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Common fixed point theorems for two weakly compatible self-mappings in cone *b*-metric spaces

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Abstract

In this paper, we establish common fixed point theorems for two weakly compatible self-mappings satisfying the contractive condition or the quasi-contractive condition in the case of a quasi-contractive constant $\lambda \in (0, 1/s)$ in cone b-metric spaces without the normal cone, where the coefficient s satisfies $s \geq 1$. The main results generalize, extend and unify several well-known comparable results in the literature.

Keywords: common fixed point; weakly compatible self-mappings; (quasi-)contractive condition; cone b-metric space

1 Introduction and preliminaries

Huang and Zhang [1] introduced the concept of a cone metric space, proved the properties of sequences on cone metric spaces and obtained various fixed point theorems for contractive mappings. The existence of a common fixed point on cone metric spaces was considered recently in [2–5]. Also, Ilic and Rakocevic [6] introduced a quasi-contraction on a cone metric space when the underlying cone was normal. Later on, Kadelburg *et al.* obtained a few similar results without the normality of the underlying cone, but only in the case of a quasi-contractive constant $\lambda \in (0, 1/2)$. However, Gajic [7] proved that result is true for $\lambda \in (0, 1)$ on a cone metric space by a new way, which answered the open question whether the result is true for $\lambda \in (0, 1)$. Recently, Hussain and Shah [8] introduced cone b-metric spaces, as a generalization of b-metric spaces and cone metric spaces, and established some important topological properties in such spaces. Following Hussain and Shah, Huang and Xu [9] obtained some interesting fixed point results for contractive mappings in cone b-metric spaces. Although Ion Marian [10] proved some common fixed point theorems in complete b-cone metric spaces, the main ways of the proof depend strongly on the nonlinear scalarization function $\xi_e : Y \to \mathbb{R}$. In the present paper, we will show common fixed point theorems for two weakly compatible self-mappings satisfying the contractive condition or quasi-contractive condition in the case of a quasi-contractive constant $\lambda \in (0, 1/s)$ in cone b-metric spaces without the assumption of normality, where the coefficient s satisfies $s \geq 1$. As consequences, our results generalize, extend and unify several well-known comparable results (see, for example, [2–7, 9–13]).

Consistent with Huang and Zhang [1], the following definitions and results will be needed in the sequel.

Let E be a real Banach space and let P be a subset of E. By θ we denote the zero element of E and by $\operatorname{int} P$ the interior of P. The subset P is called a cone if and only if:

(i) P is closed, nonempty, and $P \neq \{\theta\}$;

(ii) $a, b \in \mathbb{R}$, $a, b \geq 0$, $x, y \in P \Rightarrow ax + by \in P$;

(iii) $P \cap (-P) = \{\theta\}$.

On this basis, we define a partial ordering $\preceq$ with respect to P by $x \preceq y$ if and only if $y - x \in P$. We write $x \prec y$ to indicate that $x \preceq y$ but $x \neq y$, while $x \ll y$ stands for $y - x \in \operatorname{int} P$. Write $\|\cdot\|$ as the norm on E. The cone P is called normal if there is a number $K > 0$ such that for all $x, y \in E$, $\theta \preceq x \preceq y$ implies $\|x\| \leq K\|y\|$. The least positive number satisfying the above is called the normal constant of P. It is well known that $K \geq 1$.

In the following, we always suppose that E is a Banach space, P is a cone in E with $\operatorname{int} P \neq \emptyset$ and $\preceq$ is a partial ordering with respect to P.

Definition 1.1 [8] Let X be a nonempty set and let $s \geq 1$ be a given real number. A mapping $d : X \times X \to E$ is said to be cone b-metric if and only if for all $x, y, z \in X$ the following conditions are satisfied:

(i) $\theta \prec d(x, y)$ with $x \neq y$ and $d(x, y) = \theta$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$;

(iii) $d(x, y) \preceq s[d(x, z) + d(z, y)]$.

