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Existence of common fixed points using Bregman nonexpansive retracts and Bregman functions in Banach spaces

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Abstract
In this paper, we first introduce the concepts of Bregman nonexpansive retract and Bregman one-local retract and then use these concepts to establish the existence of common fixed points for Banach operator pairs in the framework of reflexive Banach spaces. No compactness assumption is imposed either on C or on T, where C is a closed and convex subset of a reflexive Banach space E and $T : C \to C$ is a Bregman nonexpansive mapping. We also establish the well-known De Marr theorem for a Banach operator family of Bregman nonexpansive mappings.

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1 Introduction

This paper is motivated by the recent papers [1–4]. In [3] the authors study different questions related to common fixed points of Banach operator pairs in hyperconvex spaces. In [2] the authors introduced the concept of NR-maps and then they used this concept to establish the existence of common fixed points for Banach operator pairs in the context of uniformly convex geodesic metric spaces. In our present work, using Bregman functions, we propose to consider similar questions on reflexive Banach spaces under the mildest weaker conditions we may impose. More precisely, we first introduce the concepts of Bregman NR-map and Bregman one-local retract and then use these concepts to establish the existence of common fixed points for Banach operator pairs in reflexive Banach spaces. No compactness assumption is imposed either on C or on T, where C is a closed and convex subset of a reflexive Banach space E and $T : C \to C$ is a Bregman nonexpansive mapping. For a recent survey on the existence of fixed points in geodesic spaces, we refer the readers to [1, 5].

The celebrated result on the existence of a common fixed point for a nonexpansive commutative family was first established by De Marr [6] under the assumption that C is a compact convex subset of a normed space X. In 1965, Browder [7] obtained the corresponding result under the assumption that C is a bounded, closed and convex subset of a uniformly convex Banach space X. In 1992, Khamsi *et al.* [8] established the above mentioned results for a finite as well as an arbitrary commutative family of maps in hyperconvex metric spaces. Recently, Espìnola and Hussain [9] proved De Marr's theorem in uniformly convex metric spaces of type (T). More recently, Hussain *et al.* [3] extended De Marr's result to the family of symmetric Banach operator pairs in hyperconvex metric spaces (see also [10–12]).

Throughout this paper, we denote the set of real numbers and the set of positive integers by $\mathbb{R}$ and $\mathbb{N}$, respectively. Let E be a real Banach space and let C be a nonempty subset of E. Let $T : C \to E$ be a mapping. We denote by $F(T)$ the set of fixed points of T, *i.e.*, $F(T) = \{x \in C : Tx = x\}$.

Let E be a Banach space with the norm $\|\cdot\|$ and the dual space E^*. For any $x \in E$, we denote the value of $x^* \in E^*$ at x by $\langle x, x^*\rangle$. When $\{x_n\}_{n\in\mathbb{N}}$ is a sequence in E, we denote the strong convergence of $\{x_n\}_{n\in\mathbb{N}}$ to $x \in E$ by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$. The modulus δ of the convexity of E is denoted by

$$\delta(\epsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} : \|x\| \leq 1, \|y\| \leq 1, \|x-y\| \geq \epsilon\right\}$$

for every ϵ with $0 \leq \epsilon \leq 2$. A Banach space E is said to be *uniformly convex* if $\delta(\epsilon) > 0$ for every $\epsilon > 0$. Let $S_E = \{x \in E : \|x\| = 1\}$. The norm of E is said to be *Gâteaux differentiable* if for each $x, y \in S_E$, the limit

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t} \tag{1.1}$$

exists. In this case, E is called *smooth*. If the limit (1.1) is attained uniformly in $x, y \in S_E$, then E is called *uniformly smooth*. The Banach space E is said to be *strictly convex* if $\|\frac{x+y}{2}\| < 1$ whenever $x, y \in S_E$ and $x \neq y$. It is well known that E is uniformly convex if and only if E^* is uniformly smooth. It is also known that if E is reflexive, then E is strictly convex if and only if E^* is smooth; for more details, see [13, 14].

Let E be a smooth, strictly convex and reflexive Banach space, and let J be the normalized duality mapping of E. Let C be a nonempty closed convex subset of E. The generalized projection Π_C from E onto C is denoted by

$$\Pi_C(x) = \operatorname*{argmin}_{y\in C} \phi(y, x),$$

where $\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$. If $E = H$ is a Hilbert space, then $\phi(x, y) = \|x - y\|^2$ for all $x, y \in H$.

Let E be a Banach space with the norm $\|\cdot\|$ and the dual space E^*. A function $g : E \to (-\infty, +\infty]$ is said to be *proper* if the domain $\operatorname{dom} g = \{x \in E : g(x) < \infty\}$ is nonempty. It is also called *lower semicontinuous* if $\{x \in E : g(x) \leq r\}$ is closed for all $r \in \mathbb{R}$. We say that g is *upper semicontinuous* if $\{x \in E : g(x) \geq r\}$ is closed for all $r \in \mathbb{R}$. The function g is said to be *convex* if

$$g\big(\alpha x + (1-\alpha)y\big) \leq \alpha g(x) + (1-\alpha)g(y) \tag{1.2}$$

for all $x, y \in E$ and $\alpha \in (0, 1)$. It is also said to be *strictly convex* if the strict inequality holds in (1.2) for all $x, y \in \operatorname{dom} g$ with $x \neq y$ and $\alpha \in (0, 1)$.

For any convex function $g : E \to (-\infty, +\infty]$, we denote the domain of g by $\operatorname{dom} g = \{x \in E : g(x) < \infty\}$. For any $x \in \operatorname{int}\operatorname{dom} g$ and any $y \in E$, we denote by $g^o(x, y)$ the *right-hand* *derivative* of g at x in the direction y, that is,

$$g^o(x,y) = \lim_{t\downarrow 0} \frac{g(x+ty)-g(x)}{t}. \tag{1.3}$$

The function g is said to be *Gâteaux differentiable* at x if $\lim_{t\to 0} \frac{g(x+ty)-g(x)}{t}$ exists for any y. In this case, $g^o(x,y)$ coincides with $\nabla g(x)$, the value of the *gradient* ∇g of g at x (see, for example, [9, p.12] or [13, p.508]). A convex function $g: E \to \mathbb{R}$ is said to be *Gâteaux differentiable* if it is Gâteaux differentiable everywhere. Let $g: E\to\mathbb{R}$ be a convex and Gâteaux differentiable function. Then the *Bregman distance* [15, 16] corresponding to g is the function $D_g: E\times E\to\mathbb{R}$ defined by

$$D_g(x,y) = g(x) - g(y) - \langle x-y, \nabla g(y)\rangle, \quad \forall x,y\in E. \tag{1.4}$$

It is clear that $D_g(x,y)\geq 0$ for all $x,y\in E$. In the case when E is a smooth Banach space, setting $g(x) = \|x\|^2$ for all $x\in E$, we have $\nabla g(x) = 2Jx$ for all $x\in E$, and hence

$$\begin{aligned} D_g(x,y) &= \|x\|^2 - \|y\|^2 - \langle x-y, \nabla g(y)\rangle \\ &= \|x\|^2 - \|y\|^2 - \langle x-y, 2Jy\rangle \\ &= \|x\|^2 - \|y\|^2 - \langle x, 2Jy\rangle + 2\|y\|^2 \\ &= \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2 \\ &= \phi(x,y) \end{aligned}$$

for all $x,y\in E$.

