lemma 2.1 we have

$$\int \frac{1}{N+1} \sum_{i=0}^{N} T_{t^i s} y \, d\mu_n(s) \in \overline{co} \left\{ \frac{1}{N+1} \sum_{i=0}^{N} T_{t^i}(T_s y) : s \in S \right\}. \tag{24}$$

It follows from (21), (22), (23) and (24) that

$$\begin{aligned}
T_{\mu_n} y &\in \overline{co} \left\{ \frac{1}{N+1} \sum_{i=0}^{N} T_{t^i s} y : s \in S \right\} + B_\delta \\
&\subset \overline{co} F_\delta(T_t; D) + B_\delta \subset F_\varepsilon(T_t; D),
\end{aligned}$$

for all $y \in D$ and $n \geq N_0$. Therefore,

$$\limsup_{n\to\infty} \sup_{y\in D} \left\| T_t(T_{\mu_n}y) - T_{\mu_n}y \right\| \leq \varepsilon.$$

Since $\epsilon > 0$ is arbitrary, we get (20).

Let $t \in S$ and $\epsilon > 0$. Then, there exists $\delta > 0$, which satisfies (21). From condition (B_1), (20) and Step 6, there exists $N_1 \in \mathbb{N}$ such that $\alpha_n < \frac{\delta}{4M_0}$, $T_{\mu_n}y \in F_\delta(T_t, D)$ for all $y \in D$ and $\left\| x_n - T_{\mu_n}W_n y_n \right\| < \frac{\delta}{2}$ for all $n \geq N_1$. We note that

$$\begin{aligned}
&\alpha_n \left\| f(T_{\mu_n}W_n y_n) - T_{\mu_n}W_n y_n \right\| \\
&\quad \leq \alpha_n[\left\| f(T_{\mu_n}W_n y_n) - f(p) \right\| + \left\| f(p) - p \right\| + \left\| p - T_{\mu_n}W_n y_n \right\|] \\
&\quad \leq \alpha_n[\alpha \left\| y_n - p \right\| + \left\| f(p) - p \right\| + \left\| p - y_n \right\|] \leq 2M_0\alpha_n \leq \frac{\delta}{2},
\end{aligned}$$

for all $n \geq N_1$. Therefore, we have

$$\begin{aligned}
x_{n+1} &= T_{\mu_n}W_n y_n + \alpha_n(f(T_{\mu_n}W_n y_n) - T_{\mu_n}W_n y_n) \\
&\quad + \beta_n(x_n - T_{\mu_n}W_n y_n) \\
&\in F_\delta(T_t; D) + B_{\frac{\delta}{2}} + B_{\frac{\delta}{2}} \subset F_\delta(T_t; D) + B_\delta \subset F_\varepsilon(T_t; D),
\end{aligned}$$

for all $n \geq N_1$. This shows that

$$\|x_n - T_t x_n\| \leq \varepsilon, \quad \forall n \geq N_1.$$

Since $\epsilon > 0$ is arbitrary, we get $\lim_{n\to\infty} \| x_n - T_t(x_n) \| = 0$.

Step 8. The weak ω-limit set of $\{x_n\}$, $\omega_\omega\{x_n\}$, is a subset of $\mathcal{F}$.

Proof of Step 8. Let $z \in \omega_\omega\{x_n\}$ and let $\{x_{n_m}\}$ be a subsequence of $\{x_n\}$ weakly converging to z, we need to show that $z \in \mathcal{F}$. Noting Step 5, with no loss of generality, we may assume that $\mathcal{J}^k_{n_m}x_{n_m} \rightharpoonup z, \forall k \in \{1, 2, \ldots, M\}$. At first, note that by (A_2) and given $y \in C$ and $k \in \{1, 2, ..., M\}$, we have

$$\left\langle y - \mathcal{J}^{k+1}_{n_m}x_{n_m}, \ \frac{1}{r_{k+1,n_m}}(\mathcal{J}^{k+1}_{n_m}x_{n_m} - \mathcal{J}^k_{n_m}x_{n_m}) \right\rangle \geq F_{k+1}(y, \ \mathcal{J}^{k+1}_{n_m}x_{n_m}).$$

Step 5 and condition(B_5) imply that

$$\frac{\mathcal{J}^{k+1}_{n_m}x_{n_m} - \mathcal{J}^k_{n_m}x_{n_m}}{r_{k+1,n_m}} \to 0.$$

