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Existence and approximation of solutions for system of generalized mixed variational inequalities

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Abstract

The aim of this work is to study a system of generalized mixed variational inequalities, existence and approximation of its solution using the resolvent operator technique. We further propose an algorithm which converges to its solution and common fixed points of two Lipschitzian mappings. Parallel algorithms are used, which can be used to simultaneous computation in multiprocessor computers. The results presented in this work are more general and include many previously known results as special cases.

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1 Introduction and preliminaries

Variational inequality theory was introduced by Stampacchia [1] in the early 1960s. The birth of variational inequality problem coincides with Signorini problem, see [2, p.282]. The Signorini problem consists of finding the equilibrium of a spherically shaped elastic body resting on the rigid frictionless plane. Let H be a real Hilbert space whose inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively. A variational inequality involving the nonlinear bifurcation, which characterized the Signorini problem with nonlocal friction is: find $x \in H$ such that

$$\langle Tx, y - x\rangle + \varphi(y,x) - \varphi(x,x) \geq 0, \quad \forall y \in H,$$

where $T : H \to H$ is a nonlinear operator and $\varphi(\cdot,\cdot) : H \times H \to \mathbb{R} \cup \{+\infty\}$ is a continuous bifunction.

Inequality above is called mixed variational inequality problem. It is an useful and important generalization of variational inequalities. This type of variational inequality arise in the study of elasticity with nonlocal friction laws, fluid flow through porus media and structural analysis. Mixed variational inequalities have been generalized and extended in many directions using novel and innovative techniques. One interesting problem is to find common solution of a system of variational inequalities. The existence problem for solutions of a system of variational inequalities has been studied by Husain and Tarafdar [3].

System of variational inequalities arises in double porosity models and diffusion through a composite media, description of parallel membranes, *etc.*; see [4] for details.

In this paper, we consider the following system of generalized mixed variational inequalities (SGMVI). Find $x^*, y^* \in H$ such that

$$\begin{cases} \langle \rho_1 T_1(y^*, x^*) + g_1(x^*) - g_1(y^*), x - g_1(x^*)\rangle + \varphi(x) - \varphi(g_1(x^*)) \geq 0, \\ \langle \rho_2 T_2(x^*, y^*) + g_2(y^*) - g_2(x^*), x - g_2(y^*)\rangle + \varphi(x) - \varphi(g_2(y^*)) \geq 0 \end{cases} \tag{1.1}$$

for all $x \in H$ and $\rho_1, \rho_2 > 0$, where $T_1, T_2 : H \times H \to H$ are nonlinear mappings and $g_1, g_2 : H \to H$ are any mappings.

If $T_1, T_2 : H \to H$ are univariate mappings then the problem (SGMVI) reduced to the following. Find $x^*, y^* \in H$ such that

$$\begin{cases} \langle \rho_1 T_1(y^*) + g_1(x^*) - g_1(y^*), x - g_1(x^*)\rangle + \varphi(x) - \varphi(g_1(x^*)) \geq 0, \\ \langle \rho_2 T_2(x^*) + g_2(y^*) - g_2(x^*), x - g_2(y^*)\rangle + \varphi(x) - \varphi(g_2(y^*)) \geq 0 \end{cases} \tag{1.2}$$

for all $x \in H$ and $\rho_1, \rho_2 > 0$.

If $T_1 = T_2 = T$ and $g_1 = g_2 = I$, then the problem (SGMVI) reduces to the following system of mixed variational inequalities considered by [5, 6]. Find $x^*, y^* \in H$ such that

$$\begin{cases} \langle \rho_1 T(y^*, x^*) + x^* - y^*, x - x^*\rangle + \varphi(x) - \varphi(x^*) \geq 0, \\ \langle \rho_2 T(x^*, y^*) + y^* - x^*, x - y^*\rangle + \varphi(x) - \varphi(y^*) \geq 0 \end{cases} \tag{1.3}$$

for all $x \in H$ and $\rho_1, \rho_2 > 0$.

If K is closed convex set in H and $\varphi(x) = \delta_K(x)$ for all $x \in K$, where δ_K is the indicator function of K defined by

$$\delta_K(x) = \begin{cases} 0, & \text{if } x \in K; \\ +\infty, & \text{otherwise,} \end{cases}$$

then the problem (1.1) reduces to the following system of general variational inequality problem: Find $x^*, y^* \in K$ such that

$$\begin{cases} \langle \rho_1 T_1(y^*, x^*) + g_1(x^*) - g_1(y^*), x - g_1(x^*)\rangle \geq 0, \\ \langle \rho_2 T_2(x^*, y^*) + g_2(y^*) - g_2(x^*), x - g_2(y^*)\rangle \geq 0 \end{cases} \tag{1.4}$$

for all $x \in K$ and $\rho_1, \rho_2 > 0$. The problem (1.4) with $g_1 = g_2$ has been studied by [7].

If $T_1 = T_2 = T$ and $g_1 = g_2 = I$, then the problem (1.4) reduces to the following system of general variational inequality problem. Find $x^*, y^* \in K$ such that

$$\begin{cases} \langle \rho_1 T(y^*, x^*) + x^* - y^*, x - x^*\rangle \geq 0, \\ \langle \rho_2 T(x^*, y^*) + y^* - x^*, x - y^*\rangle \geq 0 \end{cases} \tag{1.5}$$

for all $x \in K$ and $\rho_1, \rho_2 > 0$. The problem (1.5) is studied by Verma [8, 9] and Chang *et al.* [10].

In the study of variational inequalities, projection methods and its variant form has played an important role. Due to presence of the nonlinear term φ, the projection method and its variant forms cannot be extended to suggest iterative methods for solving mixed variational inequalities. If the nonlinear term φ in the mixed variational inequalities is a proper, convex and lower semicontinuous function, then the variational inequalities involving the nonlinear term φ are equivalent to the fixed point problems and resolvent equations. Hassouni and Moudafi [11] used the resolvent operator technique to study a new class of mixed variational inequalities.