The theory of fixed points with respect to Bregman distances have been studied in the last ten years and much intensively in the last four years. In [17], Bauschke and Combettes introduced an iterative method to construct the Bregman projection of a point onto a countable intersection of closed and convex sets in reflexive Banach spaces. They proved strong convergence theorem of the sequence produced by their method; for more detail, see [17, Theorem 4.7]. For some recent articles on the existence of fixed points for Bregman nonexpansive type mappings, we refer the readers to [17–26].

Let E be a Banach space, and let $g: E\to\mathbb{R}$ be a convex and Gâteaux differentiable function. Let C be a nonempty and closed convex subset of E. A mapping $T: C\to E$ is called *nonexpansive* if

$$\|Tx - Ty\| \leq \|x-y\|, \quad \forall x,y\in C. \tag{1.5}$$

The mapping $T: C\to E$ is called *Bregman nonexpansive* if

$$D_g(Tx,Ty)\leq D_g(x,y), \quad \forall x,y\in C. \tag{1.6}$$

Let us give an example of a Bregman nonexpansive mapping which is not a nonexpansive mapping (see also [27]).

Example 1.1 Let $g : \mathbb{R} \to \mathbb{R}$ be a function defined by

$$g(x) = x^{20}, \quad \forall x \in \mathbb{R}.$$

We define a mapping $T : [0, 0.9] \to [0, 0.9]$ by

$$T(x) = x^2, \quad \forall x \in [0, 0.9].$$

Then T is not a nonexpansive mapping in the sense of (1.5), but it is a Bregman nonexpansive mapping relative to D_g in the sense of (1.6). Indeed, taking $x = \frac{3}{4}$ and $y = \frac{1}{2}$, we see that T is not a nonexpansive mapping in the sense of (1.5). Now, we show that

$$D_g(Tx, Ty) \leq D_g(x, y), \quad \forall x, y \in [0, 0.9].$$

Let $x \in [0, 0.9]$ be fixed. We define a mapping $f : [0, 0.9] \to [0, 0.9]$ by

$$f(y) = D_g(Tx, Ty) - D_g(x, y), \quad \forall y \in [0, 0.9].$$

Then

$$\begin{aligned} f(y) &= g(Tx) - g(Ty) - \langle Tx - Ty, \nabla g(Ty)\rangle - \bigl[g(x) - g(y) - \langle x - y, \nabla(y)\rangle\bigr] \\ &= g(Tx) - g(Ty) - \langle Tx - Ty, g'(Ty)\rangle - \bigl[g(x) - g(y) - \langle x - y, g'(y)\rangle\bigr] \\ &= x^{40} - 20x^2y^{38} + 19y^{40} - x^{20} - 19y^{20} + 20xy^{19}. \end{aligned}$$

This implies that

$$\begin{aligned} f'(y) &= -760x^2y^{37} + 760y^{39} - 380y^{19} + 38xy^{18} \\ &= 380y^{18}\bigl[-2x^2y^{19} + 2y^{21} - y + x\bigr] \\ &= 380y^{18}\bigl[2y^{19}\bigl(y^2 - x^2\bigr) - (y - x)\bigr] \\ &= 380y^{18}(y - x)\bigl[2y^{19}(y + x) - 1\bigr]. \end{aligned}$$

Since x and y are in $[0, 0.9]$, we obtain

$$2y^{19}(y + x) - 1 < 2(0.9)^{19}(0.9 + 0.9) - 1 < 0.$$

Therefore, $f'(y) \geq 0$ if $y \leq x$ and $f'(y) \leq 0$ if $y > x$. Moreover, $f(y) = 0$ if $x = y$. Hence, $f(y) \leq 0$ for all $y \in [0, 0.9]$, which implies that

$$D_g(Tx, Ty) \leq D_g(x, y), \quad \forall x, y \in [0, 0.9].$$

In this paper we establish some common fixed point results for the Banach operator and symmetric Banach operator pairs in reflexive Banach spaces for Bregman nonexpansive mappings that generalize the concept of nonexpansivity. Our results improve and generalize many known results in the current literature; see, for example, [2].

2 Basic definitions and results

Let E be a real Banach space. Let $g : E \to \mathbb{R}$ be a convex and Gâteaux differentiable function. For any $x \in E$ and $r > 0$, we define the *Bregman ball centered at x with radius r* by

$$B(x,r) = \{y \in E : D_g(x,y) < r\}.$$

The *Bregman closed ball centered at x with radius r* is denoted by

$$\bar{B}(x,r) = \{y \in E : D_g(x,y) \le r\}.$$

Recall that a subset C of a real Banach space E is *Bregman admissible* if it is a nonempty intersection of Bregman closed balls. The class of all Bregman admissible subsets of C is denoted by $\mathcal{BA}(C)$.

Remark 2.1 Let E be a real Banach space. Let $g : E \to \mathbb{R}$ be a continuous, convex and Gâteaux differentiable function. Then, for any $x \in E$ and $r > 0$, any Bregman closed ball centered at x with radius r is $\tau(D_g)$ closed, where $\tau(D_g)$ is the topology induced by D_g on E. Indeed, suppose $\{y_n\}_{n\in\mathbb{N}} \subset \bar{B}(x,r)$ is a sequence such that $y_n \to y \in E$ as $n \to \infty$. Since g is continuous, so we have $g(y_n) \to g(y)$. This, together with the definition of the Bregman distance (see (1.4)), implies that

$$\lim_{n\to\infty} \left|D_g(x,y_n) - D_g(x,y)\right| = 0.$$

Thus we have $D_g(x,y) \le r$. We refer the readers to see some details on quasipseudometric concept in [28].

At this point we introduce some notation which will be used throughout the remainder of this work. For a subset A of E, we set

$$\begin{aligned}
&Br_x(A) = \sup\{D_g(x,y) : y \in A\}, \quad x \in E;\\
&BR(A) = \inf\{Br_x(A) : x \in A\};\\
&\text{B-diam}(A) = \sup\{D_g(x,y) : x,y \in A\};\\
&BC_A(A) = \{x \in A : Br_x(A) = BR(A)\};\\
&\operatorname{cov}(A) = \cap\{B : B \text{ is a Bregman ball and } B \supseteq A\}.
\end{aligned}$$

B-diam(A) is called the *Bregman diameter* of A, $BR(A)$ is called the *Bregman Chebyshev radius* of A, $BC_A(A)$ is called the *Bregman Chebyshev center* of A and cov(A) is called the *cover* of A.

Definition 2.1 Let $\mathcal{F}$ be a convexity structure on E.

(i) We will say that $\mathcal{F}$ is compact if any family $(A_\alpha)_{\alpha\in\Gamma}$ of elements of $\mathcal{F}$ has a nonempty intersection provided $\bigcap_{\alpha\in F} A_\alpha \neq \emptyset$ for any finite subset $F \subset \Gamma$;
(ii) We will say that $\mathcal{F}$ is normal if for any $A \in \mathcal{F}$, not reduced to one point, we have $BR(A) <$ B-diam(A).

Definition 2.2 The ordered pair (S,T) of two self-maps of a closed and convex subset C of a Banach space E is called a Banach operator pair if the set $\mathrm{Fix}(T)$ is S-invariant, namely $S(\mathrm{Fix}(T)) \subseteq \mathrm{Fix}(T)$. The ordered pair (S,T) is called nontrivially a Banach operator pair if $\mathrm{Fix}(T)$ is not empty and (S,T) is a Banach operator pair.