Since $\mathcal{J}^k_{n_m}x_{n_m} \rightharpoonup z$, from the lower semi-continuity of F_{k+1} on the second variable, we have $F_{k+1}(y, z) \leq 0$ for all $y \in C$ and for all $k \in \{0, 1, 2, ..., M - 1\}$. For t with $0 < t \leq 1$ and $y \in C$, let $y_t = ty + (1 - t)z$. Since $y \in C$ and $z \in C$, we have $y_t \in C$ and hence $F_{k+1}(y_t, z) \leq 0$. So from the convexity of F_{k+1} on second variable, we have

$$0 = F_{k+1}(y_t, y_t) \leq tF_{k+1}(y_t, y) + (1 - t)F_{k+1}(y_t, z) \leq tF_{k+1}(y_t, y) \leq F_{k+1}(y_t, y).$$

hence $F_{k+1}(y_t, y) \geq 0$. therefore, we have $F_{k+1}(z, y) \geq 0$ for all $y \in C$ and $k \in \{0, 1, 2, ..., M\text{-}1\}$. Therefore $z \in \cap_{k=1}^M \mathrm{EP}(F_k) = \mathrm{EP}(\mathcal{J})$.

Since $x_{n_m} \rightharpoonup z$, it follows by Step 7 and Lemma 2.2 that $z \in \mathrm{Fix}(T_t)$ for all $t \in S$. Therefore, $z \in \mathrm{Fix}(\phi)$. We will show $z \in \mathrm{Fix}(W)$. Assume $z \notin \mathrm{Fix}(W)$ Since

$z \in \mathrm{Fix}(\varphi) \cap \mathrm{EP}(\mathcal{J})$, by our assumption, we have $T_i z \in \mathrm{Fix}(\phi), \forall i \in \mathbb{N}$ and then $W_n z \in \mathrm{Fix}(\phi)$. Hence by Lemma 2.1, $T_{\mu_n} W_n z = W_n z$, therefore by Lemma 2.5, we get

$$T_{\mu_n} W_n \mathcal{J}_n^M z = W_n z, \quad \forall n \in \mathbb{N}. \tag{25}$$

Now, by (25), Step 6, Lemma 2.6 and Opial's condition, we have

$$\begin{aligned}
&\liminf_{n\to\infty} \|x_{n_m} - z\| \\
&\quad < \liminf_{n\to\infty} \|x_{n_m} - Wz\| \\
&\quad \le \liminf_{n\to\infty} \big(\|x_{n_m} - T_{\mu_{n_m}} W_{n_m} \mathcal{J}_{n_m}^M x_{n_m}\| \\
&\qquad + \|T_{\mu_{n_m}} W_{n_m} \mathcal{J}_{n_m}^M x_{n_m} - T_{\mu_{n_m}} W_{n_m} \mathcal{J}_{n_m}^M z\| + \|T_{\mu_{n_m}} W_{n_m} \mathcal{J}_{n_m}^M z - Wz\|\big) \\
&\quad \le \liminf_{n\to\infty} \big(\|x_{n_m} - T_{\mu_{n_m}} W_{n_m} \mathcal{J}_{n_m}^M x_{n_m}\| + \|x_{n_m} - z\| + \|W_{n_m} z - Wz\|\big) \\
&\quad \le \liminf_{n\to\infty} \|x_{n_m} - z\| .
\end{aligned}$$

This is a contradiction. So we get $z \in \mathrm{Fix}(W) = \bigcap_{i=1}^{\infty} \mathrm{Fix}(T_i)$.

Now, let us show that $z \in \mathrm{VI}(C, A) \cap \mathrm{VI}(C, B)$. Observe that,

$$\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&\quad \le \alpha_n \|f(T_{\mu_n} W_n y_n) - p\|^2 + \beta_n \|x_n - p\|^2 + \gamma_n \|T_{\mu_n} W_n \mathcal{J}_n^M y_n - p\|^2 \\
&\quad \le \alpha_n \|f(T_{\mu_n} W_n y_n) - p\|^2 + \beta_n \|x_n - p\|^2 + \gamma_n \|y_n - p\|^2 \\
&\quad = \alpha_n \|f(T_{\mu_n} W_n y_n) - p\|^2 + \beta_n \|x_n - p\|^2 + \gamma_n \|\eta_n P_C(z_n - \zeta_n A z_n) \\
&\qquad + (1-\eta_n) P_C(z_n - \delta_n B z_n) - p\|^2 \\
&\quad = \alpha_n \|f(T_{\mu_n} W_n y_n) - p\|^2 + \beta_n \|x_n - p\|^2 + \gamma_n \|\eta_n [P_C(z_n - \zeta_n A z_n) \\
&\qquad - P_C(p - \zeta_n A p)] + (1-\eta_n)[P_C(z_n - \delta_n B z_n) - P_C(p - \delta_n B p)]\|^2.
\end{aligned} \tag{26}$$