For a multivalued operator $T : H \to H$, the domain of T, the range of T and the graph of T denote by

$$D(T) = \{u \in H : T(u) \neq \emptyset\}, \qquad R(T) = \bigcup_{u \in H} T(u)$$

and

$$\operatorname{Graph}(T) = \{(u, u^*) \in H \times H : u \in D(T) \text{ and } u^* \in T(u)\},$$

respectively.

Definition 1.1 T is called *monotone* if and only if for each $u \in D(T)$, $v \in D(T)$ and $u^* \in T(u)$, $v^* \in T(v)$, we have

$$\langle v^* - u^*, v - u \rangle \geq 0.$$

T is maximal monotone if it is monotone and its graph is not properly contained in the graph of any other monotone operator.

T^{-1} is the operator defined by $v \in T^{-1}(u) \Leftrightarrow u \in T(v)$.

Definition 1.2 ([12]) For a maximal monotone operator T, the *resolvent operator* associated with T, for any $\sigma > 0$, is defined as

$$J_T(u) = (I + \sigma T)^{-1}(u), \quad \forall u \in H.$$

It is known that a monotone operator is maximal if and only if its resolvent operator is defined everywhere. Furthermore, the resolvent operator is single-valued and nonexpansive *i.e.*, $\|J_T(x) - J_T(y)\| \leq \|x - y\|$ for all $x, y \in H$. In particular, it is well known that the subdifferential $\partial\varphi$ of φ is a maximal monotone operator; see [13].

Lemma 1.3 ([12]) *For a given $z, u \in H$ satisfies the inequality*

$$\langle u - z, x - u \rangle + \sigma\varphi(x) - \sigma\varphi(u) \geq 0, \quad \forall x \in H$$

if and only if $u = J_\varphi(z)$, where $J_\varphi = (I + \sigma\partial\varphi)^{-1}$ is the resolvent operator and $\sigma > 0$ is a constant.

Using Lemma 1.3, we will establish following important relation.

Lemma 1.4 *The variational inequality problem* (1.1) *is equivalent to finding* $x^*, y^* \in H$ *such that*

$$\begin{cases} x^* = x^* - g_1(x^*) + J_\varphi(g_1(y^*) - \rho_1 T_1(y^*, x^*)), \\ y^* = y^* - g_2(y^*) + J_\varphi(g_2(x^*) - \rho_2 T_2(x^*, y^*)), \end{cases} \tag{1.6}$$

where $J_\varphi = (I + \partial\varphi)^{-1}$ *is the resolvent operator and* $\rho_1, \rho_2 > 0$.

Proof Let $x^*, y^* \in H$ be a solution of (1.1). Then for all $x \in H$, we have

$$\begin{cases} \langle \rho_1 T_1(y^*, x^*) + g_1(x^*) - g_1(y^*), x - g_1(x^*)\rangle + \varphi(x) - \varphi(g_1(x^*)) \geq 0, \\ \langle \rho_2 T_2(x^*, y^*) + g_2(y^*) - g_2(x^*), x - g_2(y^*)\rangle + \varphi(x) - \varphi(g_2(y^*)) \geq 0, \end{cases}$$

which can be written as

$$\begin{cases} \langle g_1(x^*) - (g_1(y^*) - \rho_1 T_1(y^*, x^*)), x - g_1(x^*)\rangle + \varphi(x) - \varphi(g_1(x^*)) \geq 0, \\ \langle g_2(y^*) - (g_2(x^*) - \rho_2 T_2(x^*, y^*)), x - g_2(y^*)\rangle + \varphi(x) - \varphi(g_2(y^*)) \geq 0, \end{cases}$$

using Lemma 1.3 for $\sigma = 1$, we get

$$\begin{cases} g_1(x^*) = J_\varphi(g_1(y^*) - \rho_1 T_1(y^*, x^*)), \\ g_2(y^*) = J_\varphi(g_2(x^*) - \rho_2 T_2(x^*, y^*)), \end{cases}$$

i.e.,

$$\begin{cases} x^* = x^* - g_1(x^*) + J_\varphi(g_1(y^*) - \rho_1 T_1(y^*, x^*)), \\ y^* = y^* - g_2(y^*) + J_\varphi(g_2(x^*) - \rho_2 T_2(x^*, y^*)). \end{cases}$$

This completes the proof. □

Definition 1.5 An operator $g : H \to H$ is said to be

(1) ζ*-strongly monotone* if for each $x, x' \in H$, there exists a constant $\zeta > 0$ such that

$$\langle g(x) - g(x'), x - x'\rangle \geq \zeta \|x - x'\|^2$$

for all $y, y' \in H$;

(2) η*-Lipschitz continuous* if for each $x, x' \in H$, there exists a constant $\eta > 0$ such that

$$\|g(x) - g(x')\| \leq \eta \|x - x'\|.$$

An operator $T : H \times H \to H$ is said to be

(3) *relaxed* (ω, t)*-cocoercive* with respect to the first argument if for each $x, x' \in H$, there exist constants $t > 0$ and $\omega > 0$ such that

$$\langle T(x, \cdot) - T(x', \cdot), x - x'\rangle \geq -\omega \|T(x, \cdot) - T(x', \cdot)\|^2 + t\|x - x'\|^2;$$

(4) *μ-Lipschitz continuous* with respect to the first argument if for each $x, x' \in H$, there exists a constant $\mu > 0$ such that

$$\|T(x,\cdot) - T(x',\cdot)\| \leq \mu\|x - x'\|;$$

(5) *γ-Lipschitz continuous* with respect to the second argument if for each $y, y' \in H$, there exists a constant $\gamma > 0$ such that

$$\|T(\cdot,y) - T(\cdot,y')\| \leq \gamma\|y - y'\|.$$

Lemma 1.6 ([14]) *Let $\{a_n\}$ and $\{b_n\}$ be two nonnegative real sequences satisfying the following conditions*:

$$a_{n+1} \leq (1 - d_n)a_n + b_n, \quad \forall n \geq n_0,$$

where n_0 is some nonnegative integer, $d_n \in (0,1)$ with $\sum_{n=0}^{\infty} d_n = \infty$ and $b_n = o(d_n)$, then $a_n \to 0$ as $n \to \infty$.