The pair (X, d) is called a cone b-metric space.

Example 1.2 Consider the space L_p $(0 < p < 1)$ of all real function $x(t)$ $(t \in [0,1])$ such that $\int_0^1 |x(t)|^p \, dt < \infty$. Let $X = L_p$, $E = \mathbb{R}^2$, $P = \{(x, y) \in E \mid x, y \geq 0\} \subset \mathbb{R}^2$ and $d : X \times X \to E$ such that

$$d(x, y) = \left(\alpha \left\{ \int_0^1 \left| x(t) - y(t) \right|^p dt \right\}^{\frac{1}{p}}, \beta \left\{ \int_0^1 \left| x(t) - y(t) \right|^p dt \right\}^{\frac{1}{p}} \right),$$

where $\alpha, \beta \geq 0$ are constants. Then (X, d) is a cone b-metric space with the coefficient $s = 2^{\frac{1}{p} - 1}$.

Remark 1.3 It is obvious that any cone metric space must be a cone b-metric space. Moreover, cone b-metric spaces generalize cone metric spaces, b-metric spaces and metric spaces.

Definition 1.4 [8] Let (X, d) be a cone b-metric space, $x \in X$ and $\{x_n\}$ be a sequence in X. Then

(i) $\{x_n\}$ converges to x whenever, for every $c \in E$ with $\theta \ll c$, there is a natural number N such that $d(x_n, x) \ll c$ for all $n \geq N$. We denote this by $\lim_{n \to \infty} x_n = x$ or $x_n \to x$ $(n \to \infty)$.

(ii) $\{x_n\}$ is a Cauchy sequence whenever, for every $c \in E$ with $\theta \ll c$, there is a natural number N such that $d(x_n, x_m) \ll c$ for all $n, m \geq N$.

(iii) (X, d) is a complete cone b-metric space if every Cauchy sequence is convergent.

Lemma 1.5 [8] *Let (X, d) be a cone b-metric space. The following properties are often used while dealing with cone b-metric spaces in which the cone is not necessarily normal.*

(1) *If $u \ll v$ and $v \preceq w$, then $u \ll w$;*

(2) *If $\theta \preceq u \ll c$ for each $c \in \operatorname{int} P$, then $u = \theta$;*

(3) *If $a \preceq b + c$ for each $c \in \operatorname{int} P$, then $a \preceq b$;*

(4) *If* $\theta \preceq d(x_n, x) \preceq b_n$ *and* $b_n \to \theta$, *then* $x_n \to x$;
(5) *If* $a \preceq \lambda a$, *where* $a \in P$ *and* $0 < \lambda < 1$, *then* $a = \theta$;
(6) *If* $c \in \operatorname{int} P$, $\theta \preceq a_n$ *and* $a_n \to \theta$, *then there exists* $n_0 \in \mathbb{N}$ *such that* $a_n \ll c$ *for all* $n > n_0$.

Lemma 1.6 [8] *The limit of a convergent sequence in a cone b-metric space is unique.*

Definition 1.7 [2] The mappings $f, g : X \to X$ are weakly compatible if for every $x \in X$, $fgx = gfx$ holds whenever $fx = gx$.

Definition 1.8 [3] Let f and g be self-maps of a set X. If $w = fx = gx$ for some x in X, then x is called a coincidence point of f and g, and w is called a point of coincidence of f and g.

Lemma 1.9 [3] *Let* f *and* g *be weakly compatible self-maps of a set* X. *If* f *and* g *have a unique point of coincidence* $w = fx = gx$, *then* w *is the unique common fixed point of* f *and* g.

Definition 1.10 [13] Let (X, d) be a cone metric space. A mapping $f : X \to X$ is such that, for some constant $\lambda \in (0, 1)$ and for every $x, y \in X$, there exists an element

$$u \in C(g, x, y) = \{d(gx, gy), d(gx, fx), d(gy, fy), d(gx, fy), d(gy, fx)\}$$

for which $d(fx, fy) \preceq \lambda u$ is called a g-quasi-contraction.

