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N-fixed point theorems for nonlinear contractions in partially ordered metric spaces

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Abstract

In present paper we introduce the concept of a new g-monotone mapping and define the notions of n-fixed point and n-coincidence point and prove some related theorems for nonlinear contractive mappings in partially ordered complete metric spaces. Our results are generalization of the main results of Lakshmikantham and Ćirić (Nonlinear Anal. 70:4341-4349, 2009) and include several recent developments. Moreover, we give an example to support our results.

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1 Introduction and preliminaries

The notion of a coupled fixed point is introduced by Bhaskar and Lakshmikantham [1]. Afterward Lakshmikantham and Ćirić in [2] extended this notion by defining the g-monotone property in partially ordered spaces. For other results on coupled coincidence and coupled common fixed point theory, we refer the readers to ([3–8]). Many authors obtained important results for usual coincidence and common fixed points in partially ordered spaces (see, for instance, [9–13]). Recently, Berinde and Borcut [14, 15] introduced the concept of a tripled fixed point. Other authors obtained important results in this area (see, for instance, [8, 9]). Very recently Eshaghi and Ramezani [16] introduced and investigated the concept of an n-fixed point (see also Def. 2.7 [4]).

From now, $(X, \leq, d)$ is a partially ordered complete metric space. Further, the product space $X^2 = X \times X$ has the following partial order:

$$(u, v) \leq (x, y) \quad \Leftrightarrow \quad x \geq u, \qquad y \leq v \quad \text{for all } (x, y), (u, v) \in X \times X.$$

We summarize in the following the basic notions and results established in [1, 2, 14].

Definition 1.1 (See [1]) A mapping $F : X \times X \to X$ is said to have the mixed monotone property if $F(x, y)$ is monotone non-decreasing in x and is monotone non-increasing in y, that is, for any $x, y \in X$,

$$x_1 \leq x_2 \quad \Rightarrow \quad F(x_1, y) \leq F(x_2, y) \quad \text{for } x_1, x_2 \in X,$$

$$y_1 \leq y_2 \quad \Rightarrow \quad F(x, y_2) \leq F(x, y_1) \quad \text{for } y_1, y_2 \in X.$$

Definition 1.2 (See [1]) An element $(x, y) \in X \times X$ is said to be a coupled fixed point of the mapping $F : X \times X \to X$ if $F(x, y) = x$ and $F(y, x) = y$.

Theorem 1.3 (See [1]) *Let $F : X \times X \to X$ be a mapping having the mixed monotone property on X. Assume that there exists $k \in [0, 1)$ with*

$$d\big(F(x,y),F(u,v)\big) \le \frac{k}{2}\big(d(x,u) + d(y,v)\big) \quad \text{for each } x \le u, y \ge v.$$

Also suppose either

(a) *F is continuous, or*

(b) *X has the following property:*

  (i) *If a non-decreasing sequence $\{x_n\} \to x$, then $x_n \le x$ for all n;*

  (ii) *If a non-increasing sequence $\{y_n\} \to y$, then $y_n \ge y$ for all n.*

If there exist $x_0, y_0 \in X$ such that $x_0 \le F(x_0, y_0)$ and $y_0 \ge F(y_0, x_0)$, then F has a coupled fixed point.

Inspired by Definition 1.1, Lakshmikantan and Ćirić [2] introduced the following concept of mixed g-monotone mappings.

Definition 1.4 (See [2]) Let $F : X \times X \to X$ and $g : X \to X$ be mappings. F is said to have the mixed g-monotone property if F is monotone g-non-decreasing in its first argument and is monotone g-non-increasing in its second argument, that is, for any $x, y \in X$,

$$g(x_1) \le g(x_2) \quad \Rightarrow \quad F(x_1, y) \le F(x_2, y) \quad \text{for } x_1, x_2 \in X,$$
$$g(y_1) \le g(y_2) \quad \Rightarrow \quad F(x, y_1) \ge F(x, y_2) \quad \text{for } y_1, y_2 \in X.$$

It is clear that Definition 1.4 reduces to Definition 1.1 when g is an identity mapping.

Definition 1.5 (See [2]) An element $(x, y) \in X \times X$ is called a coupled coincidence point of the mapping $F : X \times X \to X$ and $g : X \to X$ if $F(x, y) = g(x)$ and $F(y, x) = g(y)$.

Definition 1.6 (See [2]) Let $F : X \times X \to X$, $g : X \to X$ be mappings. We say that F and g are commutative if $g(F(x, y)) = F(g(x), g(y))$ for all $x, y \in X$.

Theorem 1.7 (See [2]) *Assume that there is a function $\varphi : [0, +\infty) \to [0, +\infty)$ with $\varphi(t) < t$ and $\lim_{r \to t^+} \varphi(r) < t$ for each $t > 0$, and also suppose that $F : X \times X \to X$ and $g : X \to X$ are mappings such that F has the mixed g-monotone property and*

$$d\big(F(x,y),F(u,v)\big) \le \varphi\left(\frac{d(g(x),g(u)) + d(g(y),g(v))}{2}\right)$$

for all $x, y, u, v \in X$, for which $g(x) \le g(u)$ and $g(y) \ge g(v)$.

Suppose that $F(X \times X) \subseteq g(X)$, g is continuous and commutes with F, and also suppose that either

(a) *F is continuous, or*

(b) *X has the following property:*

(i) *If a non-decreasing sequence* $\{x_n\} \to x$, *then* $x_n \leq x$ *for all* n;
(ii) *If a non-increasing sequence* $\{y_n\} \to y$, *then* $y \leq y_n$ *for all* n.

If there exist $x_0, y_0 \in X$ *such that* $g(x_0) \leq F(x_0, y_0)$ *and* $g(y_0) \geq F(y_0, x_0)$, *then there exist* $x, y \in X$ *such that* $g(x) = F(x, y)$ *and* $g(y) = F(y, x)$, *i.e.*, F *and* g *have a coupled coincidence point*.