From (26), we have

$$\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&\quad \le \alpha_n \|f(T_{\mu_n} W_n y_n) - p\|^2 + \beta_n \|x_n - p\|^2 \\
&\qquad + \gamma_n [\eta_n \|(z_n - p) - \zeta_n (A z_n - A p)\|^2 + (1-\eta_n)\|z_n - p\|^2] \\
&\quad = \alpha_n \|f(T_{\mu_n} W_n y_n) - p\|^2 + \beta_n \|x_n - p\|^2 \\
&\qquad + \gamma_n (1-\eta_n)\|z_n - p\|^2 + \gamma_n \eta_n [\|z_n - p\|^2 + \zeta_n^2 \|A z_n - A p\|^2 \\
&\qquad - 2\zeta_n \langle A z_n - A p,\ z_n - p\rangle] \\
&\quad \le \alpha_n \|f(T_{\mu_n} W_n y_n) - p\|^2 + \beta_n \|x_n - p\|^2 \\
&\qquad + \gamma_n (1-\eta_n)\|z_n - p\|^2 + \gamma_n \eta_n [\|z_n - p\|^2 + \zeta_n^2 \|A z_n - A p\|^2 \\
&\qquad - 2\zeta_n \beta \|A z_n - A p, z_n - p\|^2] \\
&\quad = \alpha_n \|f(T_{\mu_n} W_n y_n) - p\|^2 + \beta_n \|x_n - p\|^2 \\
&\qquad + \gamma_n \|z_n - p\|^2 + \zeta_n (\zeta_n - 2\beta)\|A z_n - A p\|^2 \\
&\quad \le \|x_n - p\|^2 + \alpha_n [\|f(T_{\mu_n} W_n y_n) - p\|^2 - \|x_n - p\|^2] \\
&\qquad + \zeta_n (\zeta_n - 2\beta)\|A z_n - A p\|^2,
\end{aligned}$$

which implies that

$$\begin{aligned}&-\zeta_n(\zeta_n-2\beta)\|Az_n-Ap\|^2\\&\quad\le[\|x_n-p\|+\|x_{n+1}-p\|]\,\|x_n-x_{n+1}\|\\&\quad\quad+\alpha_n[\|f(T_{\mu_n}W_ny_n)-p\|^2-\|x_n-p\|^2].\end{aligned}$$

Therefore, from step 4 and condition B_1, we obtain

$$\lim_{n\to\infty}\|Az_n-Ap\|=0. \tag{27}$$

On the other hand from (26), we have

$$\begin{aligned}&\|x_{n+1}-p\|^2\\&\quad\le\alpha_n\|f(T_{\mu_n}W_ny_n)-p\|^2+\beta_n\|x_n-p\|^2+\gamma_n\Big[\eta_n\|z_n-p\|^2\\&\quad\quad+(1-\eta_n)\|(z_n-p)-\delta_n(Bz_n-Bp)\|^2\Big]\\&\quad=\alpha_n\|f(T_{\mu_n}W_ny_n)-p\|^2+\beta_n\|x_n-p\|^2+\gamma_n\Big[\eta_n\|z_n-p\|^2\\&\quad\quad+(1-\eta_n)(\|z_n-p\|^2-2\delta_n\langle Bz_n-Bp,\ z_n-p\rangle+\delta_n^2\|Bz_n-Bp\|^2)\Big]\\&\quad\le\alpha_n\|f(T_{\mu_n}W_ny_n)-p\|^2+\beta_n\|x_n-p\|^2+\gamma_n\Big[\eta_n\|z_n-p\|^2\\&\quad\quad+(1-\eta_n)(\|z_n-p\|^2-2\delta_n\gamma\|Bz_n-Bp\|^2+\delta_n^2\|Bz_n-Bp\|^2)\Big]\\&\quad=\alpha_n\|f(T_{\mu_n}W_ny_n)-p\|^2+\beta_n\|x_n-p\|^2+\gamma_n\|z_n-p\|^2\\&\quad\quad+\delta_n(\delta_n-2\gamma)\gamma_n(1-\eta_n)\|Bz_n-Bp\|^2\\&\quad\le\|x_n-p\|^2+\alpha_n[\|f(T_{\mu_n}W_ny_n)-p\|^2-\|x_n-p\|^2]\\&\quad\quad+\delta_n(\delta_n-2\gamma)\|Bz_n-Bp\|^2\end{aligned}$$

which implies that

$$\begin{aligned}&-\delta_n(\delta_n-2\gamma)\|Bz_n-Bp\|^2\\&\quad\le[\|x_n-p\|+\|x_{n+1}-p\|]\,\|x_n-x_{n+1}\|\\&\quad\quad+\alpha_n[\|f(T_{\mu_n}W_ny_n)-p\|^2-\|x_n-p\|^2].\end{aligned}$$