2 Main results

In this section, we give some common fixed point results for two weakly compatible self-mappings satisfying the contractive condition and quasi-contractive condition in the case of a contractive constant $\lambda \in (0, 1/s)$ in cone b-metric spaces without the assumption of normality.

Theorem 2.1 *Let* (X, d) *be a cone b-metric space with the coefficient* $s \geq 1$ *and let* $a_i \geq 0$ $(i = 1, 2, 3, 4, 5)$ *be constants with* $2sa_1 + (s + 1)(a_2 + a_3) + (s^2 + s)(a_4 + a_5) < 2$. *Suppose that the mappings* $f, g : X \to X$ *satisfy the condition, for all* $x, y \in X$,

$$d(fx, fy) \preceq a_1 d(gx, gy) + a_2 d(gx, fx) + a_3 d(gy, fy) + a_4 d(gx, fy) + a_5 d(gy, fx). \tag{2.1}$$

If the range of g *contains the range of* f *and* $g(X)$ *or* $f(X)$ *is a complete subspace of* X, *then* f *and* g *have a unique point of coincidence in* X. *Moreover, if* f *and* g *are weakly compatible, then* f *and* g *have a unique common fixed point in* X.

Proof For an arbitrary $x_0 \in X$, since $f(X) \subset g(X)$, there exists an $x_1 \in X$ such that $fx_0 = gx_1$. By induction, a sequence $\{x_n\}$ can be chosen such that $fx_n = gx_{n+1}$ $(n \geq 1)$. If $gx_{n_0-1} = gx_{n_0} = fx_{n_0-1}$ for some natural number n_0, then x_{n_0-1} is a coincidence point of f and g in X. Suppose that $gx_{n-1} \neq gx_n$ for all $n \geq 1$.

Thus, by (2.1) for any $n \in \mathbb{N}$, we have

$$\begin{aligned} d(gx_{n+1}, gx_n) &= d(fx_n, fx_{n-1}) \\ &\preceq a_1 d(gx_n, gx_{n-1}) + a_2 d(gx_n, fx_n) \\ &\quad + a_3 d(gx_{n-1}, fx_{n-1}) + a_4 d(gx_n, fx_{n-1}) + a_5 d(gx_{n-1}, fx_n) \end{aligned}$$

and

$$\begin{aligned} d(gx_n, gx_{n+1}) &= d(fx_{n-1}, fx_n) \\ &\preceq a_1 d(gx_{n-1}, gx_n) + a_2 d(gx_{n-1}, fx_{n-1}) \\ &\quad + a_3 d(gx_n, fx_n) + a_4 d(gx_{n-1}, fx_n) + a_5 d(gx_n, fx_{n-1}). \end{aligned}$$

Hence

$$\begin{aligned} 2d(gx_n, gx_{n+1}) &= d(gx_{n+1}, gx_n) + d(gx_n, gx_{n+1}) \\ &\preceq (2a_1 + a_2 + a_3 + sa_4 + sa_5) d(gx_n, gx_{n-1}) \\ &\quad + (a_2 + a_3 + sa_4 + sa_5) d(gx_{n+1}, gx_n). \end{aligned}$$

Since $2sa_1 + (s+1)(a_2 + a_3) + (s^2 + s)(a_4 + a_5) < 2$, we have

$$\begin{aligned} d(gx_n, gx_{n+1}) &\preceq \frac{2a_1 + a_2 + a_3 + sa_4 + sa_5}{2 - a_2 - a_3 - sa_4 - sa_5} d(gx_n, gx_{n-1}) \\ &= k d(gx_n, gx_{n-1}) \preceq k^2 d(gx_{n-1}, gx_{n-2}) \\ &\preceq k^3 d(gx_{n-2}, gx_{n-3}) \preceq \cdots \preceq k^n d(gx_1, gx_0), \end{aligned}$$

where $k = \frac{2a_1 + a_2 + a_3 + sa_4 + sa_5}{2 - a_2 - a_3 - sa_4 - sa_5}$. Obviously, $k \in [0, \frac{1}{s})$.