Theorem 1.8 (See [2]) *In addition to the hypothesis of Theorem* 1.7, *suppose that for every* $(x, y), (x^*, y^*) \in X \times X$, *there exists* $(u, v) \in X \times X$ *such that* $(F(u, v), F(v, u))$ *is comparable to* $(F(x, y), F(y, x))$ *and* $(F(x^*, y^*), F(y^*, x^*))$. *Then* F *and* g *have a unique coupled common fixed point, i.e., there exists a unique* $(x, y) \in X \times X$ *such that*

$$x = g(x) = F(x, y), \qquad y = g(y) = F(y, x).$$

Recently, Berinde and Borcut [14] introduced the following partial order on the product space $X^3 = X \times X \times X$:

$$(x, y, z) \leq (u, v, w) \quad \Leftrightarrow \quad x \leq u, \qquad y \geq v, \qquad z \leq w,$$

where $(x, y, z), (u, v, w) \in X^3$ (see also [15]).

Definition 1.9 (See [14]) Let $F : X^3 \to X$ be a mapping. We say that F has the mixed monotone property if $F(x, y, z)$ is monotone non-decreasing in x and z, and it is monotone non-increasing in y, *i.e.*, for any $x, y, z \in X$,

$$\begin{aligned}
&x_1, x_2 \in X, \quad x_1 \leq x_2 \quad \Rightarrow \quad F(x_1, y, z) \leq F(x_2, y, z),\\
&y_1, y_2 \in X, \quad y_1 \leq y_2 \quad \Rightarrow \quad F(x, y_1, z) \geq F(x, y_2, z),\\
&yz_1, z_2 \in X, \quad z_1 \leq z_2 \quad \Rightarrow \quad F(x, y, z_1) \leq F(x, y, z_2).
\end{aligned}$$

Definition 1.10 (See [14]) An element $(x, y, z) \in X^3$ is called a tripled fixed point of $F : X^3 \to X$ if

$$F(x, y, z) = x, \qquad F(y, x, y) = y, \qquad F(z, y, x) = z.$$

Theorem 1.11 (See [14]) *Let* $F : X^3 \to X$ *have the mixed monotone property on* X. *Assume that there exist constants* $j, k, l \in [0, 1)$ *with* $j + k + l < 1$, *for which*

$$d\bigl(F(x, y, z), F(u, v, w)\bigr) \leq jd(x, u) + kd(y, v) + ld(z, w) \quad \forall x \geq u, y \leq v, z \geq w.$$

Also suppose either

(a) F *is continuous*, *or*
(b) X *has the following property*:
(i) *If a non-decreasing sequence* $\{x_n\} \to x$, *then* $x_n \leq x$ *for all* n;
(ii) *If a non-increasing sequence* $\{y_n\} \to y$, *then* $y_n \geq y$ *for all* n.

If there exist $x_0, y_0, z_0 \in X$ such that

$$x_0 \leq F(x_0, y_0, z_0), \qquad y_0 \geq F(y_0, x_0, y_0) \quad \text{and} \quad z_0 \leq F(z_0, y_0, x_0),$$

then there exist $x, y, z \in X$ such that

$$x = F(x, y, z), \qquad y = F(y, x, y), \qquad z = F(z, y, x).$$

The following concept of an n-fixed point was introduced by Eshaghi and Ramezani [16]. We suppose, as in [16], that k is a positive integer (odd or even) and that the product space $X^k = \underbrace{X \times \cdots \times X}_{k\text{-times}}$ is endowed with following partial order: for $(x_1, x_2, \ldots, x_k)$,

$$\begin{aligned}
&\bigl((y_1, y_2, \ldots, y_k) \in X^k, (x_1, x_2, \ldots, x_k) \leq (y_1, y_2, \ldots, y_k)\bigr) \\
&\quad \Longleftrightarrow \quad \biggl((x_{2i-1} \leq y_{2i-1}) \text{ for all } i \in 1, 2, \ldots, \left[\frac{k+1}{2}\right], \\
&\qquad\qquad x_{2i} \geq y_{2i} \text{ for all } i \in 1, 2, \ldots, \left[\frac{k}{2}\right]\biggr).
\end{aligned}$$

Definition 1.12 (See [16]) An element $(x_1, x_2, \ldots, x_k) \in X^k$ is called a k-fixed point of $F : X^k \to X$ if

$$x_i = F(x_i, x_{i-1}, \ldots, x_2, x_1, x_2, \ldots, x_{k-i+1}) \quad \text{for all } i \in \{1, 2, \ldots, k\}.$$

Theorem 1.13 (See [16]) *Let $F : X^k \to X$ be a continuous mapping having the mixed monotone property on X. Assume that there exist $\{j_i\}_{i \in \mathbb{Z}} \in [0,1)$ with $\sum_{i=-\infty}^{i=\infty} j_i < 1$ and $j_i = 0$ for all $i \neq \{1, 2, \ldots, k\}$ such that*

$$d\bigl(F(x_1, x_2, \ldots, x_k), F(y_1, y_2, \ldots, y_k)\bigr) \leq j_1 d(x_1, y_1) + j_2 d(x_2, y_2) + \cdots + j_k d(x_k, y_k)$$

for all $x_i, y_i \in X$ ($i \in \{1, 2, \ldots, k\}$), for which $x_{2i-1} \geq y_{2i-1}$ for all $i \in \{1, 2, \ldots, [\frac{k+1}{2}]\}$ and $x_{2i} \leq y_{2i}$ for all $i \in \{1, 2, \ldots, [\frac{k-1}{2}]([\frac{k}{2}])\}$.

If there exist $x_1^0, x_2^0, \ldots, x_k^0 \in X$ such that

$$x_{2i-1}^0 \leq F\bigl(x_{2i-1}^0, x_{2i-2}^0, \ldots, x_2^0, x_1^0, x_2^0, \ldots, x_{k-2i+2}^0\bigr)$$

for all $i \in \{1, 2, \ldots, [\frac{k+1}{2}]\}$ and

$$x_{2i}^0 \geq F\bigl(x_{2i}^0, x_{2i-1}^0, \ldots, x_2^0, x_1^0, x_2^0, \ldots, x_{k-2i+1}^0\bigr)$$

for all $i \in \{1, 2, \ldots, [\frac{k}{2}]\}$, then F has a k-fixed point.

In this paper, we present the new k-fixed point, and by defining the notion of a new g-monotone mapping, the existence of a k-coincidence point and the uniqueness of a common k-fixed point are obtained. Our definitions are thoroughly different from the ones in [14, 15].