Therefore, from step 4 and condition B_1, we obtain

$$\lim_{n\to\infty}\|Bz_n-Bp\|=0. \tag{28}$$

From (6) and (12), we have

$$\begin{aligned}&\|v_n-p\|^2\\&\quad=\|P_C(z_n-\zeta_nAz_n)-P_C(p-\zeta_nAp)\|^2\\&\quad\le\langle(z_n-\zeta_nAz_n)-(p-\zeta_nAp),v_n-p\rangle\\&\quad=\frac{1}{2}\Big[\|(z_n-\zeta_nAz_n)-(p-\zeta_nAp)\|^2+\|v_n-p\|^2\\&\quad\quad-\|(z_n-\zeta_nAz_n)-(p-\zeta_nAp)-(v_n-p)\|^2\Big]\\&\quad=\frac{1}{2}\Big[\|z_n-p\|^2+\|v_n-p\|^2-\|z_n-v_n\|^2\\&\quad\quad+2\zeta_n\langle z_n-v_n,\ Az_n-Ap\rangle-\zeta_n^2\|Az_n-Ap\|^2\Big].\end{aligned}$$

So we obtain

$$\begin{aligned}\|v_n - p\|^2 \leq\ & \|z_n - p\|^2 - \|z_n - v_n\|^2 \\ & + 2\zeta_n\langle z_n - v_n, Az_n - Ap\rangle - \zeta_n^2\|Az_n - Ap\|^2.\end{aligned} \tag{29}$$

By using the same method as (29), we have

$$\begin{aligned}\|w_n - p\|^2 \leq\ & \|z_n - p\|^2 - \|z_n - w_n\|^2 \\ & + 2\delta_n\langle z_n - w_n, Bz_n - Bp\rangle - \delta_n^2\|Bz_n - Bp\|^2.\end{aligned} \tag{30}$$

From (29), (30) and definition of y_n, we have,

$$\begin{aligned}&\|y_n - p\|^2 \\ &= \|\eta_n[P_C(z_n - \zeta_n Az_n) - p] \\ &\quad +(1-\eta_n)[P_C(z_n - \delta_n Bz_n) - p]\|^2 \\ &= \|\eta_n(v_n - p) + (1-\eta_n)(w_n - p)\|^2 \\ &\leq \eta_n\|v_n - p\|^2 + (1-\eta_n)\|w_n - p\|^2 \\ &\leq \eta_n[\|z_n - p\|^2 - \|z_n - v_n\|^2 + 2\zeta_n\langle z_n - v_n,\ Az_n - Ap\rangle \\ &\quad -\zeta_n^2\|Az_n - Ap\|^2] + (1-\eta_n)[\|z_n - p\|^2 - \|z_n - w_n\|^2 \\ &\quad +2\delta_n\langle z_n - w_n,\ Bz_n - Bp\rangle - \delta_n^2\|Bz_n - Bp\|^2] \\ &\leq \|z_n - p\|^2 + \eta_n[-\|z_n - v_n\|^2 + 2\zeta_n\|z_n - v_n\|\ \|Az_n - Ap\| \\ &\quad -\zeta_n^2\|Az_n - Ap\|^2] + (1-\eta_n)[-\|z_n - w_n\|^2 \\ &\quad +2\delta_n\|z_n - w_n\|\ \|Bz_n - Bp\| - \delta_n^2\|Bz_n - Bp\|^2]\end{aligned} \tag{31}$$