Obviously, a commuting pair (S,T) is a Banach operator pair but not conversely in general; see [4–16, 29–34].

Let $A: E \to 2^{E^*}$ be a set-valued mapping. We define the domain and range of A by $\operatorname{dom} A = \{x \in E : Ax \neq \emptyset\}$ and $\operatorname{ran} A = \bigcup_{x\in E} Ax$, respectively. The graph of A is denoted by $G(A) = \{(x,x^*) \in E \times E^* : x^* \in Ax\}$. The mapping $A \subset E \times E^*$ is said to be *monotone* [35] if $\langle x-y, x^*-y^*\rangle \geq 0$ whenever $(x,x^*),(y,y^*) \in A$. It is also said to be *maximal monotone* [36] if its graph is not contained in the graph of any other monotone operator on E. If $A \subset E \times E^*$ is maximal monotone, then we can show that the set $A^{-1}0 = \{z \in E : 0 \in Az\}$ is closed and convex. For a proper, lower semicontinuous and convex function $g: E \to (-\infty,+\infty]$, the *subdifferential* ∂g of g is defined by

$$\partial g(x) = \bigl\{x^* \in E^* : g(x) + \langle y-x, x^*\rangle \leq g(y), \forall y \in E\bigr\} \tag{2.1}$$

for all $x \in E$. It is well known that $\partial g \subset E \times E^*$ is maximal monotone [37, 38]. For any proper, lower semicontinuous and convex function $g: E \to (-\infty,+\infty]$, the *conjugate function* g^* of g is defined by

$$g^*(x^*) = \sup_{x\in E}\bigl\{\langle x, x^*\rangle - g(x)\bigr\}$$

for all $x^* \in E^*$. It is well known that $g(x) + g^*(x^*) \geq \langle x, x^*\rangle$ for all $(x,x^*) \in E \times E^*$. It is also known that $(x,x^*) \in \partial g$ is equivalent to

$$g(x) + g^*(x^*) = \langle x, x^*\rangle. \tag{2.2}$$

We also know that if $g: E \to (-\infty,+\infty]$ is a proper, lower semicontinuous and convex function, then $g^*: E^* \to (-\infty,+\infty]$ is a proper, weak* lower semicontinuous and convex function; see [14] for more details on convex analysis. Let $g: E \to \mathbb{R}$ be a convex function. The function g is also said to be *Fréchet differentiable* at $x \in E$ (see, for example, [29, p.13] or [30, p.508]) if for all $\epsilon > 0$, there exists $\delta > 0$ such that $\|y-x\| \leq \delta$ implies that

$$\bigl|g(y) - g(x) - \langle y-x, \nabla g(x)\rangle\bigr| \leq \epsilon \|y-x\|.$$

A convex function $g: E \to \mathbb{R}$ is said to be *Fréchet differentiable* if it is Fréchet differentiable everywhere. It is well known that if a continuous convex function $g: E \to \mathbb{R}$ is Gâteaux differentiable, then ∇g is norm-to-weak* continuous (see, for example, [29, Proposition 1.1.10]). Also, it is known that if g is Fréchet differentiable, then ∇g is norm-to-norm continuous (see, [30, p.508]). The mapping ∇g is said to be *weakly sequentially continuous* if $x_n \rightharpoonup x$ implies that $\nabla g(x_n) \rightharpoonup^* \nabla g(x)$ (for more details, see [29, Theorem 3.2.4] or [30, p.508]). The function g is said to be *strongly coercive* if

$$\lim_{\|x_n\|\to\infty} \frac{g(x_n)}{\|x_n\|} = \infty.$$

It is also said to be *bounded on bounded subsets* if $g(U)$ is bounded for each bounded subset U of E.

Remark 2.2 Let E be a real Banach space. Let $g : E \to \mathbb{R}$ be a Gâteaux differentiable function which is bounded on bounded subsets. Let A be a bounded subset of E. Then $\text{B-diam}(A) = \sup\{D_g(x,y) : x, y \in A\} < \infty$. Indeed, the function g is bounded on bounded subsets of E and, thus, ∇g is also bounded on bounded subsets of E^* (see, for example, [29, Proposition 1.1.11] for more details). This implies that there exist positive real numbers M_1, M_2 and M_3 such that

$$\sup\{|g(x)| : x \in A\} \le M_1, \qquad \sup\{\|x\| : x \in A\} \le M_2$$

and

$$\sup\{\|\nabla g(z)\| : z \in A\} \le M_3.$$

It follows that for any $x, y \in A$,

$$\begin{aligned} D_g(x,y) &= g(x) - g(y) - \langle x - y, \nabla g(y)\rangle \\ &\le |g(x)| + |g(y)| + \|x - y\| \|\nabla g(y)\| \\ &\le 2M_1 + 2M_2M_3. \end{aligned}$$

Therefore, $\text{B-diam}(A) = \sup\{D_g(x,y) : x, y \in A\} < \infty$.

The following definition is slightly different from that in Butnariu and Iusem [29].

Definition 2.3 [30] Let E be a Banach space. The function $g : E \to \mathbb{R}$ is said to be a Bregman function if the following conditions are satisfied:

(1) g is continuous, strictly convex and Gâteaux differentiable;
(2) the set $\{y \in E : D_g(x,y) \le r\}$ is bounded for all $x \in E$ and $r > 0$.

The following lemma follows from Butnariu and Iusem [29] and Zălinscu [39].

Lemma 2.1 *Let E be a reflexive Banach space and let $g : E \to \mathbb{R}$ be a strongly coercive Bregman function. Then*

(1) $\nabla g : E \to E^*$ *is one-to-one, onto and norm-to-weak* continuous;*
(2) $\langle x - y, \nabla g(x) - \nabla g(y)\rangle = 0$ *if and only if* $x = y$;
(3) $\{x \in E : D_g(x,y) \le r\}$ *is bounded for all* $y \in E$ *and* $r > 0$;
(4) $\operatorname{dom} g^* = E^*$, g^* *is Gâteaux differentiable and* $\nabla g^* = (\nabla g)^{-1}$.

Let E be a Banach space and let C be a nonempty and convex subset of E. Let $g : E \to \mathbb{R}$ be a convex and Gâteaux differentiable function. Then we know from [40] that for $x \in E$ and $x_0 \in C$, $D_g(x_0, x) = \min_{y \in C} D_g(y, x)$ if and only if

$$\langle y - x_0, \nabla g(x) - \nabla g(x_0)\rangle \le 0, \quad \forall y \in C. \tag{2.3}$$

Further, if C is a nonempty, closed and convex subset of a reflexive Banach space E and $g: E \to \mathbb{R}$ is a strongly coercive Bregman function, then for each $x \in E$, there exists a unique $x_0 \in C$ such that

$$D_g(x_0, x) = \min_{y \in C} D_g(y, x).$$

The *Bregman projection* proj_C^g from E onto C is defined by $\operatorname{proj}_C^g(x) = x_0$ for all $x \in E$. It is also well known that proj_C^g has the following property:

$$D_g\bigl(y, \operatorname{proj}_C^g x\bigr) + D_g\bigl(\operatorname{proj}_C^g x, x\bigr) \le D_g(y, x) \tag{2.4}$$

for all $y \in C$ and $x \in E$ (see [29] for more details).