Several iterative algorithms have been devised to study existence and approximation of different classes of variational inequalities. Most of them are sequential iterative methods, when we implement such algorithms on computers, then only one processor is used at a time. Availability of multiprocessor computers enabled researchers to develop iterative algorithms having the parallel characteristics. Lions [15] studied a parallel algorithm for a solution of parabolic variational inequalities. Bertsekas and Tsitsiklis [16, 17] developed parallel algorithm using the metric projection. Recently, Yang *et al.* [7] studied parallel projection algorithm for a system of nonlinear variational inequalities.

2 Existence and convergence

Lemma 1.4 established the equivalence between the fixed-point problem and the variational inequality problem (1.1). Using this equivalence in this section, we construct a parallel iterative algorithm to approximate the solution of the problem (1.1) and study the convergence of the sequence generated by the algorithm.

Algorithm 2.1 For arbitrary chosen points $x_0, y_0 \in H$, compute the sequences $\{x_n\}$ and $\{y_n\}$ such that

$$\begin{cases} x_{n+1} = x_n - g_1(x_n) + J_\varphi(g_1(y_n) - \rho_1 T_1(y_n, x_n)), \\ y_{n+1} = y_n - g_2(y_n) + J_\varphi(g_2(x_n) - \rho_2 T_2(x_n, y_n)), \end{cases} \tag{2.1}$$

where $J_\varphi = (I + \partial\varphi)^{-1}$ is the resolvent operator and ρ_1, ρ_2 is positive real numbers.

Theorem 2.2 *Let H be a real Hilbert space. Let $T_i : H \times H \to H$ and $g_i : H \to H$ be mappings such that T_i is relaxed (ω_i, t_i)-cocoercive, μ_i-Lipschitz continuous with respect to the first argument, γ_i-Lipschitz continuous with respect to the second argument and g_i is η_i-Lipschitz continuous, ζ_i-strongly monotone mapping for $i = 1, 2$. Assume that the following* *assumptions hold*:

$$\left|\rho_1 - \frac{t_i - \gamma_i(1-\kappa) - \omega_i\mu_i^2}{(\mu_i^2 - \gamma_i^2)}\right| < \frac{\sqrt{(\omega_i\mu_i^2 + \gamma_i(1-\kappa) - t_i)^2 - (\mu_i^2 - \gamma_i^2)\kappa(2-\kappa)}}{(\mu_i^2 - \gamma_i^2)},$$

$$\left|\omega_i\mu_i^2 + \gamma_i(1-\kappa) - t_i\right| > \sqrt{\left(\mu_i^2 - \gamma_i^2\right)\kappa(2-\kappa)},$$

where $\kappa = \sum_{i=1}^{2}\sqrt{1 - 2\zeta_i + \eta_i^2} < 1$.

Then there exist $x^*, y^* \in H$, *which solves the problem* (1.1). *Moreover, the iterative sequences* $\{x_n\}$ *and* $\{y_n\}$ *generated by the Algorithm* 2.1 *converges to* x^* *and* y^*, *respectively.*

Proof Using (2.1), we have

$$\begin{aligned}
&\|x_{n+1} - x_n\| \\
&\quad = \big\|x_n - g_1(x_n) + J_\varphi\big(g_1(y_n) - \rho_1 T_1(y_n, x_n)\big) \\
&\qquad - \big[x_{n-1} - g_1(x_{n-1}) + J_\varphi\big(g_1(y_{n-1}) - \rho_1 T_1(y_{n-1}, x_{n-1})\big)\big]\big\| \\
&\quad \le \big\|x_n - x_{n-1} - \big(g_1(x_n) - g_1(x_{n-1})\big)\big\| \\
&\qquad + \big\|J_\varphi\big(g_1(y_n) - \rho_1 T_1(y_n, x_n)\big) - J_\varphi\big(g_1(y_{n-1}) - \rho_1 T_1(y_{n-1}, x_{n-1})\big)\big\| \\
&\quad \le \big\|x_n - x_{n-1} - \big(g_1(x_n) - g_1(x_{n-1})\big)\big\| \\
&\qquad + \big\|g_1(y_n) - g_1(y_{n-1}) - \rho_1\big(T_1(y_n, x_n) - T_1(y_{n-1}, x_{n-1})\big)\big\| \\
&\quad \le \big\|x_n - x_{n-1} - \big(g_1(x_n) - g_1(x_{n-1})\big)\big\| + \big\|\big(g_1(y_n) - g_1(y_{n-1})\big) - (y_n - y_{n-1})\big\| \\
&\qquad + \big\|y_n - y_{n-1} - \rho_1\big(T_1(y_n, x_n) - T_1(y_{n-1}, x_n)\big)\big\| \\
&\qquad + \rho_1\big\|T_1(y_{n-1}, x_n) - T_1(y_{n-1}, x_{n-1})\big\|.
\end{aligned} \tag{2.2}$$

Since T_1 is relaxed (ω_1, t_1)-cocoercive and μ_1-Lipschitz continuous in the first argument, we have

$$\begin{aligned}
&\big\|y_n - y_{n-1} - \rho_1\big(T_1(y_n, x_n) - T_1(y_{n-1}, x_n)\big)\big\|^2 \\
&\quad = \|y_n - y_{n-1}\|^2 - 2\rho_1\big\langle T_1(y_n, x_n) - T_1(y_{n-1}, x_n), y_n - y_{n-1}\big\rangle \\
&\qquad + \rho_1^2\big\|T_1(y_n, x_n) - T_1(y_{n-1}, x_n)\big\|^2 \\
&\quad \le \|y_n - y_{n-1}\|^2 + 2\rho_1\omega_1\big\|T_1(y_n, x_n) - T_1(y_{n-1}, x_n)\big\|^2 \\
&\qquad - 2\rho_1 t_1\|y_n - y_{n-1}\|^2 + \rho_1^2\big\|T_1(y_n, x_n) - T_1(y_{n-1}, x_n)\big\|^2 \\
&\quad \le \|y_n - y_{n-1}\|^2 + 2\rho_1\omega_1\mu_1^2\|y_n - y_{n-1}\|^2 - 2\rho_1 t_1\|y_n - y_{n-1}\|^2 + \rho_1^2\mu_1^2\|y_n - y_{n-1}\|^2 \\
&\quad = \big(1 + 2\rho_1\omega_1\mu_1^2 - 2\rho_1 t_1 + \rho_1^2\mu_1^2\big)\|y_n - y_{n-1}\|^2.
\end{aligned} \tag{2.3}$$