Thus, setting any positive integers m and n, we have

$$\begin{aligned} d(gx_n, gx_{n+m}) &\preceq sd(gx_n, gx_{n+1}) + sd(gx_{n+1}, gx_{n+m}) \\ &\preceq sd(gx_n, gx_{n+1}) + s^2 d(gx_{n+1}, gx_{n+2}) + s^2 d(gx_{n+2}, gx_{n+m}) \\ &\preceq sd(gx_n, gx_{n+1}) + s^2 d(gx_{n+1}, gx_{n+2}) + s^3 d(gx_{n+2}, gx_{n+3}) \\ &\quad + \cdots + s^{m-1} d(gx_{n+m-2}, gx_{n+m-1}) + s^{m-1} d(gx_{n+m-1}, gx_{n+m}) \\ &\preceq sd(gx_n, gx_{n+1}) + s^2 d(gx_{n+1}, gx_{n+2}) + s^3 d(gx_{n+2}, gx_{n+3}) \\ &\quad + \cdots + s^{m-1} d(gx_{n+m-2}, gx_{n+m-1}) + s^m d(gx_{n+m-1}, gx_{n+m}) \\ &\preceq \left(sk^n + s^2 k^{n+1} + \cdots + s^m k^{n+m-1}\right) d(gx_1, gx_0) \\ &= \frac{sk^n[1 - (sk)^m]}{1 - sk} d(gx_1, gx_0) \\ &\preceq \frac{sk^n}{1 - sk} d(gx_1, gx_0). \end{aligned}$$

Since $k \in [0, 1/s)$, we notice that $\frac{sk^n}{1-sk} d(gx_1, gx_0) \to \theta$ as $n \to \infty$ for any $m \in \mathbb{N}_+$. By Lemma 1.5, for any $c \in \operatorname{int} P$, we can choose $n_0 \in \mathbb{N}$ such that $\frac{sk^n}{1-sk} d(gx_1, gx_0) \ll c$ for all $n > n_0$. Thus, for each $c \in \operatorname{int} P$, $d(gx_{n+m}, gx_n) \ll c$ for all $n > n_0$, $m \geq 1$. Therefore $\{gx_n\}$ is a Cauchy sequence in $g(X)$.

If $g(X) \subset X$ is complete, there exist $q \in g(X)$ and $p \in X$ such that $gx_n \to q$ as $n \to \infty$ and $gp = q$. (If $f(X)$ is complete, there exists $q \in f(X)$ such that $fx_n \to q$ as $n \to \infty$. Since $f(X) \subset g(X)$, we can find $p \in X$ such that $gp = q$.)

Now, from (2.1) we show that $fp = q$,

$$\begin{aligned} d(gx_{n+2}, fp) &= d(fx_{n+1}, fp) \\ &\preceq a_1 d(gx_{n+1}, q) + a_2 d(gx_{n+1}, gx_{n+2}) \\ &\quad + a_3 d(q, fp) + a_4 d(gx_{n+1}, fp) + a_5 d(q, gx_{n+2}). \end{aligned}$$

Similarly,

$$\begin{aligned} d(fp, gx_{n+2}) &= d(fp, fx_{n+1}) \\ &\preceq a_1 d(q, gx_{n+1}) + a_2 d(q, fp) \\ &\quad + a_3 d(gx_{n+1}, gx_{n+2}) + a_4 d(q, gx_{n+2}) + a_5 d(gx_{n+1}, fp), \end{aligned}$$

thus, we have

$$\begin{aligned} 2d(gx_{n+2}, fp) &\preceq 2a_1 d(gx_{n+1}, q) + (a_2 + a_3) d(gx_{n+1}, gx_{n+2}) + (a_2 + a_3) d(q, fp) \\ &\quad + (a_4 + a_5) d(gx_{n+1}, fp) + (a_4 + a_5) d(q, gx_{n+2}) \\ &\preceq (2sa_1 + sa_2 + sa_3 + a_4 + a_5) d(gx_{n+2}, q) \\ &\quad + (sa_2 + sa_3 + sa_4 + sa_5) d(gx_{n+2}, fp) \\ &\quad + (2sa_1 + a_2 + a_3 + sa_4 + sa_5) d(gx_{n+1}, gx_{n+2}). \end{aligned}$$