Let E be a Banach space and $B_r := \{z \in E : \|z\| \le r\}$ for all $r > 0$. Then a function $g: E \to \mathbb{R}$ is said to be *uniformly convex on bounded subsets* ([39, pp.203-221]) if $\rho_r(t) > 0$ for all $r, t > 0$, where $\rho_r : [0, +\infty) \to [0, \infty]$ is defined by

$$\rho_r(t) = \inf_{x,y \in B_r, \|x-y\|=t, \alpha \in (0,1)} \frac{\alpha g(x) + (1-\alpha)g(y) - g(\alpha x + (1-\alpha)y)}{\alpha(1-\alpha)}$$

for all $t \ge 0$. The function ρ_r is called the gage of uniform convexity of g. The function g is also said to be *uniformly smooth on bounded subsets* ([39, pp.207-221]) if $\lim_{t \downarrow 0} \frac{\sigma_r(t)}{t} = 0$ for all $r > 0$, where $\sigma_r : [0, +\infty) \to [0, \infty]$ is defined by

$$\sigma_r(t) = \sup_{x \in B_r, y \in S_E, \alpha \in (0,1)} \frac{\alpha g(x + (1-\alpha)ty) + (1-\alpha)g(x - \alpha ty) - g(x)}{\alpha(1-\alpha)}$$

for all $t \ge 0$.

The function g is said to be *uniformly convex* if the function $\delta_g : [0, +\infty) \to [0, +\infty]$, defined by

$$\delta_g(t) := \sup\left\{\frac{1}{2}g(x) + \frac{1}{2}g(y) - g\left(\frac{x+y}{2}\right) : \|y - x\| = t\right\},$$

satisfies that $\lim_{t \downarrow 0} \frac{\sigma_r(t)}{t} = 0$. Let $g: E \to (-\infty, +\infty]$ be a convex and Gâteaux differentiable function. Recall that, in view of [29, Section 1.2, p.17], the function g is called *totally convex* at a point $x \in \operatorname{int} \operatorname{dom} g$ if its *modulus of total convexity* at x, that is, the function $\nu_g : \operatorname{int} \operatorname{dom} g \times [0, +\infty) \to [0, +\infty)$ defined by

$$\nu_g(x, t) := \inf\bigl\{D_g(y, x) : y \in \operatorname{int} \operatorname{dom} g, \|y - x\| = t\bigr\},$$

is positive whenever $t > 0$. The function g is called *totally convex* when it is *totally convex* at every point $x \in \operatorname{int} \operatorname{dom} g$. Moreover, the function g is called *totally convex on bounded subsets* if $\nu_g(x, t) > 0$ for any bounded subset X of E and for any $t > 0$, where the *modulus of total convexity of the function* g on the set X is the function $\nu_g : \operatorname{int} \operatorname{dom} g \times [0, +\infty) \to [0, +\infty)$ defined by

$$\nu_g(X, t) := \inf\bigl\{\nu_g(x, t) : x \in X \cap \operatorname{int} \operatorname{dom} g\bigr\}.$$

It is well known that any uniformly convex function is totally convex, but the converse is not true in general (see [29, Section 1.3, p.30]).

It is also well known that g is totally convex on bounded sets if and only if the function g is uniformly convex on bounded sets (see [41, Theorem 2.10, p.9]).

Examples of totally convex functions can be found, for instance, in [29, 41].

Let E be a Banach space and let $g : E \to \mathbb{R}$ be a convex and Gâteaux differentiable function. Then the Bregman distance [15, 16] does not satisfy the well-known properties of a metric, but it does have the following important property, which is called the *three point identity* [42]:

$$D_g(x,z) = D_g(x,y) + D_g(y,z) + \langle x - y, \nabla g(y) - \nabla g(z)\rangle, \quad \forall x,y,z \in E. \tag{2.5}$$

In particular, it can be easily seen that

$$D_g(x,y) = -D_g(y,x) + \langle x - y, \nabla g(x) - \nabla g(y)\rangle, \quad \forall x,y \in E. \tag{2.6}$$

Indeed, by letting $z = x$ in (2.5) and taking into account that $D_g(x,x) = 0$, we get the desired result.

We will need the following important result; for the proof, we refer to ([29, p.67]).

Lemma 2.2 *Let E be a Banach space and let $g : E \to \mathbb{R}$ be a Gâteaux differentiable function which is uniformly convex on bounded sets. Let $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ be bounded sequences in E. Then the following assertions are equivalent*:

(1) $\lim_{n\to\infty} D_g(x_n,y_n) = 0$;
(2) $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Remark 2.3 Let E be a Banach space and let $g : E \to \mathbb{R}$ be a convex and Gâteaux differentiable function. Let C be a closed and convex subset of E. Then, in view of Lemma 2.2, any Bregman nonexpansive mapping $T : C \to C$ is continuous.

Let l^∞ denote the Banach space of bounded real sequences with the supremum norm. It is well known that there exists a bounded linear functional μ on l^∞ such that the following three conditions hold:

(1) If $\{t_n\}_{n\in\mathbb{N}} \in l^\infty$ and $t_n \geq 0$ for every $n \in \mathbb{N}$, then $\mu(t_n) \geq 0$;
(2) If $t_n = 1$ for every $n \in \mathbb{N}$, then $\mu(t_n) = 1$;
(3) $\mu(\{t_{n+1}\}) = \mu(\{t_n\})$ for all $\{t_n\}_{n\in\mathbb{N}} \in l^\infty$.

Such a functional μ is called a Banach limit and the value of μ at $\{t_n\}_{n\in\mathbb{N}} \in l^\infty$ is denoted by $\mu_n t_n$ (see, for example, [13]).

3 Common fixed points for Banach operator pairs

Let E be a Banach space and let $g : E \to \mathbb{R}$ be a convex and Gâteaux differentiable function. Let C be a closed and convex subset of a real Banach space E. A mapping $T : C \to E$ is said to be *Bregman quasi-nonexpansive* [17] if $F(T) \neq \emptyset$ and

$$D_g(p, Tx) \leq D_g(p,x), \quad \forall x \in C, p \in F(T).$$

Let C and D be nonempty subsets of a real Banach space E with $D \subset C$. A mapping $R_D : C \to D$ is said to be *sunny* if

$$R_D\big(R_D x + t(x - R_D x)\big) = R_D x$$

for each $x \in E$ and $t \geq 0$. A mapping $R_D : C \to D$ is said to be a *retraction* if $R_D x = x$ for each $x \in C$.

The following result was proved in [24].

Lemma 3.1 *Let E be a reflexive Banach space and let $g : E \to \mathbb{R}$ be a convex, continuous, strongly coercive and Gâteaux differentiable function which is bounded on bounded subsets and uniformly convex on bounded subsets. Let C be a nonempty, closed and convex subset of E. Let $T : C \to E$ be a Bregman quasi-nonexpansive mapping. Then $F(T)$ is closed and convex.*

Corollary 3.1 *Let E be a reflexive Banach space and let $g : E \to \mathbb{R}$ be a convex, continuous, strongly coercive and Gâteaux differentiable function which is bounded on bounded sets and uniformly convex on bounded sets. Let C be a nonempty, closed and convex subset of E and let $T : C \to E$ be a Bregman nonexpansive mapping. If $F(T) \neq \emptyset$, then it is closed and convex.*

Using ideas in [43], we can prove the following result.