2 The main results

Definition 2.1 Let X be a non-empty set, and let $F : X^k \to X$ be a given mapping ($k \geq 2$). An element $(x_1, x_2, x_3, \ldots, x_k) \in X^k$ is said to be a k-fixed point of the mapping F if

$$\begin{aligned}
&F(x_1, x_2, \ldots, x_{k-1}, x_k) = x_1, \\
&F(x_2, x_3, \ldots, x_k, x_1) = x_2, \\
&\vdots \\
&F(x_k, x_1, x_2, \ldots, x_{k-1}) = x_k.
\end{aligned}$$

Definition 2.2

Let X be a non-empty set, and let $g : X \to X$ and $F : X^k \to X$ ($k \geq 2$) be two given mappings. F is said to have the new g-monotone property if F is monotone g-non-decreasing in its first argument. That is, for any $(x_1, x_2, \ldots, x_k), (y_1, y_2, \ldots, y_k) \in X^k$,

$$g(x_1) \leq g(y_1) \quad \Rightarrow \quad F(x_1, x_2, \ldots, x_k) \leq F(y_1, y_2, \ldots, y_k).$$

Definition 2.3 Let X be a non-empty set, and let $g : X \to X$ and $F : X^k \to X$ ($k \geq 2$) be two given mappings. An element $(x_1, x_2, \ldots, x_k) \in X^k$ is called a k-coincidence point of $F : X^k \to X$ and $g : X \to X$ if

$$\begin{aligned}
&g(x_1) = F(x_1, x_2, \ldots, x_{k-1}, x_k), \\
&g(x_2) = F(x_2, x_3, \ldots, x_k, x_1), \\
&\vdots \\
&g(x_k) = F(x_k, x_1, x_2, \ldots, x_{k-1}).
\end{aligned}$$

Note that if g is an identity mapping, then Definition 2.3 reduces to Definition 2.1.

Definition 2.4 Let X be a non-empty set, and let $g : X \to X$ and $F : X^k \to X$ ($k \geq 2$) be two given mappings. We say F and g are commutative if

$$g\big(F(x_1, x_2, \ldots, x_k)\big) = F\big(g(x_1), g(x_2), \ldots, g(x_k)\big) \quad \text{for all } x_1, x_2, \ldots, x_k \in X.$$

Theorem 2.5 *Let $(X, \leq, d)$ be a partially ordered complete metric space, and let $F : X^k \to X$ and $g : X \to X$ be two given mappings such that F has a new g-monotone property, g is continuous, $F(X^k) \subset g(X)$ and g commutes with F. Assume that there exists a continuous function $\varphi : [0, +\infty) \to [0, +\infty)$, satisfying*

(i) $\varphi(t) < t$ *for* $t > 0$ *and* $\varphi(0) = 0$;

(ii) $\lim_{r \to t^+} \varphi(r) < t$ *for each* $t > 0$,

such that

$$d\big(F(x_1, \ldots, x_k), F(y_1, \ldots, y_k)\big) \leq \varphi\left(\frac{d(g(x_1), g(y_1)) + \cdots + d(g(x_k), g(y_k))}{k}\right) \tag{2.1}$$

for all x_j, y_j ($j \in \{1,2,\dots,k\}$) so that $g(x_{2i-1}) \le g(y_{2i-1})$ for all $i \in \{1,2,\dots,[\frac{k+1}{2}]\}$ and $g(y_{2i}) \le g(x_{2i})$ for all $i \in \{1,2,\dots,[\frac{k}{2}]\}$, and suppose there exist $x_1^0, x_2^0, \dots, x_k^0 \in X$ such that

$$\begin{aligned} &g\bigl(x_{2i-1}^0\bigr) \le F\bigl(x_{2i-1}^0, x_{2i}^0, \dots, x_k^0, x_1^0, \dots, x_{2i-2}^0\bigr) \quad \text{for all } i \in \left\{1,2,\dots,\left[\frac{k+1}{2}\right]\right\}, \\ &g\bigl(x_{2i}^0\bigr) \ge F\bigl(x_{2i}^0, x_{2i+1}^0, \dots, x_k^0, x_1^0, x_2^0, \dots, x_{2i-1}^0\bigr) \quad \text{for all } i \in \left\{1,2,\dots,\left[\frac{k}{2}\right]\right\}. \end{aligned} \tag{2.2}$$

Also suppose that either

(a) *F is continuous, or*

(b) *X has the following property:*

  (i) *If a non-decreasing sequence $\{x_n\} \to x$, then $x_n \le x$ for all n;*

  (ii) *If a non-increasing sequence $\{y_n\} \to y$, then $y_n \ge y$ for all n.*

Then there exist $x_1, x_2, \dots, x_k \in X$ such that

$$g(x_i) = F(x_i, x_{i+1}, \dots, x_n, x_1, x_2, \dots, x_{i-1}) \quad \text{for all } i \in \{1,2,\dots,k\}. \tag{2.3}$$

That is, F and g have a k-coincidence point.

Proof Since $F(X^k) \subset g(X)$, we can find an element $x_i^n \in X$ such that

$$g\bigl(x_i^n\bigr) = F\bigl(x_i^{n-1}, x_{i+1}^{n-1}, \dots, x_k^{n-1}, x_1^{n-1}, \dots, x_{i-1}^{n-1}\bigr) \quad \text{for all } i \in \{1,2,\dots,k\}. \tag{2.4}$$

We claim that

$$\begin{aligned} &g\bigl(x_{2i-1}^{n-1}\bigr) \le g\bigl(x_{2i-1}^n\bigr) \quad \text{for all } i \in \left\{1,2,\dots,\left[\frac{k+1}{2}\right]\right\} \quad \text{and} \\ &g\bigl(x_{2i}^{n-1}\bigr) \ge g\bigl(x_{2i}^n\bigr) \quad \text{for all } i \in \left\{1,2,\dots,\left[\frac{k}{2}\right]\right\}. \end{aligned} \tag{2.5}$$

We prove (2.5) by induction. Note that by (2.2), (2.4) we have

$$\begin{aligned} &g\bigl(x_{2i-1}^0\bigr) \le F\bigl(x_{2i-1}^0, x_{2i}^0, \dots, x_k^0, x_1^0, \dots, x_{2i-2}^0\bigr) = g\bigl(x_{2i-1}^1\bigr) \\ &\qquad \text{for all } i \in \left\{1,2,\dots,\left[\frac{k+1}{2}\right]\right\} \end{aligned}$$

and

$$g\bigl(x_{2i}^0\bigr) \ge F\bigl(x_{2i}^0, x_{2i+1}^0, \dots, x_k^0, x_1^0, \dots, x_{2i-1}^0\bigr) = g\bigl(x_{2i}^1\bigr) \quad \text{for all } i \in \left\{1,2,\dots,\left[\frac{k}{2}\right]\right\}.$$

Suppose that (2.5) is true for some n.