Since g_1 is η_1-Lipschitz continuous and ζ_1-strongly monotone,

$$\begin{aligned}
&\big\|x_n - x_{n-1} - \big(g_1(x_n) - g_1(x_{n-1})\big)\big\|^2 \\
&\quad = \|x_n - x_{n-1}\|^2 - 2\big\langle g_1(x_n) - g_1(x_{n-1}), x_n - x_{n-1}\big\rangle
\end{aligned}$$

$$\begin{aligned} &+ \| g_1(x_n) - g_1(x_{n-1}) \|^2 \\ &\le (1 - 2\zeta_1 + \eta_1^2) \|x_n - x_{n-1}\|^2. \end{aligned} \tag{2.4}$$

Similarly,

$$\| y_n - y_{n-1} - (g_1(y_n) - g_1(y_{n-1})) \|^2 \le (1 - 2\zeta_1 + \eta_1^2) \|y_n - y_{n-1}\|^2. \tag{2.5}$$

By γ_1-Lipschitz continuity of T_1 with respect to second argument,

$$\| T_1(y_{n-1}, x_n) - T_1(y_{n-1}, x_{n-1}) \| \le \gamma_1 \|x_n - x_{n-1}\|. \tag{2.6}$$

It follows from (2.2)-(2.6) that

$$\|x_{n+1} - x_n\| \le (\psi_1 + \rho_1\gamma_1)\|x_n - x_{n-1}\| + (\psi_1 + \theta_1)\|y_n - y_{n-1}\|, \tag{2.7}$$

where $\psi_1 = \sqrt{1 - 2\zeta_1 + \eta_1^2}$ and $\theta_1 = \sqrt{1 + 2\rho_1\omega_1\mu_1^2 - 2\rho_1 t_1 + \rho_1^2\mu_1^2}$.

Similarly, we get

$$\|y_{n+1} - y_n\| \le (\psi_2 + \theta_2)\|x_n - x_{n-1}\| + (\psi_2 + \rho_2\gamma_2)\|y_n - y_{n-1}\|, \tag{2.8}$$

where $\psi_2 = \sqrt{1 - 2\zeta_2 + \eta_2^2}$ and $\theta_2 = \sqrt{1 + 2\rho_2\omega_2\mu_2^2 - 2\rho_2 t_2 + \rho_2^2\mu_2^2}$.

Now (2.7) and (2.8) imply

$$\begin{aligned} \|x_{n+1} - x_n\| + \|y_{n+1} - y_n\| &\le (\psi_1 + \psi_2 + \theta_2 + \rho_1\gamma_1)\|x_n - x_{n-1}\| \\ &\quad + (\psi_1 + \psi_2 + \theta_1 + \rho_2\gamma_2)\|y_n - y_{n-1}\| \\ &\le \Theta(\|x_n - x_{n-1}\| + \|y_n - y_{n-1}\|), \end{aligned}$$

where $\Theta = \max\{(\psi_1 + \psi_2 + \theta_2 + \rho_1\gamma_1), (\psi_1 + \psi_2 + \theta_1 + \rho_2\gamma_2)\} < 1$ by assumption. Hence $\{x_n\}$ and $\{y_n\}$ are both Cauchy sequences in H, and $\{x_n\}$ converges to $x^* \in H$ and $\{y_n\}$ converges to $y^* \in H$. Since g_1, g_2, T_1, T_2 and J_φ are all continuous, we have

$$\begin{cases} x^* = x^* - g_1(x^*) + J_\varphi(g_1(y^*) - \rho_1 T_1(y^*, x^*)), \\ y^* = y^* - g_2(y^*) + J_\varphi(g_2(x^*) - \rho_2 T_2(x^*, y^*)). \end{cases}$$

The result follows from Lemma 1.4. This completes the proof. □

If $T_1, T_2 : H \to H$ are univariate mappings, then the Algorithm 2.1 reduces to the following.

Algorithm 2.3 For arbitrary chosen points $x_0, y_0 \in H$, compute the sequences $\{x_n\}$ and $\{y_n\}$ such that

$$\begin{cases} x_{n+1} = x_n - g_1(x_n) + J_\varphi(g_1(y_n) - \rho_1 T_1(y_n)), \\ y_{n+1} = y_n - g_2(y_n) + J_\varphi(g_2(x_n) - \rho_2 T_2(x_n)), \end{cases}$$

where $J_\varphi = (I + \partial\varphi)^{-1}$ is the resolvent operator and ρ_1, ρ_2 is positive real numbers.

Theorem 2.4 *Let H be a real Hilbert space. Let $T_i, g_i : H \to H$ be mappings such that T_i is relaxed (ω_i, t_i)-cocoercive, μ_i-Lipschitz continuous and g_i is η_i-Lipschitz continuous, ζ_i-strongly monotone mapping for $i = 1, 2$. Assume that the following assumptions hold*:

$$\left|\rho_1 - \frac{t_i - \omega_i \mu_i^2}{\mu_i^2}\right| < \frac{\sqrt{(\omega_i \mu_i^2 - t_i)^2 - \mu_i^2 \kappa(2-\kappa)}}{\mu_i^2},$$
$$\left|\omega_i \mu_i^2 - t_i\right| > \mu_i \sqrt{\kappa(2-\kappa)},$$

where $\kappa = \sum_{i=1}^{2} \sqrt{1 - 2\zeta_i + \eta_i^2} < 1$.

Then there exist $x^*, y^* \in H$, which solves the problem (1.2). *Moreover the iterative sequences $\{x_n\}$ and $\{y_n\}$ generated by the Algorithm* 2.3 *converges to x^* and y^*, respectively.*

3 Relaxed algorithm and approximation solvability

Lemma 1.4 implies that the system of general mixed variational inequality problem (1.1) is equivalent to the fixed-point problem. This alternative equivalent formulation is very useful for a numerical point of view. In this section, we construct a relaxed iterative algorithm for solving the problem (1.1) and study the convergence of the iterative sequence generated by the algorithm.