Theorem 3.1 *Let E be a reflexive Banach space and let $g : E \to \mathbb{R}$ be a convex, continuous, strongly coercive and Gâteaux differentiable function which is bounded on bounded subsets and uniformly convex on bounded subsets. Let C be a nonempty, closed and convex subset of E and let $T : C \to C$ be a mapping. Let $\{x_n\}_{n\in\mathbb{N}}$ be a bounded sequence of C and let μ be a mean on l^∞. Suppose that*

$$\mu_n D_g(x_n, Ty) \leq \mu_n D_g(x_n, y)$$

for all $y \in C$. Then T has a fixed point in C.

Proof Let μ be a mean on l^∞ and $\{x_n\}_{n\in\mathbb{N}}$ be a bounded sequence in C. Define a mapping $h : E^* \to \mathbb{R}$ by

$$h(x^*) = \mu_n\langle x_n, x^*\rangle, \quad x^* \in E^*.$$

Since μ is linear, so is h. Observe that

$$\begin{aligned} \big|h(x^*)\big| &= \big|\mu_n\langle x_n, x^*\rangle\big| \\ &\leq \|\mu\| \sup_{n\in\mathbb{N}} \big|\langle x_n, x^*\rangle\big| \\ &\leq \|\mu\| \sup_{n\in\mathbb{N}} \|x_n\| \, \|x^*\| \\ &= \sup_{n\in\mathbb{N}} \|x_n\| \, \|x^*\| \end{aligned}$$

for all $x^* \in E^*$. This implies that h is a linear and continuous real-valued mapping on E^*. Since E is reflexive, then there exists a unique element $z \in E$ such that

$$h(x^*) = \mu_n\langle x_n, x^*\rangle = \langle z, x^*\rangle, \quad x^* \in E^*.$$

We claim that $z \in C$. If not, then by the separation theorem [13] there exists $y^* \in E^*$ such that

$$\langle z, y^*\rangle < \inf_{y \in C}\langle y, y^*\rangle.$$

Since $\{x_n\}_{n\in\mathbb{N}} \subset C$, we conclude that

$$\langle z, y^*\rangle < \inf_{y \in C}\langle y, y^*\rangle \leq \inf_{n\in\mathbb{N}}\langle x_n, y^*\rangle \leq \mu_n\langle x_n, x^*\rangle = \langle z, x^*\rangle.$$

This is a contradiction. Thus we have $z \in C$. In view of (2.5), for any $y \in C$ and $n \in \mathbb{N}$, we deduce that

$$D_g(x_n, y) = D_g(x_n, Ty) + D_g(Ty, y) + \langle x_n - Ty, \nabla g(Ty) - \nabla g(y)\rangle.$$

Thus we have, for any $y \in C$, that

$$\begin{aligned}\mu_n D_g(x_n, y) &= \mu_n D_g(x_n, Ty) + \mu_n D_g(Ty, y) + \mu_n\langle x_n - Ty, \nabla g(Ty) - \nabla g(y)\rangle \\ &= \mu_n D_g(x_n, Ty) + D_g(Ty, y) + \langle z - Ty, \nabla g(Ty) - \nabla g(y)\rangle.\end{aligned}$$

By the assumption, we have that

$$\mu_n D_g(x_n, Ty) \leq \mu_n D_g(x_n, y)$$

for all $y \in C$. This implies that

$$\mu_n D_g(x_n, y) \leq \mu_n D_g(x_n, y) + D_g(Ty, y) + \langle z - Ty, \nabla g(Ty) - \nabla g(y)\rangle \tag{3.1}$$

for all $y \in C$. Putting $y = z$ in (3.1) and taking into account (2.6), we see that

$$\begin{aligned}0 &\leq D_g(Tz, z) + \langle z - Tz, \nabla g(Tz) - \nabla g(z)\rangle \\ &= -D_g(z, Tz) + \langle z - Tz, \nabla g(z) - \nabla g(Tz)\rangle + \langle z - Ty, \nabla g(Tz) - \nabla g(z)\rangle \\ &= -D_g(z, Tz).\end{aligned}$$

Then we have $0 \leq -D_g(z, Tz)$, which implies that $D_g(z, Tz) = 0$. In view of Lemma 2.2, we conclude that $Tz = z$, which completes the proof. □

Remark 3.1 Let g and T be as in Example 1.1. Let $x \in [0, 0.9]$ be fixed. Then $\{T^n x\}_{n\in\mathbb{N}}$ is a bounded sequence in $[0, 0.9]$. Set $x_n := T^n x$ for $n = 1, 2, \ldots$. It is obvious that T satisfies all the aspects of the hypothesis of Theorem 3.1, so it has a fixed point.

Corollary 3.2 *Let E be a reflexive Banach space and let $g : E \to \mathbb{R}$ be a convex, continuous, strongly coercive and Gâteaux differentiable function which is bounded on bounded subsets and uniformly convex on bounded subsets. Let C be a nonempty, closed and convex subset of E and let $T : C \to C$ be a mapping. Suppose that there exist $x \in C$ and a Banach limit μ such that $\{T^n x\}_{n\in\mathbb{N}}$ is bounded and*

$$\mu_n D_g\big(T^n x, Ty\big) \le \mu_n D_g\big(T^n x, y\big)$$

for all $y \in C$. Then T has a fixed point.

Corollary 3.3 *Let E be a reflexive Banach space and let $g : E \to \mathbb{R}$ be a convex, continuous, strongly coercive and Gâteaux differentiable function which is bounded on bounded subsets and uniformly convex on bounded subsets. Let C be a nonempty, closed and convex subset of E and let $T : C \to C$ be a Bregman nonexpansive mapping. Suppose that there exists $x \in C$ such that $\{T^n x\}_{n\in\mathbb{N}}$ is bounded. Then T has a fixed point.*

Proof Let μ a Banach limit on l^∞ and $x \in C$ be such that $\{T^n x\}_{n\in\mathbb{N}}$ is bounded. Then we have

$$\mu_n D_g\big(T^n x, Ty\big) = \mu_n D_g\big(T^{n+1} x, Ty\big) \le \mu_n D_g\big(T^n x, y\big)$$

for all $y \in C$. In view of Corollary 3.2, we deduce that $F(T) \neq \emptyset$, which completes the proof. □

Corollary 3.4 *Let E be a reflexive Banach space and let $g : E \to \mathbb{R}$ be a convex, continuous, strongly coercive and Gâteaux differentiable function which is bounded on bounded sets and uniformly convex on bounded sets. Let C be a nonempty, bounded, closed and convex subset of E and let $T : C \to C$ be a Bregman nonexpansive mapping. Then T has a fixed point.*

Definition 3.1 Let A and C be nonempty subsets of a real Banach space E with $A \subset C$. We say that A is a Bregman nonexpansive retract of C if there exists a Bregman nonexpansive map $R : C \to A$ such that $R(a) = a$ for every $a \in A$.

Definition 3.2 Let C be a nonempty, closed and convex subset of a real Banach space E. The mapping $T : C \to C$ is called Bregman NR-map if $\mathrm{Fix}(T)$ is a Bregman nonexpansive retract of C.