Due to the new g-monotone property of F, for $i \in \{1,2,\dots,[\frac{k+1}{2}]\}$, we have

$$\begin{aligned} g\bigl(x_{2i-1}^n\bigr) &= F\bigl(x_{2i-1}^{n-1}, x_{2i}^{n-1}, \dots, x_k^{n-1}, x_1^{n-1}, \dots, x_{2i-2}^{n-1}\bigr) \\ &\le F\bigl(x_{2i-1}^n, x_{2i}^n, \dots, x_k^n, x_1^n, \dots, x_{2i-2}^n\bigr) = g\bigl(x_{2i-1}^{n+1}\bigr), \end{aligned}$$

and for $i \in \{1, 2, \dots, [\frac{k}{2}]\}$, we have

$$\begin{aligned} g(x_{2i}^n) &= F(x_{2i}^{n-1}, x_{2i+1}^{n-1}, \dots, x_k^{n-1}, x_1^{n-1}, \dots, x_{2i-1}^{n-1}) \\ &\geq F(x_{2i}^n, x_{2i+1}^n, \dots, x_k^n, x_1^n, \dots, x_{2i-1}^n) = g(x_{2i}^{n+1}). \end{aligned}$$

Thus (2.5) is true. We denote

$$\delta_n := d(g(x_1^n), g(x_1^{n+1})) + d(g(x_2^n), g(x_2^{n+1})) + \cdots + d(g(x_k^n), g(x_k^{n+1})).$$

We will show that

$$\delta_{n+1} \leq k\left(\varphi\left(\frac{\delta_n}{k}\right)\right). \tag{2.6}$$

By (2.1), (2.3) and (2.5), we have

$$\begin{aligned} d(g(x_i^{n+1}), g(x_i^{n+2})) &= d(F(x_i^n, x_{i+1}^n, \dots, x_k^n, x_1^n, \dots, x_{i-1}^n), \\ &\quad F(x_i^{n+1}, x_{i+1}^{n+1}, \dots, x_k^{n+1}, x_1^{n+1}, \dots, x_{i-1}^{n+1})) \\ &\leq \varphi((d(g(x_i^n), g(x_i^{n+1})) + \cdots + d(g(x_k^n), g(x_k^{n+1})) \\ &\quad + d(g(x_1^n), g(x_1^{n+1})) + \cdots + d(g(x_{i-1}^n), g(x_{i-1}^{n+1})))/k) \\ &\leq \varphi\left(\frac{\delta_n}{k}\right). \end{aligned}$$

Summing, we get

$$\delta_{n+1} = d(g(x_1^{n+1}), g(x_1^{n+2})) + d(g(x_2^{n+1}), g(x_2^{n+2})) + \cdots + d(g(x_k^{n+1}), g(x_k^{n+2})) \leq k\varphi\left(\frac{\delta_n}{k}\right).$$

If for some n we have $\delta_n = 0$, then $\delta_{n+1} = 0$; otherwise, $\delta_n > 0$ for all $n \in \mathbb{N}$, then

$$\delta_{n+1} \leq k\varphi\left(\frac{\delta_n}{k}\right) < k\frac{\delta_n}{k} = \delta_n.$$

Hence $\{\delta_n\}$ is a non-increasing sequence which is bounded below ($0 \leq \delta_n$), then there exits some $\delta \geq 0$ such that

$$\lim_{n\to\infty} \delta_n = \delta.$$

We will show that $\delta = 0$. If for some n, $\delta_n = 0$, it is obvious; otherwise, suppose that $\delta > 0$. Keeping in mind that $\lim_{r\to t^+} \varphi(r) < t$ (for all $t > 0$) and taking the limit as $\delta_n \to \delta$ of both sides of (2.6), we have

$$\delta = \lim_{n\to\infty} \delta_{n+1} < \lim_{n\to\infty} k\varphi\left(\frac{\delta_n}{k}\right) = \lim_{\delta_n\to\delta} k\varphi\left(\frac{\delta_n}{k}\right) < k\varphi\left(\frac{\delta}{k}\right) < k\frac{\delta}{k} = \delta,$$

which is contradiction. Thus $\delta = 0$, that is,

$$\lim_{n\to\infty} d(g(x_1^n), g(x_1^{n+1})) + d(g(x_2^n), g(x_2^{n+1})) + \cdots + d(g(x_k^n), g(x_k^{n+1})) = 0. \tag{2.7}$$

Now, we will show that $\{g(x_i^n)\}_{n\in\mathbb{N}}$ for all $i \in \{1,2,\dots,k\}$ is a Cauchy sequence. Suppose, on the contrary, that at least one of $\{g(x_i^n)\}$ ($i \in \{1,2,\dots,k\}$) is not Cauchy. So, there exists $\epsilon > 0$ for which we can find sub-sequences $\{g(x_i^{n(l)})\}$, $\{g(x_i^{m(l)})\}$ of $\{g(x_i^n)\}$ with $n(l) > m(l) \geq l$ such that

$$\sum_{i=1}^{k} d\big(g\big(x_i^{n(l)}\big), g\big(x_i^{m(l)}\big)\big) \geq \epsilon. \tag{2.8}$$

We can choose $n(l)$, corresponding to $m(l)$, such that it is the smallest integer satisfying (2.8) and $n(l) > m(l) \geq l$. Hence

$$\sum_{i=1}^{k} d\big(g\big(x_i^{n(l)-1}\big), g\big(x_i^{m(l)}\big)\big) < \epsilon. \tag{2.9}$$

Due to (2.8), (2.9) and by using the triangle inequality, we have

$$\begin{aligned}\epsilon \leq t_l &:= \sum_{i=1}^{k} d\big(g\big(x_i^{n(l)}\big), g\big(x_i^{m(l)}\big)\big) \\ &\leq \sum_{i=1}^{k}\big(d\big(g\big(x_i^{n(l)}\big), g\big(x_i^{n(l)-1}\big)\big)\big) + \sum_{i=1}^{k}\big(d\big(g\big(x_i^{n(l)-1}\big), g\big(x_i^{m(l)}\big)\big)\big) \\ &< \sum_{i=1}^{k}\big(d\big(g\big(x_i^{n(l)}\big), g\big(x_i^{n(l)-1}\big)\big) + \epsilon.\end{aligned} \tag{2.10}$$

Taking $l \to \infty$ in (2.10) and using (2.7), we have

$$\epsilon < \liminf_{l\to\infty}\left(\sum_{i=1}^{k} d\big(g\big(x_i^{n(l)}\big), g\big(x_i^{n(l)-1}\big)\big)\right) + \epsilon = \epsilon.$$

That is a contradiction. Therefore $\{g(x_i^n)\}$ for all $i \in \{1,2,\dots,k\}$ are Cauchy sequences.