By (31), we have

$$\begin{aligned}&||x_{n+1} - p||^2 \\ &\leq \alpha_n||f(T_{\mu_n}W_n y_n) - p||^2 + \beta_n||x_n - p||^2 + \gamma_n||T_{\mu_n}W_n\mathcal{J}_n^M y_n - p||^2 \\ &\leq \alpha_n||f(T_{\mu_n}W_n y_n) - p||^2 + \beta_n||x_n - p||^2 + \gamma_n||z_n - p||^2 \\ &\quad +\gamma_n\eta_n[-||z_n - v_n||^2 + 2\zeta_n||z_n - v_n||\ ||Az_n - Ap|| \\ &\quad -\zeta_n^2||Az_n - Ap||^2] + \gamma_n(1-\eta_n)[-||z_n - w_n||^2 \\ &\quad +2\delta_n||\ ||z_n - w_n||\ ||Bz_n - Bp|| - \delta_n^2||Bz_n - Bp||^2] \\ &\leq \alpha_n||f(T_{\mu_n}W_n y_n) - p||^2 + \beta_n||x_n - p||^2 + \gamma_n||x_n - p||^2 \\ &\quad -\gamma_n\eta_n||z_n - v_n||^2 + \gamma_n\eta_n[2\zeta_n||z_n - v_n||\ ||Az_n - Ap|| \\ &\quad -\zeta_n^2||Az_n - Ap||^2] - \gamma_n(1-\eta_n)||z_n - w_n||^2 \\ &\quad +\gamma_n(1-\eta_n)[2\delta_n||\ ||z_n - w_n||\ ||Bz_n - Bp|| - \delta_n^2||Bz_n - Bp||^2] \\ &= ||x_n - p||^2 + \alpha_n[||f(T_{\mu_n}W_n y_n) - p||^2 - ||x_n - p||^2] \\ &\quad -\gamma_n\eta_n||z_n - v_n||^2 + \gamma_n\eta_n[2\zeta_n||z_n - v_n||\ ||Az_n - Ap|| \\ &\quad -\zeta_n^2||Az_n - Ap||^2] - \gamma_n(1-\eta_n)||z_n - w_n||^2 \\ &\quad +\gamma_n(1-\eta_n)[2\delta_n||\ ||z_n - w_n||\ ||Bz_n - Bp|| - \delta_n^2||Bz_n - Bp||^2],\end{aligned}$$

which implies that

$$\begin{aligned}&\gamma_n\eta_n||z_n - v_n||^2\\ &\quad\leq [||x_n - p|| + ||x_{n+1} - p||]||x_{n+1} - x_n||\\ &\qquad + \alpha_n[||f(T_{\mu_n}W_n y_n) - p||^2 - ||x_n - p||^2]\\ &\qquad + \gamma_n\eta_n[2\zeta_n||z_n - v_n||\ ||Az_n - Ap|| - \zeta_n^2||Az_n - Ap||^2]\\ &\qquad + \gamma_n(1-\eta_n)[2\delta_n||\ ||z_n - w_n||\ ||Bz_n - Bp|| - \delta_n^2||Bz_n - Bp||^2],\end{aligned}$$

and

$$\begin{aligned}&\gamma_n(1-\eta_n)||z_n - w_n||^2\\ &\quad\leq [||x_n - p|| + ||x_{n+1} - p||]||x_{n+1} - x_n||\\ &\qquad + \gamma_n\eta_n[2\zeta_n||z_n - v_n||\ ||Az_n - Ap|| - \zeta_n^2||Az_n - Ap||^2]\\ &\qquad + \gamma_n(1-\eta_n)[2\delta_n||\ ||z_n - w_n||\ ||Bz_n - Bp|| - \delta_n^2||Bz_n - Bp||^2].\end{aligned}$$

Therefore, from $0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < 1$, condition B_1, step 4, (27) and (28) we get

$$\lim_{n\to\infty} ||z_n - v_n|| = 0 \quad \textit{and} \quad \lim_{n\to\infty} ||z_n - w_n|| = 0. \tag{32}$$

Let $U : H \to 2^H$ be a set-valued mapping is defined by

$$Ux = \begin{cases} Ax + N_C x, & x \in C,\\ \emptyset, & x \notin C,\end{cases}$$

where $N_C x$ is the normal cone to C at $x \in C$. Since A is monotone. Thus U is maximal monotone see [1]. Let $(x, y) \in G(U)$, hence $y - Ax \in N_C x$ and since $v_n = P_C(z_n - \zeta_n Az_n)$ therefore, $\langle x - v_n, y - Ax\rangle \geq 0$. On the other hand from (7), we have