Algorithm 3.1 For arbitrary chosen points $x_0, y_0 \in H$, compute the sequences $\{x_n\}$ and $\{y_n\}$ such that

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n(x_n - g_1(x_n) + J_\varphi(g_1(y_n) - \rho_1 T_1(y_n, x_n))), \\ y_{n+1} = (1-\beta_n)y_n + \beta_n(y_n - g_2(y_n) + J_\varphi(g_2(x_n) - \rho_2 T_2(x_n, y_n))), \end{cases} \tag{3.1}$$

where $J_\varphi = (I + \partial\varphi)^{-1}$ is the resolvent operator, $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $[0, 1]$ and ρ_1, ρ_2 is positive real numbers.

We first prove a result, which will be helpful to prove main result of this section.

Lemma 3.2 *Let H be a real Hilbert space. Let $\{x_n\}$ and $\{y_n\}$ be sequences in H such that*

$$\left\|x_{n+1} - x^*\right\| + \left\|y_{n+1} - y^*\right\| \leq \max\{(1-t_n), (1-s_n)\}\left(\left\|x_n - x^*\right\| + \left\|y_n - y^*\right\|\right) \tag{3.2}$$

for some $x^*, y^* \in H$, where $\{s_n\}$ and $\{t_n\}$ are sequences in $(0, 1)$ such that $\sum_{n=0}^{\infty} t_n = \infty$ and $\sum_{n=0}^{\infty} s_n = \infty$. Then $\{x_n\}$ and $\{y_n\}$ converges to x^* and y^*, respectively.

Proof Now, define the norm $\|\cdot\|_1$ on $H \times H$ by

$$\left\|(x, y)\right\|_1 = \|x\| + \|y\|, \quad \forall (x, y) \in H \times H.$$

Then $(H \times H, \|\cdot\|_1)$ is a Banach space. Hence, (3.2) implies that

$$\left\|(x_{n+1}, y_{n+1}) - (x^*, y^*)\right\|_1 \leq \max\{(1-t_n), (1-s_n)\}\left\|(x_n, y_n) - (x^*, y^*)\right\|_1.$$

Using Lemma 1.6, we get

$$\lim_{n\to\infty}\left\|(x_n,y_n)-\left(x^*,y^*\right)\right\|_1=0.$$

Therefore, sequences $\{x_n\}$ and $\{y_n\}$ converges to x^* and y^*, respectively. This completes the proof. □

We now present the approximation solvability of the problem (1.1).

Theorem 3.3 *Let H be a real Hilbert space H. Let $T_i : H \times H \to H$ and $g_i : H \to H$ be mappings such that T_i is relaxed (ω_i, t_i)-cocoercive, μ_i-Lipschitz continuous with respect to the first argument, γ_i-Lipschitz continuous with respect to the second argument and g_i is η_i-Lipschitz continuous, ζ_i-strongly monotone mapping for $i = 1,2$. Suppose that $x^*, y^* \in H$ be a solution of the problem* (1.1) *and $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in* $[0,1]$. *Assume that the following assumptions hold*:

(i) $0 < \Theta_{1n} = \alpha_n(1-(\psi_1+\rho_1\gamma_1)) - \beta_n(\psi_2+\theta_2) < 1$,

(ii) $0 < \Theta_{2n} = \beta_n(1-(\psi_2+\rho_2\gamma_2)) - \alpha_n(\psi_1+\theta_1) < 1$,

(iii) $\sum_{n=0}^{\infty}\Theta_{1n} = \infty$ *and* $\sum_{n=0}^{\infty}\Theta_{2n} = \infty$,

where

$$\theta_i=\sqrt{1+2\rho_i\omega_i\mu_i^2-2\rho_1 t_i+\rho_i^2\mu_i^2},\qquad \psi_i=\sqrt{1-2\zeta_i+\eta_i^2},\quad i=1,2.$$

Then the sequences $\{x_n\}$ and $\{y_n\}$ generated by the Algorithm 3.1 *converges to x^* and y^*, respectively.*

Proof From Theorem 2.2 the problem (1.1) has a solution (x^*, y^*) in H. By Lemma 1.4, we have

$$\begin{cases} x^* = x^* - g_1(x^*) + J_\varphi(g_1(y^*) - \rho_1 T_1(y^*, x^*)), \\ y^* = y^* - g_2(y^*) + J_\varphi(g_2(x^*) - \rho_2 T_2(x^*, y^*)). \end{cases} \tag{3.3}$$

To prove the result, we first evaluate $\|x_{n+1}-x^*\|$ for all $n \ge 0$. Using (3.1), we obtain

$$\begin{aligned}
&\left\|x_{n+1}-x^*\right\| \\
&\quad\le \left\|(1-\alpha_n)x_n+\alpha_n\left(x_n-g_1(x_n)+J_\varphi\left(g_1(y_n)-\rho_1T_1(y_n,x_n)\right)\right)-x^*\right\| \\
&\quad\le (1-\alpha_n)\left\|x_n-x^*\right\|+\alpha_n\left\|x_n-x^*-\left(g_1(x_n)-g_1\left(x^*\right)\right)\right\| \\
&\qquad+\alpha_n\left\|J_\varphi\left(g_1(y_n)-\rho_1T_1(y_n,x_n)\right)-J_\varphi\left(g_1\left(x^*\right)-\rho_1T_1\left(y^*,x^*\right)\right)\right\| \\
&\quad\le (1-\alpha_n)\left\|x_n-x^*\right\|+\alpha_n\left\|x_n-x^*-\left(g_1(x_n)-g_1\left(x^*\right)\right)\right\| \\
&\qquad+\alpha_n\left\|g_1(y_n)-g_1\left(y^*\right)-\left(y_n-y^*\right)\right\| \\
&\qquad+\alpha_n\left\|y_n-y^*-\rho_1\left(T_1(y_n,x_n)-T_1\left(y^*,x_n\right)\right)\right\| \\
&\qquad+\alpha_n\rho_1\left\|T_1\left(y^*,x_n\right)-T_1\left(y^*,x^*\right)\right\|.
\end{aligned} \tag{3.4}$$