Since X is a complete metric space, there exist $x_1, x_2, \dots, x_k \in X$ such that

$$\lim_{n\to\infty} g\big(x_i^n\big) = x_i \tag{2.11}$$

for all $i \in \{1,2,\dots,k\}$. Due to the continuity of g, (2.11) implies that

$$\lim_{n\to\infty} g\big(g\big(x_i^n\big)\big) = g(x_i) \quad \text{for all } i \in \{1,2,\dots,k\}. \tag{2.12}$$

By (2.4) and the commutativity of F and g, we have

$$\begin{aligned} g\big(g\big(x_i^{n+1}\big)\big) &= g\big(F\big(x_i^n, x_{i+1}^n, \dots, x_k^n, x_1^n, \dots, x_{i-1}^n\big)\big) \\ &= F\big(g\big(x_i^n\big), g\big(x_{i+1}^n\big), \dots, g\big(x_k^n\big), g\big(x_1^n\big), \dots, g\big(x_{i-1}^n\big)\big)\end{aligned} \tag{2.13}$$

for all $i \in \{1,2,\dots,k\}$. We will show that

$$F(x_i, x_{i+1}, \dots, x_k, x_1, x_2, \dots, x_{i-1}) = g(x_i). \tag{2.14}$$

We consider the following two cases.

Case I: The assumption (a) holds. Then by (2.4), (2.13) and (2.11), we have

$$\begin{aligned} g(x_i) &= \lim_{n\to\infty} g\big(g\big(x_i^{n+1}\big)\big) = \lim_{n\to\infty} g\big(F\big(x_i^n, x_{i+1}^n, \ldots, x_k^n, x_1^n, \ldots, x_{i-1}^n\big)\big) \\ &= \lim_{n\to\infty} F\big(g\big(x_i^n\big), g\big(x_{i+1}^n\big), \ldots, g\big(x_k^n\big), g\big(x_1^n\big), \ldots, g\big(x_{i-1}^n\big)\big) \\ &= F(x_i, x_{i+1}, \ldots, x_k, x_1, \ldots, x_{i-1}) \end{aligned}$$

for all $i \in \{1,2,\ldots,k\}$. Thus (2.14) is proved.

Case II: The assumption (b) holds. Since $\{g(x_{2i-1}^n)\}$ is non-decreasing for all $i \in \{1,2,\ldots,[\frac{k+1}{2}]\}$ and $g(x_{2i-1}^n) \to x_{2i-1}$, and also $\{g(x_{2i}^n)\}$ is non-increasing for all $i \in \{1,2,\ldots,[\frac{k}{2}]\}$ and $g(x_{2i}^n) \to x_{2i}$, then by assumption (b) we have

$$\begin{aligned} &g\big(x_{2i-1}^n\big) \le x_{2i-1} \quad \text{for all } i \in \left\{1,2,\ldots,\left[\frac{k+1}{2}\right]\right\}, \\ &g\big(x_{2i}^n\big) \ge x_{2i} \quad \text{for all } i \in \left\{1,2,\ldots,\left(\left[\frac{k}{2}\right]\right)\right\} \end{aligned}$$

for all n. Thus by (2.13), (2.1) and the triangle inequality,

$$\begin{aligned} &d\big(g(x_i), F(x_i, x_{i+1}, \ldots, x_k, x_1, x_2, \ldots, x_{i-1})\big) \\ &\quad \le d\big(g(x_i), g\big(g\big(x_i^{n+1}\big)\big)\big) + d\big(g\big(g\big(x_i^{n+1}\big)\big), F(x_i, x_{i+1}, \ldots, x_k, x_1, x_2, \ldots, x_{i-1})\big) \\ &\quad = d\big(g(x_i), g\big(g\big(x_i^{n+1}\big)\big)\big) + d\big(g\big(F\big(x_i^n, x_{i+1}^n, \ldots, x_k^n, x_1^n, x_2^n, \ldots, x_{i-1}^n\big)\big), \\ &\qquad F(x_i, x_{i+1}, \ldots, x_k, x_1, x_2, \ldots, x_{i-1})\big) \\ &\quad = d\big(g(x_i), g\big(g\big(x_i^{n+1}\big)\big)\big) + d\big(F\big(g\big(x_i^n\big), g\big(x_{i+1}^n\big), \ldots, g\big(x_k^n\big), g\big(x_1^n\big), g\big(x_2^n\big), \ldots, g\big(x_{i-1}^n\big)\big), \\ &\qquad F(x_i, x_{i+1}, \ldots, x_k, x_1, x_2, \ldots, x_{i-1})\big) \\ &\quad \le d\big(g(x_i), g\big(g\big(x_i^{n+1}\big)\big)\big) + \varphi\bigg(\frac{1}{k}\big[d\big(g\big(g\big(x_i^n\big)\big), g(x_i)\big) + d\big(g\big(g\big(x_{i+1}^n\big)\big), g(x_{i+1})\big) \\ &\qquad + \cdots + d\big(g\big(g\big(x_k^n\big)\big), g(x_k)\big) + d\big(g\big(g\big(x_1^n\big)\big), g(x_1)\big) + \cdots + d\big(g\big(g\big(x_{i-1}^n\big)\big), g(x_{i-1})\big)\big]\bigg) \end{aligned}$$

for all $i \in \{1,2,\ldots,k\}$. Taking the limit as $n \to \infty$, by (2.12) and the fact that $\varphi(0) = 0$, we get $d(g(x_i), F(x_i, x_{i+1}, \ldots, x_k, x_1, \ldots, x_{i-1})) \le 0$. Thus

$$g(x_i) = F(x_i, x_{i+1}, \ldots, x_k, x_1, x_2, \ldots, x_{i-1}) \quad \text{for all } i \in \{1,2,\ldots,k\}.$$