$$\langle x - v_n, v_n - (z_n - \zeta_n Az_n)\rangle \geq 0,$$

i.e.,

$$\left\langle x - v_n, \frac{v_n - z_n}{\zeta_n} + Az_n\right\rangle \geq 0$$

Therefore, we have

$$\begin{aligned}&\langle x - v_{n_i}, y\rangle\\ &\quad\geq \langle x - v_{n_i}, Ax\rangle\\ &\quad\geq \langle x - v_{n_i}, Ax\rangle - \left\langle x - v_{n_i}, \frac{v_{n_i} - z_{n_i}}{\zeta_{n_i}} + Az_{n_i}\right\rangle\\ &\quad= \left\langle x - v_{n_i}, Ax - \frac{v_{n_i} - z_{n_i}}{\zeta_{n_i}} - Az_{n_i}\right\rangle\\ &\quad= \langle x - v_{n_i}, Ax - Av_{n_i}\rangle + \langle x - v_{n_i}, Av_{n_i} - Az_{n_i}\rangle - \left\langle x - v_{n_i}, \frac{v_{n_i} - z_{n_i}}{\zeta_{n_i}}\right\rangle\\ &\quad\geq \langle x - v_{n_i}, Av_{n_i} - Az_{n_i}\rangle - \left\langle x - v_{n_i}, \frac{v_{n_i} - z_{n_i}}{\zeta_{n_i}}\right\rangle\\ &\quad\geq \langle x - v_{n_i}, Av_{n_i} - Az_{n_i}\rangle - ||x - v_{n_i}||\left\|\frac{v_{n_i} - z_{n_i}}{\zeta_{n_i}}\right\|.\end{aligned}$$

From (32), we get $\lim_{i\to\infty}||v_{n_i} - z_{n_i}|| = 0$. Noting that $x_{n_i} \rightharpoonup z$ and A *is* $\frac{1}{\beta}$-lipschitzian, we obtain

$$\langle x - z, y\rangle \geq 0. \tag{33}$$

Since U is maximal monotone, we have $z \in U^{-1}0$, and hence $z \in \mathrm{VI}(C, A)$. Let $V : H \to 2^H$ be a set-valued mapping is defined by

$$Vx = \begin{cases} Bx + N_C x, & x \in C, \\ \emptyset, & x \notin C, \end{cases}$$

where $N_C x$ is the normal cone to C at $x \in C$. Since B is monotone. Thus U is maximal monotone see [1]. Repeating the same argument as above, we can derive $z \in \mathrm{VI}(C, B)$. Therefore, $z \in \mathcal{F}$.

Step 9. There exists a unique $x^* \in C$ such that

$$\limsup_{n\to\infty}\langle f(x^*) - x^*, x_n - x^*\rangle \leq 0.$$

Proof of Step 9. Note that f is a contraction mapping with coefficient $\alpha \in (0, 1)$. Then $||P_{\mathcal{F}}f(x) - P_{\mathcal{F}}f(y)|| \leq ||f(x) - f(y)|| \leq \alpha||x - y||$ for all $x, y \in H$. Therefore $P_{\mathcal{F}}$ is a contraction of H into itself, which implies that there exists a unique element $x^* \in H$ such that $x^* = P_{\mathcal{F}}f(x^*)$. at the same time, we note that $x^* \in C$. Using Lemma 2.3, we have

$$\langle f(x^*) - x^*, x^* - z\rangle \geq 0, \quad \forall z \in \mathcal{F}. \tag{34}$$

We can choose a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle f(x^*) - x^*, x_n - x^*\rangle = \lim_{k\to\infty}\langle f(x^*) - x^*, x_{n_k} - x^*\rangle.$$

Since $\{x_{n_k}\}$ is bounded, therefore, $\{x_{n_k}\}$ has subsequence $\{x_{n_{k_j}}\}$ such that $x_{n_{k_j}} \rightharpoonup z^*$. With no loss of generality, we may assume that $x_{n_k} \rightharpoonup z^*$. Applying Step 8 and (34), we have

$$\limsup_{n\to\infty}\langle f(x^*) - x^*,\ x_n - x^*\rangle = \langle f(x^*) - x^*,\ z^* - x^*\rangle \leq 0.$$

Step 10, The sequences $\{x_n\}$ converges strongly to x^*, which is obtained in Steep 9.