Theorem 3.2 *Let E be a reflexive Banach space and let $g : E \to \mathbb{R}$ be a convex, continuous, strongly coercive and Gâteaux differentiable function which is bounded on bounded sets and uniformly convex on bounded sets. Let C be a nonempty, bounded, closed and convex subset of E. Let $T : C \to C$ be a continuous Bregman NR-map. Let $S : C \to C$ be a Bregman nonexpansive mapping such that (S, T) is a Banach operator pair. Then $F(S, T)$ is not empty.*

Proof Since the retract of a nonempty space is nonempty, $\mathrm{Fix}(T)$ is nonempty and is closed as T is continuous. Since T is a Bregman NR-map, then there exists a Bregman nonexpansive retract $R : C \to \mathrm{Fix}(T)$. Since (S, T) is a Banach operator pair, then $S(\mathrm{Fix}(T)) \subset \mathrm{Fix}(T)$.

Hence $S \circ R : C \to C$ is a Bregman nonexpansive map such that $S \circ R(C) \subset \mathrm{Fix}(T)$. Corollary 3.4 implies the existence of a fixed point of $S \circ R$. Clearly, such a fixed point is a fixed point of S which belongs to $\mathrm{Fix}(T)$. Hence $\mathrm{Fix}(T) \cap \mathrm{Fix}(S) = F(S, T)$ is not empty. □

Example 3.1 Let E be a reflexive and smooth Banach space and let C be a closed and convex subset of E such that $0 \in C$. Let $T : C \to E$ be defined as

$$T(x) = -x, \quad x \in C.$$

Then T is a Bregman quasi-nonexpansive mapping with $g(x) = \frac{1}{2}\|x\|^2$, $\nabla g(x) = Jx$ for all $x \in C$ and $F(T) = \{0\}$. Indeed, it is clear that

$$\|Tx\| \leq \|x\|, \quad \forall x \in C.$$

This implies that

$$g(Tx) - g(x) = \frac{1}{2}\|Tx\|^2 - \frac{1}{2}\|x\|^2 \leq \bigl\langle 0, \nabla g(Tx) - \nabla g(x)\bigr\rangle = \bigl\langle p, \nabla g(Tx) - \nabla g(x)\bigr\rangle$$

for all $p \in F(T)$. Then we have

$$\|p\|^2 + \|Tx\|^2 - 2\bigl\langle p, \nabla g(Tx)\bigr\rangle \leq \|p\|^2 + \|x\|^2 - 2\bigl\langle p, \nabla g(x)\bigr\rangle.$$

This means that

$$D_g(p, Tx) \leq D_g(p, x),$$

for all $p \in F(T)$ and $x \in C$. Hence, T is a Bregman quasi-nonexpansive mapping. Define a mapping $R : C \to \{0\}$ by

$$R(x) = 0, \quad \forall x \in C.$$

Then T is a Bregman NR-map.

Assume now that $h : E \to \mathbb{R}$ is a lower semicontinuous function satisfying the following conditions:

(i) h is totally convex on bounded sets;
(ii) h, as well as its Fenchel conjugate h^*, are defined and (Gâteaux) differentiable on E and E^*, respectively;
(iii) h' is uniformly continuous and h^* is bounded on bounded sets.

Let $A : \mathrm{dom}\,A \to E^*$ be an operator and Ω be a nonempty subset of $\mathrm{dom}\,A$ such that $0 \in \Omega$, $A(0) = 0$ and $C \subset \mathrm{dom}\,A$. For any $\alpha \in (0, \infty)$, we define the operator $A_\alpha^h : \mathrm{dom}\,A \to E$ by

$$A_\alpha^h x = h^{*\prime}\bigl(h'(x) - \alpha Ax\bigr).$$

It is worth mentioning that $Ax = 0$ if and only if $x \in \mathrm{dom}\,A$ is a fixed point of A_α^h. The operator A is said to be inverse-strongly-monotone relative to h on the set Ω if there exist a real number $\alpha > 0$ and a vector $z \in \Omega$ such that

$$\bigl\langle Ay, A_\alpha^h y - z\bigr\rangle \geq 0, \quad \forall y \in \Omega.$$

If we set $S := A^h_\alpha$, then S is a Bregman nonexpansive mapping (for more details, see [41]). It is clear that T and S satisfy all the aspects of the hypothesis of Theorem 3.2 and T and S have a common fixed point.

Remark 3.2 Let E be a reflexive Banach space and let $g : E \to \mathbb{R}$ be a convex, continuous, strongly coercive and Gâteaux differentiable function which is bounded on bounded subsets and uniformly convex on bounded subsets. Let C be a nonempty, bounded, closed and convex subset of E and let $T : C \to C$ be a Bregman nonexpansive mapping. Then, in view of Corollary 3.4 and Lemma 3.1, $\operatorname{Fix}(T)$ is not empty and closed convex which implies that $\operatorname{Fix}(T)$ is a Bregman nonexpansive retract of C. Thus T is a Bregman NR-map.

Theorem 3.3 *Let E be a reflexive Banach space and let $g : E \to \mathbb{R}$ be a convex, continuous, strongly coercive and Gâteaux differentiable function which is bounded on bounded subsets and uniformly convex on bounded subsets. Let T and S be two Bregman nonexpansive self-mappings defined on a closed and convex subset C of E. If (S, T) is a Banach operator pair and $T(C)$ is bounded, then* $\operatorname{Fix}(T) \cap \operatorname{Fix}(S) \neq \emptyset$.

Proof Let $K = \overline{\operatorname{conv}}(T(C))$. Then $T : K \to K$ and K is nonempty and bounded. In view of Corollary 3.4, the fixed point set $\operatorname{Fix}(T)$ of T is nonempty and bounded. Since (S, T) is a Banach operator pair, $S : \operatorname{Fix}(T) \to \operatorname{Fix}(T)$. By Corollary 3.4, S has a fixed point in $\operatorname{Fix}(T)$ as required. □

The following slight extension of Theorem 3.3 can be proved easily.

Theorem 3.4 *Let E be a reflexive Banach space and let $g : E \to \mathbb{R}$ be a convex, continuous, strongly coercive and Gâteaux differentiable function which is bounded on bounded subsets and uniformly convex on bounded subsets. Let C be a nonempty, closed and convex subset of E. Let X be a normed space and T and S be two Bregman nonexpansive self-mappings defined on a closed convex set $C \subset E$. If (S, T) is a Banach operator pair, and if $\overline{T^n(C)}$ is bounded for some $n \in \mathbb{N}$, then* $\operatorname{Fix}(T) \cap \operatorname{Fix}(S) \neq \emptyset$.