Since T_1 is relaxed (ω_1, t_1)-cocoercive and μ_1-Lipschitz mapping with respect to the first argument, we have

$$\begin{aligned}
&\|y_n - y^* - \rho_1(T_1(y_n, x_n) - T_1(y^*, x_n))\|^2 \\
&\quad = \|y_n - y^*\|^2 - 2\rho_1\langle T_1(y_n, x_n) - T_1(y^*, x_n), y_n - y^*\rangle \\
&\qquad + \rho_1^2\|T_1(y_n, x_n) - T_1(y^*, x_n)\|^2 \\
&\quad \le \|y_n - y^*\|^2 + 2\rho_1\omega_1\|T_1(y_n, x_n) - T_1(y^*, x_n)\|^2 \\
&\qquad - 2\rho_1 t_1\|y_n - y^*\|^2 + \rho_1^2\|T_1(y_n, x_n) - T_1(y^*, x_n)\|^2 \\
&\quad \le (1 + 2\rho_1\omega_1\mu_1^2 - 2\rho_1 t_1 + \rho_1^2\mu_1^2)\|y_n - y^*\|^2.
\end{aligned} \tag{3.5}$$

Since g_1 is η_1-Lipschitz continuous and ζ_1-strongly monotone,

$$\begin{aligned}
&\|x_n - x^* - (g_1(x_n) - g_1(x^*))\|^2 \\
&\quad = \|x_n - x^*\|^2 - 2\langle g_1(x_n) - g_1(x^*), x_n - x^*\rangle + \|g_1(x_n) - g_1(x^*)\|^2 \\
&\quad \le (1 - 2\zeta_1 + \eta_1^2)\|x_n - x^*\|^2.
\end{aligned} \tag{3.6}$$

Similarly, we have

$$\|y_n - y^* - (g_1(y_n) - g_1(y^*))\|^2 \le (1 - 2\zeta_1 + \eta_1^2)\|y_n - y^*\|^2. \tag{3.7}$$

By γ_1-Lipschitz continuity of T_1 with respect to second argument,

$$\|T_1(y^*, x_n) - T_1(y^*, x^*)\| \le \gamma_1\|x_n - x^*\|. \tag{3.8}$$

By (3.4)-(3.8), we have

$$\|x_{n+1} - x^*\| \le [1 - \alpha_n + \alpha_n(\psi_1 + \rho_1\gamma_1)]\|x_n - x^*\| + \alpha_n(\psi_1 + \theta_1)\|y_n - y^*\|, \tag{3.9}$$

where $\psi_1 = \sqrt{1 - 2\zeta_1 + \eta_1^2}$ and $\theta_1 = \sqrt{1 + 2\rho_1\omega_1\mu_1^2 - 2\rho_1 t_1 + \rho_1^2\mu_1^2}$.

Similarly, we have

$$\|y_{n+1} - y^*\| \le \beta_n(\psi_2 + \theta_2)\|x_n - x^*\| + [1 - \beta_n + \beta_n(\psi_2 + \rho_2\gamma_2)]\|y_n - y^*\|, \tag{3.10}$$

where $\psi_2 = \sqrt{1 - 2\zeta_2 + \eta_2^2}$ and $\theta_2 = \sqrt{1 + 2\rho_2\omega_2\mu_2^2 - 2\rho_2 t_2 + \rho_2^2\mu_2^2}$.

Now (3.9) and (3.10) imply

$$\begin{aligned}
&\|x_{n+1} - x^*\| + \|y_{n+1} - y^*\| \\
&\quad \le [1 - (\alpha_n(1 - (\psi_1 + \rho_1\gamma_1)) - \beta_n(\psi_2 + \theta_2))]\|x_n - x^*\| \\
&\qquad + [1 - (\beta_n(1 - (\psi_2 + \rho_2\gamma_2)) - \alpha_n(\psi_1 + \theta_1))]\|y_n - y^*\| \\
&\quad \le \max\{(1 - \Theta_{1n}), (1 - \Theta_{2n})\}(\|x_n - x^*\| + \|y_n - y^*\|),
\end{aligned}$$

where

$$\Theta_{1n} = \alpha_n\bigl(1 - (\psi_1 + \rho_1\gamma_1)\bigr) - \beta_n(\psi_2 + \theta_2),$$
$$\Theta_{2n} = \beta_n\bigl(1 - (\psi_2 + \rho_2\gamma_2)\bigr) - \alpha_n(\psi_1 + \theta_1).$$

By the assumptions and Lemma 3.2, we get that the sequences $\{x_n\}$ and $\{y_n\}$ converges to x^* and y^*, respectively. This completes the proof. □

Remark 3.4 Theorem 3.3 extend and generalize the main result in [5], which itself is a extension and improvement of the main result in Chang *et al.* [10].

If $T_1, T_2 : H \to H$ are univariate mappings, then the Algorithm 3.1 reduces to the following.

Algorithm 3.5 For arbitrary chosen points $x_0, y_0 \in H$, compute the sequences $\{x_n\}$ and $\{y_n\}$ such that

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n(x_n - g_1(x_n) + J_\varphi(g_1(y_n) - \rho_1 T_1(y_n))), \\ y_{n+1} = (1-\beta_n)y_n + \beta_n(y_n - g_2(y_n) + J_\varphi(g_2(x_n) - \rho_2 T_2(x_n))), \end{cases}$$

where $J_\varphi = (I + \partial\varphi)^{-1}$ is the resolvent operator, $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $[0,1]$ and ρ_1, ρ_2 is positive real numbers.

As a consequence of Theorem 3.3, we have following result.