Hence we proved that F and g have a k-coincidence point. □

Corrollary 2.6 *Let $F : X^k \to X$ and $g : X \to X$ be a continuous mapping such that F has a new g-monotone property, $F(X^k) \subset g(X)$ and g commutes with F. Assume that there exists $l \in [0,1)$ with*

$$d\big(F(x_1, x_2, \ldots, x_k), F(y_1, y_2, \ldots, y_k)\big) \le \frac{l}{k}\big[d\big(g(x_1), g(y_1)\big) + \cdots + d\big(g(x_k), g(y_k)\big)\big]$$

for all x_j, y_j ($j \in \{1,2,\dots,k\}$) *which* $g(x_{2i-1}) \leq g(y_{2i-1})$ *for all* $i \in \{1,2,\dots,[\frac{k+1}{2}]\}$ *and* $g(y_{2i}) \leq g(x_{2i})$ *for all* $i \in \{1,2,\dots,[\frac{k}{2}]\}$, *and suppose that there exist* $x_1^0, x_2^0, \dots, x_k^0 \in X$ *such that*

$$g(x_{2i-1}^0) \leq F(x_{2i-1}^0, x_{2i}^0, \dots, x_k^0, x_1^0, \dots, x_{2i-2}^0) \quad \text{for all } i \in \left\{1,2,\dots,\left[\frac{k+1}{2}\right]\right\},$$

$$g(x_{2i}^0) \geq F(x_{2i}^0, x_{2i+1}^0, \dots, x_k^0, x_1^0, x_2^0, \dots, x_{2i-1}^0) \quad \text{for all } i \in \left\{1,2,\dots,\left[\frac{k}{2}\right]\right\},$$

and suppose either

(a) F *is continuous*, *or*

(b) X *has the following property*:

  (i) *If a non-decreasing sequence* $\{x_n\} \to x$, *then* $x_n \leq x$ *for all* n;

  (ii) *If a non-increasing sequence* $\{y_n\} \to y$, *then* $y \leq y_n$ *for all* n.

Then there exist $x_1, x_2, \dots, x_k \in X$ *such that*

$$g(x_i) = F(x_i, x_{i+1}, \dots, x_n, x_1, x_2, \dots, x_{i-1}) \quad \text{for all } i \in \{1,2,\dots,k\}.$$

That is, F *and* g *have a* k*-coincidence point.*

Proof It follows from Theorem 2.5 by putting $\varphi(t) = l \cdot t$ for $l \in [0,1)$. □

Example 2.7 Let $X = \mathbb{R}$, $d(x,y) = |x-y|$, $k \in \mathbb{N}$, $k > 1$, and let $F : X^k \to X$ be defined by

$$F(x_1, x_2, \dots, x_k) = \frac{(k+1)\sum_{j=1}^{k-1}(-1)^{j-1}x_j + k(-1)^{k-1}x_k + 1}{2k(k+1)}$$

for all $x_1, x_2, \dots, x_k \in X$. It is easy to check that F satisfies Corollary 2.6 by taking $\varphi(t) = \frac{t}{2}$ and $g = id_X$. If k is an odd positive integer, then

$$\underbrace{\left(\frac{1}{k(2k+1)}, \frac{1}{k(2k+1)}, \dots, \frac{1}{k(2k+1)}, \dots, \frac{1}{k(2k+1)}\right)}_{k\text{-times}}$$

is the k-fixed point of F, and if k is an even positive integer, then

$$\underbrace{\left(\frac{1}{2k(k+1)-1}, \frac{1}{2k(k+1)-1}, \dots, \frac{1}{2k(k+1)-1}, \dots, \frac{1}{2k(k+1)-1}\right)}_{k\text{-times}}$$

is the k-fixed point of F.

Theorem 2.8 *In addition to the hypothesis of Theorem* 2.5, *suppose that for every*

$$(x_1, x_2, \dots, x_k), (x_1^*, x_2^*, \dots, x_k^*) \in X^k,$$

there exists $(u_1, u_2, \dots, u_k) \in X^k$ *such that*

$$\bigl(F(u_1, \dots, u_k), F(u_2, \dots, u_k, u_1), \dots, F(u_i, \dots, u_k, u_1, \dots, u_{i-1}), \dots, F(u_k, u_1, \dots, u_{k-1})\bigr)$$

is comparable to

$$\begin{aligned}&\big(F(x_1,\ldots,x_k),F(x_2,\ldots,x_k,x_1),\ldots,\\&\quad F(x_i,\ldots,x_k,x_1,\ldots,x_{i-1}),\ldots,F(x_k,x_1,\ldots,x_{k-1})\big)\end{aligned}$$

and

$$\begin{aligned}&\big(F\big(x_1^*,\ldots,x_k^*\big),F\big(x_2^*,\ldots,x_k^*,x_1^*\big),\ldots,\\&\quad F\big(x_i^*,\ldots,x_k^*,x_1^*,\ldots,x_{i-1}^*\big),\ldots,F\big(x_k^*,x_1^*,\ldots,x_{k-1}^*\big)\big).\end{aligned}$$

Then F and g have a unique k-coincidence point, which is a fixed point of $g: X \to X$ and a k-fixed point of $F: X^k \to X$. That is, there exists a unique $(x_1,x_2,\ldots,x_k) \in X^k$ such that

$$x_i = g(x_i) = F(x_i,x_{i+1},\ldots,x_k,x_1,x_2,\ldots,x_{i-1}) \quad \textit{for all } i \in \{1,2,\ldots,k\}.$$

Proof By Theorem 2.5, the set of k-coincidence fixed points is nonempty. Now, suppose $(x_1,x_2,\ldots,x_k)$ and $(x_1^*,x_2^*,\ldots,x_k^*)$ are two coincidence fixed points of F and g, that is,

$$\begin{aligned}&g(x_i) = F(x_i,x_{i+1},\ldots,x_k,x_1,x_2,\ldots,x_{i-1}) \quad \text{for all } i \in \{1,2,\ldots,k\},\\&g\big(x_i^*\big) = F\big(x_i^*,x_{i+1}^*,\ldots,x_k^*,x_1^*,x_2^*,\ldots,x_{i-1}^*\big) \quad \text{for all } i \in \{1,2,\ldots,k\}.\end{aligned}$$