Proof of Step 10. We have

$$\begin{aligned}
&||x_{n+1} - x^*||^2 \\
&\quad = ||\alpha_n(f(T_{\mu_n}W_n y_n) - x^*) + \beta_n(x_n - x^*) + \gamma_n(T_{\mu_n}W_n y_n - x^*)||^2 \\
&\quad \leq ||\beta_n(x_n - x^*) + \gamma_n(T_{\mu_n}W_n y_n - x^*)||^2 + 2\alpha_n\langle f(T_{\mu_n}W_n y_n) - x^*,\ x_{n+1} - x^*\rangle \\
&\quad \leq [\beta_n||x_n - x^*|| + \gamma_n||T_{\mu_n}W_n y_n - x^*||]^2 \\
&\qquad +2\alpha_n\langle f(T_{\mu_n}W_n y_n) - f(x^*),\ x_{n+1} - x^*\rangle + 2\alpha_n\langle f(x^*) - x^*,\ x_{n+1} - x^*\rangle \\
&\quad \leq [\beta_n||x_n - x^*|| + \gamma_n||y_n - x^*||]^2 \\
&\qquad +2\alpha_n\alpha||y_n - x^*||\ ||x_{n+1} - x^*|| + 2\alpha_n\langle f(x^*) - x^*,\ x_{n+1} - x^*\rangle \\
&\quad \leq [\beta_n||x_n - x^*|| + \gamma_n||x_n - x^*||]^2 \\
&\qquad +2\alpha_n\alpha||x_n - x^*||\ ||x_{n+1} - x^*|| + 2\alpha_n\langle f(x^*) - x^*,\ x_{n+1} - x^*\rangle \\
&\quad = (1 - \alpha_n)^2||x_n - x^*||^2 + \alpha_n\alpha[||x_n - x^*||^2 + ||x_{n+1} - x^*||^2] \\
&\qquad +2\alpha_n\langle f(x^*) - x^*,\ x_{n+1} - x^*\rangle
\end{aligned}$$

Which implies that

$$\begin{aligned}||x_{n+1} - x^*||^2 &\\ \leq \frac{(1-\alpha_n)^2 + \alpha_n\alpha}{1-\alpha_n\alpha}||x_n - x^*||^2 + \frac{2\alpha_n}{1-\alpha_n\alpha}\langle f(x^*) - x^*,\ x_{n+1} - x^*\rangle &\\ = \left(1 - \alpha_n\frac{2-2\alpha}{1-\alpha_n\alpha}\right)||x_n - x^*||^2 + \alpha_n\tau_n,\end{aligned} \tag{35}$$

where

$$\tau_n = \frac{\alpha_n}{1-\alpha_n\alpha}||x_n - x^*||^2 + \frac{2}{1-\alpha_n\alpha}\langle f(x^*) - x^*,\ x_{n+1} - x^*\rangle,$$

By Step 9, and condition (B_1), we get $\limsup_{n\to\infty}\tau_n \leq 0$. Now applying Lemma 2.8 to (35), we conclude that $x_n \to x^*$. Consequently, from $||J^{F_k}_{r_{k,n}}x_n - x^*|| \leq ||x_n - x^*||$, we have $J^{F_k}_{r_{k,n}}x_n \to x^*$, for all $k \in \{1, 2, ..., M\}$.

Corollary 3.2. (see Yao et al. [8]) *Let C be a nonempty closed convex subset of a real Hilbert space H, F a bi-functions from* $C\times C$ *into* $\mathbb{R}$ *which satisfy* (A_1) - (A_4) *and* $\{T_i\}_{i=1}^{\infty}$ *an infinite family of nonexpansive mapping of C into C such that* $\cap_{i=1}^{\infty}\mathrm{Fix}(T_i)\cap \mathrm{EP}(F) \neq \emptyset$. *Let* $\{\alpha_n\}$, $\{\beta_n\}$ *and* $\{\gamma_n\}$ *are three sequences in* $(0, 1)$ *such that* $\alpha_n + \beta_n + \gamma_n = 1$ *and* $\{r_n\} \subset (0, \infty)$. *Suppose the following conditions are satisfied:*

(B_1) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty}\alpha_n = \infty$,

(B_2) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$,

(B_3) $\liminf_{n\to\infty} r_n > 0$ *and* $\lim_{n\to\infty}(r_{n+1} - r_n) = 0$.

Let f be a contraction of C into itself with coefficient $\alpha \in (0, 1)$ *and given* $x_1 \in C$ *arbitrarily. Then the sequence* $\{x_n\}$ *generated by*

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n W_n J^F_{r_n}x_n,\ n \geq 1.$$

converge strongly to $x^* \in \cap_{i=1}^{\infty}\mathrm{Fix}(T_i)\cap\mathrm{EP}(F)$, *where* $x^* \in P_{\cap_{i=1}^{\infty}\mathrm{Fix}(T_i)\cap\mathrm{EP}(F)}f(x^*)$.