Corollary 3.5 *Let E be a reflexive Banach space and let $g : E \to \mathbb{R}$ be a convex, continuous, strongly coercive and Gâteaux differentiable function which is bounded on bounded subsets and uniformly convex on bounded subsets. Let C be a nonempty, bounded, closed and convex subset of E. Let $T : C \to C$ be Bregman nonexpansive. Let $S : C \to C$ be a Bregman nonexpansive mapping such that (S, T) is a Banach operator pair. Then $F(S, T)$ is not empty.*

Corollary 3.6 *Let E be a reflexive Banach space and let $g : E \to \mathbb{R}$ be a convex, continuous, strongly coercive and Gâteaux differentiable function which is bounded on bounded subsets and uniformly convex on bounded subsets. Let C be a nonempty, closed and convex subset of E. Let $T : C \to C$ be a Bregman nonexpansive map such that $T(C)$ is bounded and $T(C) \subset$* $\operatorname{Fix}(T)$. *Let $S : C \to C$ be a Bregman nonexpansive mapping such that (S, T) is nontrivially a Banach operator pair. Then* $\operatorname{Fix}(S) \cap \operatorname{Fix}(T)$ *is not empty.*

Corollary 3.7 *Let E be a reflexive Banach space and let $g : E \to \mathbb{R}$ be a convex, continuous, strongly coercive and Gâteaux differentiable function which is bounded on bounded* *subsets and uniformly convex on bounded subsets. Let C be a nonempty, closed and convex subset of E. Let $S, T : C \to C$ be a nontrivially Banach operator pair such that* $\mathrm{Fix}(T)$ *is bounded and S is a Bregman nonexpansive map. Assume that $T : C \to \mathrm{Fix}(T)$ is a Bregman nonexpansive map. Then* $\mathrm{Fix}(S) \cap \mathrm{Fix}(T)$ *is not empty.*

Theorem 3.5 *Let E be a reflexive Banach space and let $g : E \to \mathbb{R}$ be a convex, continuous, strongly coercive and Gâteaux differentiable function which is bounded on bounded subsets and uniformly convex on bounded subsets. Let C be a nonempty, closed and convex subset of E which has the property that every Bregman nonexpansive mapping of $C \to C$ is Bregman NR-map. Suppose $T : C \to C$ is a mapping for which T^n is Bregman nonexpansive for some $n \in \mathbb{N}$, and suppose the restriction of T to* $\mathrm{Fix}(T^n)$ *is also Bregman nonexpansive. Then* $\mathrm{Fix}(T)$ *is a nonempty Bregman nonexpansive retract of C. Consequently, if $S : C \to C$ is Bregman nonexpansive and if (S, T) is a Banach operator pair, then* $\mathrm{Fix}(T) \cap \mathrm{Fix}(S)$ *is a nonempty Bregman nonexpansive retract of C.*

Proof By assumption, there exists a Bregman nonexpansive retraction R_1 of C onto $\mathrm{Fix}(T^n)$. Consequently, $T \circ R_1$ is a Bregman nonexpansive mapping of C into C, so $\mathrm{Fix}(T \circ R_1)$ is a nonempty Bregman nonexpansive retract of C. But $x \in \mathrm{Fix}(T \circ R_1) \Leftrightarrow x \in \mathrm{Fix}(T) \cap \mathrm{Fix}(T^n)$, and by Lemma 1 [44]

$$x \in \mathrm{Fix}(T) \cap \mathrm{Fix}\big(T^n\big) \quad \Leftrightarrow \quad x \in \mathrm{Fix}\big(T^{n+1}\big) \cap \mathrm{Fix}\big(T^n\big) = \mathrm{Fix}(T).$$

Therefore there is a Bregman nonexpansive retraction R_2 of C onto $\mathrm{Fix}(T)$. So, $S \circ R_2$ is a Bregman nonexpansive mapping of C into $\mathrm{Fix}(T)$. Therefore $\mathrm{Fix}(S \circ R_2) = \mathrm{Fix}(S) = \mathrm{Fix}(S) \cap \mathrm{Fix}(T)$ is a nonempty Bregman nonexpansive retract of C. □

We might observe that in the above theorem it is not necessary that T be Bregman nonexpansive. The only facts needed for the proof is that $\mathrm{Fix}(T^n)$ be a Bregman nonexpansive retract of C.

4 Fixed point of Banach operator family

Definition 4.1 Let C be a closed and convex subset of a real Banach space E and let T and S be two self-maps on C. The pair (S, T) is called a symmetric Banach operator pair if both (S, T) and (T, S) are Banach operator pairs, *i.e.*, $T(\mathrm{Fix}(S)) \subseteq \mathrm{Fix}(S)$ and $S(\mathrm{Fix}(T)) \subseteq \mathrm{Fix}(T)$.

It is easy to see that the pair (S, T) is a symmetric Banach operator pair if and only if T and S are commuting on $\mathrm{Fix}(T) \cup \mathrm{Fix}(S)$.

Definition 4.2 A subset A of a Banach space E is said to be a 1-*local Bregman retract* of E if for every family $\{B_i : i \in I\}$ of Bregman closed balls centered in A with nonempty intersection, it is the case that $A \cap (\bigcap_{i \in I} B_i) \neq \emptyset$. It is immediate that each Bregman nonexpansive retract of E is a 1-local Bregman retract (but not conversely).

Definition 4.3 Let C be a closed and convex subset of a real Banach space E and let $\mathcal{T}$ be a family of mappings defined on C. Then the family $\mathcal{T}$ has a common fixed point if it is the fixed point of each member of $\mathcal{T}$. The family $\mathcal{T}$ is called a Banach operator family if any two of maps in the family form a symmetric Banach operator pair.

Theorem 4.1 *Let E be a reflexive Banach space and let $g : E \to \mathbb{R}$ be a convex, continuous, strongly coercive and Gâteaux differentiable function which is bounded on bounded subsets and uniformly convex on bounded subsets. Let C be a nonempty, closed and convex subset of E and let $\mathcal{H}$ be a nonempty family of Bregman nonexpansive maps of C into itself. If $\mathcal{H}$ is a Banach operator family and there exists $T \in \mathcal{H}$ such that $\overline{T(C)}$ is compact, then $\mathcal{H}$ has a common fixed point in C.*

Proof Let $K = \overline{\operatorname{conv}}(T(C))$. It suffices to show that each finite subfamily of $\mathcal{H}$ has a nonempty common fixed point set in K. The full conclusion then follows from the compactness of K. Let $\{T_1, T_2, \dots, T_n\}$ be a finite subfamily of $\mathcal{H}$. As above, $\operatorname{Fix}(T)$ is nonempty. Since (T_1, T) is a Banach operator pair, $T_1 : \operatorname{Fix}(T) \to \operatorname{Fix}(T)$. By Corollary 3.4, T_1 has a fixed point in $\operatorname{Fix}(T)$. Since (T_2, T_1) is a Banach operator pair, $T_2 : \operatorname{Fix}(T_1) \to \operatorname{Fix}(T_1)$. Proceeding in a step by step way, we conclude $\operatorname{Fix}(T) \cap \operatorname{Fix}(T_1) \cap \cdots \cap \operatorname{Fix}(T_n) \neq \emptyset$. □

Theorem 4.2 *Let E be a reflexive Banach space and let $g : E \to \mathbb{R}$ be a convex, continuous, strongly coercive and Gâteaux differentiable function which is bounded on bounded subsets and uniformly convex on bounded subsets. Let C be a nonempty, bounded, closed and convex subset of E such that $\mathcal{BA}(C)$ is compact and normal. Let $\mathcal{T}$ be a family of Bregman nonexpansive mappings $T_1, T_2, \dots, T_n$, $T_i : C \to C$. Assume that any two mappings from $\mathcal{T}$ form a symmetric Banach operator pair. Then the family $\mathcal{T}$ has a common fixed point. Moreover, the common fixed point set* $\operatorname{Fix}(\mathcal{T})$ *is a* 1-*local Bregman retract of C.*