We will show that

$$g(x_i) = g\big(x_i^*\big) \quad \text{for all } i \in \{1,2,\ldots,k\}. \tag{2.15}$$

By assumption, there exists $(u_1,u_2,\ldots,u_k) \in X^k$ such that

$$\begin{aligned}&\big(F(u_1,\ldots,u_k),F(u_2,\ldots,u_k,u_1),\ldots,\\&\quad F(u_i,\ldots,u_k,u_1,\ldots,u_{i-1}),\ldots,F(u_k,u_1,\ldots,u_{k-1})\big)\end{aligned}$$

is comparable with

$$\begin{aligned}&\big(F(x_1,\ldots,x_k),F(x_2,\ldots,x_k,x_1),\ldots,\\&\quad F(x_i,\ldots,x_k,x_1,x_2,\ldots,x_{i-1}),\ldots,F(x_k,x_1,\ldots,x_{k-1})\big)\end{aligned}$$

and

$$\begin{aligned}&\big(F\big(x_1^*,\ldots,x_k^*\big),F\big(x_2^*,\ldots,x_k^*,x_1^*\big),\ldots,\\&\quad F\big(x_i^*,\ldots,x_k^*,x_1^*,\ldots,x_{i-1}^*\big),\ldots,F\big(x_k^*,x_1^*,\ldots,x_{k-1}^*\big)\big).\end{aligned}$$

Let $u_i^0 := u_i$ for all $i \in \{1,2,\ldots,k\}$.

Since $F(X^k) \subset g(X)$, we can choose $u_i^1 \in X$ such that $g(u_i^1) = F(u_i^0,u_{i+1}^0,\ldots,u_k^0,u_1^0,\ldots,u_{i-1}^0)$ for all $i \in \{1,2,\ldots,k\}$. By a similar reason as in the proof of Theorem 2.5, we can inductively define sequences $\{g(u_i^n)\}_{n\in\mathbb{N}}$ for all $i\in\{1,2,\dots,k\}$ such that for all $n\in\mathbb{N}\cup\{0\}$,

$$g\bigl(u_i^{n+1}\bigr)=F\bigl(u_i^n,u_{i+1}^n,\dots,u_k^n,u_1^n,\dots,u_{i-1}^n\bigr)\quad\text{for all } i\in\{1,2,\dots,k\}.$$

In addition, let $x_i^0:=x_i$ and $x_i^{*0}:=x_i^*$ for all $i\in\{1,2,\dots,k\}$ and, in the same way, define the sequences $\{g(x_i^n)\}_{n\in\mathbb{N}}$ and $\{g(x_i^{*n})\}_{n\in\mathbb{N}}$ for all $i\in\{1,2,\dots,k\}$. Since

$$\begin{aligned}&\bigl(F(x_1,\dots,x_k),F(x_2,\dots,x_k,x_1),\dots,F(x_i,\dots,x_k,x_1,\dots,x_{i-1}),\dots,F(x_k,x_1,\dots,x_{k-1})\bigr)\\&\quad=\bigl(g\bigl(x_1^1\bigr),g\bigl(x_2^1\bigr),\dots,g\bigl(x_i^1\bigr),\dots,g\bigl(x_k^1\bigr)\bigr)\end{aligned}$$

and

$$\begin{aligned}&\bigl(F(u_1,\dots,u_k),F(u_2,\dots,u_k,u_1),\dots,F(u_i,\dots,u_k,u_1,\dots,u_{i-1}),\dots,F(u_k,u_1,\dots,u_{k-1})\bigr)\\&\quad=\bigl(g\bigl(u_1^1\bigr),g\bigl(u_2^1\bigr),\dots,g\bigl(u_i^1\bigr),\dots,g\bigl(u_k^1\bigr)\bigr)\end{aligned}$$

are comparable, then

$$\begin{aligned}&g\bigl(x_{2i-1}^1\bigr)\le g\bigl(u_{2i-1}^1\bigr)\quad\text{for all } i\in\left\{1,2,\dots,\left[\frac{k+1}{2}\right]\right\},\\&g\bigl(x_{2i}^1\bigr)\ge g\bigl(u_{2i}^1\bigr)\quad\text{for all } i\in\left\{1,2,\dots,\left[\frac{k}{2}\right]\right\}.\end{aligned}$$

Now, for all $n\in\mathbb{N}$, we have

$$\begin{aligned}&g(x_{2i-1})=g\bigl(x_{2i-1}^1\bigr)\le g\bigl(u_{2i-1}^1\bigr)\le g\bigl(u_{2i-1}^2\bigr)\le\cdots\le g\bigl(u_{2i-1}^n\bigr)\quad\left(i\in\left\{1,2,\dots,\left[\frac{k+1}{2}\right]\right\}\right),\\&g(x_{2i})=g\bigl(x_{2i}^1\bigr)\ge g\bigl(u_{2i}^1\bigr)\ge g\bigl(u_{2i}^2\bigr)\ge\cdots\ge g\bigl(u_{2i}^n\bigr)\quad\left(i\in\left\{1,2,\dots,\left[\frac{k}{2}\right]\right\}\right).\end{aligned}$$

Then $(g(x_1),g(x_2),\dots,g(x_k))$ and $(g(u_1^n),g(u_2^n),\dots,g(u_k^n))$ are comparable for all $n\in\mathbb{N}$.