Corollary 3.6 *Let H be a real Hilbert space H. Let $T_i, g_i : H \to H$ be mappings such that T_i is relaxed (ω_i, t_i)-cocoercive, μ_i-Lipschitz continuous and g_i is η_i-Lipschitz continuous, ζ_i-strongly monotone mapping for $i = 1, 2$. Suppose that $x^*, y^* \in H$ be a solution of the problem* (1.2) *and $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in* $[0,1]$. *Assume that the following assumptions hold*:

(i) $0 < \Theta_{1n} = \alpha_n(1 - \psi_1) - \beta_n(\psi_2 + \theta_2) < 1$,
(ii) $0 < \Theta_{2n} = \beta_n(1 - \psi_2) - \alpha_n(\psi_1 + \theta_1) < 1$,
(iii) $\sum_{n=0}^{\infty} \Theta_{1n} = \infty$ *and* $\sum_{n=0}^{\infty} \Theta_{2n} = \infty$,

where

$$\theta_i = \sqrt{1 + 2\rho_i\omega_i\mu_i^2 - 2\rho_1 t_i + \rho_i^2\mu_i^2}, \qquad \psi_i = \sqrt{1 - 2\zeta_i + \eta_i^2}, \quad i = 1, 2.$$

Then the sequences $\{x_n\}$ and $\{y_n\}$ generated by the Algorithm 3.5 *converges to x^* and y^*, respectively.*

4 Algorithms for common element

Now, we consider, the approximation solvability of the system (1.1) which is also a common fixed point of two Lipschitzian mappings. We propose a relaxed two-step algorithm, which can be applied to the approximation of solution of the problem (1.1) and common fixed point of two Lipschitzian mappings.

Algorithm 4.1 For arbitrary chosen points $x_0, y_0 \in H$, compute the sequences $\{x_n\}$ and $\{y_n\}$ such that

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n S_1(x_n - g_1(x_n) + J_\varphi(g_1(y_n) - \rho_1 T_1(y_n, x_n))), \\ y_{n+1} = (1-\beta_n)y_n + \beta_n S_2(y_n - g_2(y_n) + J_\varphi(g_2(x_n) - \rho_2 T_2(x_n, y_n))), \end{cases} \tag{4.1}$$

where $J_\varphi = (I + \partial\varphi)^{-1}$ is the resolvent operator, $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $[0,1]$ and ρ_1, ρ_2 be positive real numbers.

Let $F(S_i)$ denote the set of fixed points of the mapping S_i, *i.e.*, $F(S_i) = \{x \in H : S_i x = x\}$, $\operatorname{Fix}(S) = \bigcap_{i=1}^{2} F(S_i)$ and $\mathcal{SOL}\,(1.1)$ the set of solutions of the problem (1.1).

Theorem 4.2 *Let H be a real Hilbert space H. Let $T_i : H \times H \to H$ and $g_i : H \to H$ be mappings such that T_i is relaxed (ω_i, t_i)-cocoercive, μ_i-Lipschitz continuous with respect to the first argument, γ_i-Lipschitz continuous with respect to the second argument and g_i is η_i-Lipschitz continuous, ζ_i-strongly monotone mapping for $i = 1, 2$. Let $S_i : H \to H$ be ϑ_i-Lipschitzian mapping for $i = 1, 2$ with* $\operatorname{Fix}(S) \neq \emptyset$, $\{\alpha_n\}$, $\{\beta_n\}$ *are sequences in* $[0,1]$. *Assume that the following assumptions hold*:

(i) $0 < \Theta_{1n} = \alpha_n \vartheta(1 - (\psi_1 + \rho_1\gamma_1)) - \beta_n \vartheta(\psi_2 + \theta_2) < 1$,
(ii) $0 < \Theta_{2n} = \beta_n \vartheta(1 - (\psi_2 + \rho_2\gamma_2)) - \alpha_n \vartheta(\psi_1 + \theta_1) < 1$,
(iii) $\sum_{n=0}^{\infty} \Theta_{1n} = \infty$ *and* $\sum_{n=0}^{\infty} \Theta_{2n} = \infty$,

where $\vartheta = \max\{\vartheta_1, \vartheta_2\}$ *and*

$$\theta_i = \sqrt{1 + 2\rho_i\omega_i\mu_i^2 - 2\rho_1 t_i + \rho_i^2\mu_i^2}, \qquad \psi_i = \sqrt{1 - 2\zeta_i + \eta_i^2}, \quad i = 1, 2.$$

If $\mathcal{SOL}\,(1.1) \cap \operatorname{Fix}(S) \neq \emptyset$, *then the sequences* $\{x_n\}$ *and* $\{y_n\}$ *generated by the Algorithm* 4.1 *converges to* x^* *and* y^*, *respectively, such that* $(x^*, y^*) \in \mathcal{SOL}\,(1.1)$ *and* $\{x^*, y^*\} \in \operatorname{Fix}(S)$.

Proof Let us have $(x^*, y^*) \in \mathcal{SOL}\,(1.1)$ and $\{x^*, y^*\} \in \operatorname{Fix}(S)$. By Lemma 1.4, we have

$$\begin{cases} x^* = x^* - g_1(x^*) + J_\varphi(g_1(y^*) - \rho_1 T_1(y^*, x^*)), \\ y^* = y^* - g_2(y^*) + J_\varphi(g_2(x^*) - \rho_2 T_2(x^*, y^*)). \end{cases}$$

Also since $\{x^*, y^*\} \in \operatorname{Fix}(S)$, we have

$$\begin{cases} x^* = S_1(x^* - g_1(x^*) + J_\varphi(g_1(y^*) - \rho_1 T_1(y^*, x^*))), \\ y^* = S_2(y^* - g_2(y^*) + J_\varphi(g_2(x^*) - \rho_2 T_2(x^*, y^*))). \end{cases}$$