Proof. Take $A = B = 0$, $\phi = \{I\}$, $F_1 = F$ and $F_k = 0$ for $k \in \{2, ..., M\}$ in Theorem 3.1, then we have $T_{\mu_n} = I$ and $y_n = z_n = J^k_{r_m}x_n$. So from Theorem 3.1 the sequence $\{x_n\}$ converges strongly to $x^* \in \cap_{i=1}^{\infty}\mathrm{Fix}(T_i)\cap\mathrm{EP}(F)$, where $x^* \in P_{\cap_{i=1}^{\infty}\mathrm{Fix}(T_i)\cap\mathrm{EP}(F)}f(x^*)$.

Corollary 3.3. *Let C be a nonempty closed convex subset of a real Hilbert space H,* $\mathcal{J} = \{F_k : k = 1, 2, \ldots, M\}$ *be a finite family of bi-functions from* $C \times C$ *into* $\mathbb{R}$ *which satisfy* (A_1)-(A_4), *T a nonexpansive mappings on C such that* $\mathrm{Fix}(T)\cap\mathrm{EP}(\mathcal{J}) \neq \emptyset$. *Let* $\{\alpha_n\}$, $\{\beta_n\}$ *and* $\{\gamma_n\}$ *are three sequences in* $(0, 1)$ *such that* $\alpha_n + \beta_n + \gamma_n = 1$ *and* $\{r_{k,n}\}_{k=1}^{M}$ *be sequences in* $(0, \infty)$. *Suppose the following conditions are satisfied:*

(B_1) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty}\alpha_n = \infty$,

(B_2) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$,

(B_3) $\liminf_{n\to\infty} r_{k,n} > 0$ *and* $\lim_{n\to\infty}(r_{k,n+1} - r_{k,n}) = 0$ *for* $k \in \{1, 2, ..., M\}$.

Let f be a contraction of H into itself and given $x_1 \in H$ *arbitrarily. If the sequences* $\{x_n\}$ *generated iteratively by*

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n\frac{1}{n}\sum_{k=0}^{\infty}\left(\frac{n-1}{n}\right)^k T^k J^{F_M}_{r_{M,n}}\ldots J^{F_2}_{r_{2n}}, J^{F_1}_{r_{1n}}, x_n, \quad n \geq 1.$$

Then, sequences $\{x_n\}$ *and* $\{J_{r_k,n}^{F_k} x_n\}_{k=1}^{M}$ *converge strongly to* $x^* \in \text{Fix}(T) \cap \text{EP}(\mathcal{J})$, *where* $x^* = P_{\text{Fix}(T)\cap\text{EP}(\mathcal{J})} f(x^*)$.

Proof. Let $S = \{0, 1, ...\}$, $\phi = \{T^i : i \in S\}$ and $T^0 = I$. For $f = (x_0, x_1, ...) \in B(S)$, define

$$\mu_n(f) = \frac{1}{n}\sum_{k=0}^{\infty}\left(\frac{n-1}{n}\right)^k x_k, \quad \forall n \in \mathbb{N}.$$

Then $\{\mu_n\}$ is a regular sequence of means on $B(S)$ such that $\lim_{n\to\infty} \| \mu_{n+} - \mu_n \| = 0$; for more details, see [34]. Next for each $x \in H$ and $n \in \mathbb{N}$, we have

$$T_{\mu_n} x = \frac{1}{n}\sum_{k=0}^{\infty}\left(\frac{n-1}{n}\right)^k T^k x.$$

Take $A = B = 0$, $T_i = I$ for all $i \in \mathbb{N}$ in Theorem 3.1 then we have $y_n = z_n$ and $W_n = I$ for all $n \in \mathbb{N}$. Therefore, it follows from Theorem 3.1 that the sequences $\{x_n\}$ and $\{J_{r_k,n}^{F_k} x_n\}_{k=1}^{M}$ converge strongly, as $n \to \infty$ to a point $x^* \in \text{Fix}(T) \cap \text{EP}(\mathcal{J})$, where $x^* = P_{\text{Fix}(T)\cap\text{EP}(\mathcal{J})} f(x^*)$.

Remark **3.4**. Theorem 3.1 improve [8, Theorem 1.2] in the following aspects.

(a) Our iterative process (11) is more general than Yao et al. process (14) because it can be applied to solving the problem of finding a common element of the set of solutions of systems of equilibrium problems and systems of variational inequalities.

(b) Our iterative process (11) is very diffident from Yao et al. process (14) because there are left amenable semigroup of nonexpansive mappings.

(c) Our method of proof is very different from the on in Yao et al. [8] for example we use Corollary 1.1 and Theorem 1.2 of Bruck [33] fore the proof of Theorem 3.1.

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Competing interests
The authors declare that they have no competing interests.

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