It follows from (2.1) that

$$\begin{aligned}&d\bigl(g(x_{2i-1}),g\bigl(u_{2i-1}^{n+1}\bigr)\bigr)\\&\quad=d\bigl(F(x_{2i-1},x_{2i},\dots,x_k,x_1,\dots,x_{2i-2}),F\bigl(u_{2i-1}^n,u_{2i}^n,\dots,u_k^n,u_1^n,\dots,u_{2i-2}^n\bigr)\bigr)\\&\quad\le\varphi\bigl(\bigl(d\bigl(g(x_{2i-1}),g\bigl(u_{2i-1}^n\bigr)\bigr)+\cdots+d\bigl(g(x_k),g\bigl(u_k^n\bigr)\bigr)+d\bigl(g(x_1),g\bigl(u_1^n\bigr)\bigr)\\&\qquad+\cdots+d\bigl(g(x_{2i-2}),g\bigl(u_{2i-2}^n\bigr)\bigr)\bigr)/k\bigr)\end{aligned}$$

for all $i\in\{1,2,\dots,[\frac{k+1}{2}]\}$ and

$$\begin{aligned}&d\bigl(g(x_{2i}),g\bigl(u_{2i}^{n+1}\bigr)\bigr)\\&\quad=d\bigl(F(x_{2i},x_{2i+1},\dots,x_k,x_1,\dots,x_{2i-1}),F\bigl(u_{2i}^n,u_{2i+1}^n,\dots,u_k^n,u_1^n,\dots,u_{2i-1}^n\bigr)\bigr)\\&\quad\le\varphi\bigl(\bigl(d\bigl(g(x_{2i}),g\bigl(u_{2i}^n\bigr)\bigr)+\cdots+d\bigl(g(x_k),g\bigl(u_k^n\bigr)\bigr)+d\bigl(g(x_1),g\bigl(u_1^n\bigr)\bigr)\\&\qquad+\cdots+d\bigl(g(x_{2i-1}),g\bigl(u_{2i-1}^n\bigr)\bigr)\bigr)/k\bigr)\end{aligned}$$

for all $i\in\{1,2,\dots,([\frac{k}{2}])\}$.

Summing, we get

$$\begin{aligned}&\frac{1}{k}\bigl[d\bigl(g(x_1),g\bigl(u_1^{n+1}\bigr)\bigr)+d\bigl(g(x_2),g\bigl(u_2^{n+1}\bigr)\bigr)+\cdots+d\bigl(g(x_k),g\bigl(u_k^{n+1}\bigr)\bigr)\bigr]\\&\quad\leq\frac{k}{k}\varphi\biggl(\frac{d(g(x_1),g(u_1^n))+d(g(x_2),g(u_2^n))+\cdots+d(g(x_k),g(u_k^n))}{k}\biggr).\end{aligned}$$

It follows that

$$\begin{aligned}&\frac{1}{k}\bigl[d\bigl(g(x_1),g\bigl(u_1^{n+1}\bigr)\bigr)+d\bigl(g(x_2),g\bigl(u_2^{n+1}\bigr)\bigr)+\cdots+d\bigl(g(x_k),g\bigl(u_k^{n+1}\bigr)\bigr)\bigr]\\&\quad\leq\varphi^n\biggl(\frac{d(g(x_1),g(u_1))+d(g(x_2),g(u_2))+\cdots+d(g(x_k),g(u_k))}{k}\biggr)\end{aligned}\tag{2.16}$$

for all $n\geq 1$. Note that $\varphi(0)=0$, $\varphi(t)<t$, $\lim_{r\to t^+}\varphi(r)<t$ for $t>0$ imply that $\lim_{n\to\infty}\varphi^n(t)=0$ for all $t>0$. Hence from (2.16) we have

$$\lim_{n\to\infty}d\bigl(g(x_i),g\bigl(u_i^{n+1}\bigr)\bigr)=0\quad\text{for all }i\in\{1,2,\ldots,k\}.\tag{2.17}$$

Similarly, one can prove that

$$\lim_{n\to\infty}d\bigl(g\bigl(x_i^*\bigr),g\bigl(u_i^{n+1}\bigr)\bigr)=0\quad\text{for all }i\in\{1,2,\ldots,k\}.\tag{2.18}$$

It follows from (2.17), (2.18) and the triangle inequality that

$$d\bigl(g(x_i),g\bigl(x_i^*\bigr)\bigr)\leq d\bigl(g(x_i),g\bigl(u_i^{n+1}\bigr)\bigr)+d\bigl(g\bigl(u_i^{n+1}\bigr),g\bigl(x_i^*\bigr)\bigr)\to 0,$$

as $n\to\infty$ for all $i\in\{1,2,\ldots,k\}$. Hence $g(x_i)=g(x_i^*)$, therefore (2.15) is proved.

Since $g(x_i)=F(x_i,x_{i+1},\ldots,x_k,x_1,x_2,\ldots,x_{i-1})$ for all $i\in\{1,2,\ldots,k\}$, by the commutativity of F and g, we have

$$\begin{aligned}g\bigl(g(x_i)\bigr)&=g\bigl(F(x_i,x_{i+1},\ldots,x_k,x_1,x_2,\ldots,x_{i-1})\bigr)\\&=F\bigl(g(x_i),g(x_{i+1}),\ldots,g(x_k),g(x_1),\ldots,g(x_{i-1})\bigr).\end{aligned}\tag{2.19}$$

Denote $g(x_i)=y_i$ for all $i\in\{1,2,\ldots,k\}$. From (2.19), we have

$$g(y_i)=g\bigl(g(x_i)\bigr)=F(y_i,y_{i+1},\ldots,y_k,y_1,\ldots,y_{i-1})\quad\text{for all }i\in\{1,2,\ldots,k\}.\tag{2.20}$$

Hence $(y_1,y_2,\ldots,y_k)$ is a k-coincidence point of F and g.

It follows from (2.15) and $x_i^*=y_i$ that

$$g(y_i)=g(x_i)\quad\text{for all }i\in\{1,2,\ldots,k\}.$$

This means that

$$g(y_i)=y_i\quad\text{for all }i\in\{1,2,\ldots,k\}.$$

Now, from (2.20) we have

$$y_i = g(y_i) = F(y_i, y_{i+1}, \ldots, y_k, y_1, \ldots, y_{i-1}) \quad \text{for all } i \in \{1, 2, \ldots, k\}.$$

Hence, $(y_1, y_2, \ldots, y_k)$ is a k-fixed point of F and a fixed point of g.

To prove the uniqueness of the fixed point, assume that $(z_1, z_2, \ldots, z_k)$ is another k-fixed point. Then by (2.15) we have

$$z_i = g(z_i) = g(y_i) = y_i \quad \text{for all } i \in \{1, 2, \ldots, k\}.$$

Thus $(z_1, z_2, \ldots, z_k) = (y_1, y_2, \ldots, y_k)$. This completes the proof. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

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