Proof First, let us prove that $F = \operatorname{Fix}(\mathcal{T})$ is not empty. Using Corollary 3.4, we know that $\operatorname{Fix}(T_1)$ is not empty. Since $\operatorname{Fix}(T_1)$ is a 1-local Bregman retract [4] of C, by a similar argument as in [4], we conclude that $\mathcal{A}(\operatorname{Fix}(T_1))$ is compact and normal. On the other hand, we have $T_2(\operatorname{Fix}(T_1)) \subset \operatorname{Fix}(T_1)$ because any two mappings from $\mathcal{T}$ form a symmetric Banach operator pair. Hence T_2 has a fixed point in $\operatorname{Fix}(T_1)$. If we restrict ourselves to $\operatorname{Fix}(T_1, T_2)$, the common fixed point set of T_1 and T_2, then one can prove in an identical argument that T_3 has a fixed point in $\operatorname{Fix}(T_1, T_2)$. Step by step, we can prove that the common fixed point set $\operatorname{Fix}(\mathcal{T})$ of $T_1, \dots, T_n$ is not empty. The same argument, used to prove that the fixed point set of a Bregman nonexpansive map is a 1-local Bregman retract, can be reproduced here to prove that $\operatorname{Fix}(\mathcal{T})$ is a 1-local Bregman retract. □

Theorem 4.3 *Let E be a reflexive Banach space and let $g : E \to \mathbb{R}$ be a convex, continuous, strongly coercive and Gâteaux differentiable function which is bounded on bounded subsets and uniformly convex on bounded subsets. Let C be a nonempty, bounded, closed and convex subset of E such that $\mathcal{BA}(C)$ is compact and normal. Let $\mathcal{T}$ be a family of Bregman nonexpansive mappings $(T_i)_{i \in I}$, $T_i : C \to C$. Assume that any two mappings from $\mathcal{T}$ form a symmetric Banach operator pair. Then the family $\mathcal{T}$ has a common fixed point. Moreover, the common fixed point set* $\operatorname{Fix}(\mathcal{T})$ *is a* 1-*local Bregman retract of C.*

Proof $\Gamma = 2^I = \{\beta \subset I : \beta \textit{ is finite and nonempty}\}$. It is obvious that Γ is downward directed (the order on Γ is the set inclusion). Theorem 4.2 implies that for every $\beta \in \Gamma$, the set F_β of a common fixed point set of the mappings T_i, $i \in \beta$, is a nonempty 1-local Bregman retract of C. Clearly, the family $(F_\beta)_{\beta \in \Gamma}$ is decreasing. Using the remark following Theorem 6 [4], we deduce that $\bigcap_{\beta \in \Gamma} F_\beta$ is nonempty and is a 1-local Bregman retract of C. □

Lemma 4.1 *Let E be a reflexive Banach space and let $g : E \to \mathbb{R}$ be a convex, continuous, strongly coercive and Gâteaux differentiable function which is bounded on bounded subsets and uniformly convex on bounded subsets. Let C be a nonempty, closed and convex subset of E such that $\mathcal{BA}(C)$ is compact and normal. Let $\mathcal{T}$ be a family of Bregman nonexpansive mappings defined on C. Let τ be a topology on C for which the closed balls are τ-closed. Assume that there exists a bounded subset $A \subset C$ with $\delta = \text{B-diam}(A)$ such that $C = \bigcap_{a\in A} \bar{B}(a,\delta)$, is 1-local Bregman retract of C, $A \subset \overline{T(A)}^{\tau}$, for any $T \in \mathcal{T}$, where $\overline{T(A)}^{\tau}$ is the τ-closure of $T(A)$. Assume that any two mappings from $\mathcal{T}$ form a symmetric Banach operator pair. Then the family $\mathcal{T}$ has a common fixed point.*

Proof Denote by $\delta = \text{B-diam}(A)$. Consider the subset

$$C = \bigcap_{a\in A} \bar{B}(a,\delta).$$

Clearly, we have $A \subset C$. Let $T \in \mathcal{T}$, then

$$T(C) \subset \bigcap_{a\in A} \bar{B}\big(T(a),\delta\big)$$

since T is Bregman nonexpansive. This implies

$$T(A) \subset \bigcap_{c\in T(C)} \bar{B}(c,\delta).$$

Since the Bregman closed balls are τ-closed, we get

$$\overline{T(A)}^{\tau} \subset \bigcap_{c\in T(C)} \bar{B}(c,\delta).$$

Our assumption implies

$$A \subset \bigcap_{c\in T(C)} \bar{B}(c,\delta).$$

Hence

$$T(C) \subset \bigcap_{a\in A} \bar{B}(a,\delta) = C.$$

Since C is bounded and is 1-local Bregman retract of C, so $\mathcal{BA}(C)$ is compact and normal and the theorem above implies that $\mathcal{T}$ has a common fixed point. □

Definition 4.4 Let C be nonempty, closed and convex subset of a Banach space E. Let $\mathcal{T}$ be a family of mappings defined on C. The family $\mathcal{T}$ is called a semigroup if $S \circ T \in \mathcal{T}$ whenever $S, T \in \mathcal{T}$. We will call the semigroup $\mathcal{T}$ an invertible semigroup if and only if any element in $\mathcal{T}$ is invertible and $T^{-1} \in \mathcal{T}$ for any $T \in \mathcal{T}$. For any $x \in C$, define the orbit of x by

$$\mathcal{T}(x) = \big\{T(x); T \in \mathcal{T}\big\}.$$

Theorem 4.4 *Let C be a nonempty, closed and convex subset of a Banach space E such that $\mathcal{BA}(C)$ is compact and normal. Let $\mathcal{T}$ be an invertible semigroup of isometric mappings defined on H such that any two mappings from $\mathcal{T}$ form a symmetric Banach operator pair. Assume that $C = \bigcap_{a\in A} \bar{B}(a,\delta)$ is 1-local Bregman retract of C, where $A = \mathcal{T}(x)$ and $\delta =$ B-diam(A). Then the family $\mathcal{T}$ has a common fixed point if and only if $\bigcap_{T\in\mathcal{T}} T(C)$ is not empty and $\mathcal{T}$-orbits are bounded.*

Proof Clearly, if $\mathcal{T}$ has a fixed point, then we have $\bigcap_{T\in\mathcal{T}} T(C)$ is not empty and $\mathcal{T}$-orbits are bounded. So, let us assume that $\bigcap_{T\in\mathcal{T}} T(C)$ is not empty and $\mathcal{T}$-orbits are bounded. Let $x \in \bigcap_{T\in\mathcal{T}} T(C)$. The orbit $A = \mathcal{T}(x)$ is bounded. Note that $T(A) = A$ for any $A \in \mathcal{T}$. Indeed, by the definition of the orbit $\mathcal{T}(x)$, we have $T(A) \subset A$. Let $a \in A$, then there exists $S \in \mathcal{T}$ such that $a = S(x)$. Clearly, we have $a = T(T^{-1} \circ S(x))$. Since $T^{-1} \circ S \in \mathcal{T}$, we conclude that $a \in T(A)$. Next we consider the admissible subset $C = \bigcap_{a\in A} \bar{B}(a,\delta)$, where $\delta =$ B-diam(A). Obviously, $A \subset C$ and C is a bounded and 1-local Bregman retract of C. As in the proof of the lemma above, one will easily show that $T(C) \subset C$ for any $T \in \mathcal{T}$. So, from Theorem 4.3, we conclude that $\mathcal{T}$ has a common fixed point and its fixed point set Fix$(\mathcal{T})$ is 1-local Bregman retract of C. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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