To prove the result, we first evaluate $\|x_{n+1} - x^*\|$ for all $n \geq 0$. Using (4.1), we obtain

$$\begin{aligned} &\|x_{n+1} - x^*\| \\ &\quad \leq \|(1-\alpha_n)x_n + \alpha_n S_1(x_n - g_1(x_n) + J_\varphi(g_1(y_n) - \rho_1 T_1(y_n, x_n))) - x^*\| \\ &\quad \leq (1-\alpha_n)\|x_n - x^*\| \\ &\qquad + \alpha_n \|S_1(x_n - g_1(x_n) + J_\varphi(g_1(y_n) - \rho_1 T_1(y_n, x_n))) - S_1 x^*\| \end{aligned}$$

$$\begin{aligned}
&\le (1-\alpha_n)\|x_n - x^*\| + \alpha_n\vartheta_1\|x_n - x^* - (g_1(x_n) - g_1(x^*))\| \\
&\quad + \alpha_n\vartheta_1\|J_\varphi(g_1(y_n) - \rho_1 T_1(y_n, x_n)) - J_\varphi(g_1(y^*) - \rho_1 T_1(y^*, x^*))\| \\
&\le (1-\alpha_n)\|x_n - x^*\| + \alpha_n\vartheta_1\|x_n - x^* - (g_1(x_n) - g_1(x^*))\| \\
&\quad + \alpha_n\vartheta_1\|g_1(y_n) - g_1(y^*) - (y_n - y^*)\| \\
&\quad + \alpha_n\vartheta_1\|y_n - y^* - \rho_1(T_1(y_n, x_n) - T_1(y^*, x_n))\| \\
&\quad + \alpha_n\vartheta_1\rho_1\|T_1(y^*, x_n) - T_1(y^*, x^*)\|. \qquad (4.2)
\end{aligned}$$

Using the arguments as in the proof of Theorem 3.3, from (4.2) we get that

$$\|x_{n+1} - x^*\| \le [1 - \alpha_n + \alpha_n\vartheta_1(\psi_1 + \rho_1\gamma_1)]\|x_n - x^*\| + \alpha_n\vartheta_1(\psi_1 + \theta_1)\|y_n - y^*\|, \qquad (4.3)$$

where $\psi_1 = \sqrt{1 - 2\zeta_1 + \eta_1^2}$ and $\theta_1 = \sqrt{1 + 2\rho_1\omega_1\mu_1^2 - 2\rho_1 t_1 + \rho_1^2\mu_1^2}$.

Similarly, we get

$$\|y_{n+1} - y^*\| \le \beta_n\vartheta_2(\psi_2 + \theta_2)\|x_n - x^*\| + [1 - \beta_n + \beta_n\vartheta_2(\psi_2 + \rho_2\gamma_2)]\|y_n - y^*\|, \qquad (4.4)$$

where $\psi_2 = \sqrt{1 - 2\zeta_2 + \eta_2^2}$ and $\theta_2 = \sqrt{1 + 2\rho_2\omega_2\mu_1^2 - 2\rho_2 t_2 + \rho_2^2\mu_2^2}$.

Adding (4.3) and (4.4), taking $\vartheta = \max\{\vartheta_1, \vartheta_2\}$ we get

$$\begin{aligned}
&\|x_{n+1} - x^*\| + \|y_{n+1} - y^*\| \\
&\quad \le [1 - (\alpha_n(1 - \vartheta(\psi_1 + \rho_1\gamma_1)) - \beta_n\vartheta(\psi_2 + \theta_2))]\|x_n - x^*\| \\
&\qquad + [1 - (\beta_n(1 - \vartheta(\psi_2 + \rho_2\gamma_2)) - \alpha_n\vartheta(\psi_1 + \theta_1))]\|y_n - y^*\| \\
&\quad \le \max\{(1 - \Theta_{1n}), (1 - \Theta_{2n})\}(\|x_n - x^*\| + \|y_n - y^*\|),
\end{aligned}$$

where

$$\begin{aligned}
\Theta_{1n} &= \alpha_n(1 - \vartheta(\psi_1 + \rho_1\gamma_1)) - \vartheta\beta_n(\psi_2 + \theta_2), \\
\Theta_{2n} &= \beta_n(1 - \vartheta(\psi_2 + \rho_2\gamma_2)) - \vartheta\alpha_n(\psi_1 + \theta_1).
\end{aligned}$$

By the assumptions and Lemma 3.2, we get that the sequences $\{x_n\}$ and $\{y_n\}$ converges to x^* and y^*, respectively. This completes the proof. □

A mapping $S : H \to H$ is said to be *asymptotically λ-strictly pseudocontractive* [18] if there exist a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\|S^n x - S^n y\|^2 \le k_n^2\|x - y\|^2 + \lambda\|(x - S^n x) - (y - S^n y)\|^2$$

for some $\lambda \in (0, 1)$, for all $x, y \in H$ and $n \ge 1$.

Kim and Xu [19] proved that, if $S : H \to H$ is an asymptotically λ-strictly pseudocontractive mapping, then S^n is a Lipschitzian mapping with Lipschitz constant

$$L_n = \frac{\lambda + \sqrt{1 + (k_n^2 - 1)(1 - \lambda)}}{1 - \lambda}$$

for each integer $n > 1$.

Also if $x^* \in F(S)$, then $x^* \in F(S^n)$ for all integer $n \geq 1$.

Assume that $S_i : H \to H$ is asymptotically λ_i-strictly pseudocontractive mappings for $i = 1, 2$ with $\bigcap_{i=1}^2 F(S_i) \neq \emptyset$. Now generate sequence $\{x_n\}$ and $\{y_n\}$ by Algorithm 4.1 with $S_1 := S_1^j$ and $S_2 := S_2^k$ for some integer $j, k > 1$. Theorem 4.2 can be applied to study approximate solvability of the problem (1.1) and common fixed points of two asymptotically strictly pseudocontractive mappings.

A mapping $S : H \to H$ is said to be *asymptotically nonexpansive* [20] if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that $\|S^n x - S^n y\| \leq k_n \|x - y\|$ for all $x, y \in K$ and $n \geq 1$. Clearly every asymptotically nonexpansive mapping is an asymptotically 0-strictly pseudocontractive mapping. Theorem 4.2 can be applied to study approximate solvability of the problem (1.1) and common fixed points of two asymptotically nonexpansive mappings.

Remark 4.3 An important feature of the algorithms used in the paper is its suitability for implementing on multiprocessor computer. Assume that $\{x_n\}$ and $\{y_n\}$ are given, in order to get the new iterative point; we can set one processor of computer to compute $\{x_{n+1}\}$ and other processor to compute $\{y_{n+1}\}$, *i.e.*, $\{x_{n+1}\}$ and $\{y_{n+1}\}$ are computed parallel, which will take less time then computing $\{x_{n+1}\}$ and $\{y_{n+1}\}$ in a sequence using a single processor; we refer [16, 17, 21–23] and references therein for more examples and ideas of the parallel iterative methods.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

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