Since $0 \leq a_2 + a_3 + a_4 + a_5 < 2/s$, by the triangular inequality, it follows that

$$\begin{aligned} d(gx_{n+2}, fp) &\preceq \frac{2sa_1 + sa_2 + sa_3 + a_4 + a_5}{2 - sa_2 - sa_3 - sa_4 - sa_5} d(gx_{n+2}, q) \\ &\quad + \frac{2sa_1 + a_2 + a_3 + sa_4 + sa_5}{2 - sa_2 - sa_3 - sa_4 - sa_5} d(gx_{n+1}, gx_{n+2}). \end{aligned}$$

Since $\{gx_n\}$ is a Cauchy sequence and $gx_n \to q$ $(n \to \infty)$, for any $c \in \operatorname{int} P$, we can choose $n_1 \in \mathbb{N}$ such that for all $n \geq n_1$,

$$d(gx_{n+1}, gx_{n+2}) \ll \frac{(2 - sa_2 - sa_3 - sa_4 - sa_5)c}{2(2sa_1 + a_2 + a_3 + sa_4 + sa_5)}$$

and

$$d(gx_{n+2}, q) \ll \frac{(2 - sa_2 - sa_3 - sa_4 - sa_5)c}{2(2sa_1 + sa_2 + sa_3 + a_4 + a_5)}.$$

Thus, for any $c \in \operatorname{int} P$, $d(gx_{n+2}, fp) \ll c$ for all $n \geq n_1$. Therefore, by Lemma 1.5, we have $fp = q = gp$.

Assume that there exist u, w in X such that $fu = gu = w$.

$$\begin{aligned} d(gu, gp) &= d(fu, fp) \\ &\preceq a_1 d(gu, gp) + a_2 d(fu, gu) + a_3 d(fp, gp) + a_4 d(fp, gu) + a_5 d(fu, gp) \\ &= (a_1 + a_4 + a_5) d(gu, gp). \end{aligned}$$

Since $0 \leq a_1 + a_4 + a_5 < 1$, by Lemma 1.5, we can obtain that $d(gu, gp) = \theta$, *i.e.*, $w = gu = gp = q$. Moreover, the mappings f and g are weakly compatible, by Lemma 1.9, we know that q is the unique common fixed point of f and g. □

Example 2.2 Let $E = C^1_{\mathbb{R}}([0,1])$, $P = \{\varphi \in E : \varphi \geq 0\} \subset E$, $X = [1, \infty)$ and $d(x,y) = |x-y|^2 e^t$. Then (X,d) is a cone b-metric space with the coefficient $s = 2$, but it is not a cone metric space. We consider the functions $f, g : X \to X$ defined by $fx = \frac{1}{6}\ln x + 1$, $gx = \ln x + 1$. Hence

$$\begin{aligned} d(fx,fy) &= \left|\frac{1}{6}\ln x + 1 - \frac{1}{6}\ln y - 1\right|^2 e^t \\ &\preceq \left|\frac{1}{6}\ln x + \frac{1}{6}\ln y\right|^2 e^t \\ &= \left|\frac{1}{5}\left(\ln x - \frac{1}{6}\ln y\right) + \frac{1}{5}\left(\ln y - \frac{1}{6}\ln x\right)\right|^2 e^t \\ &\preceq \frac{2}{25}\left|\ln x - \frac{1}{6}\ln y\right|^2 e^t + \frac{2}{25}\left|\ln y - \frac{1}{6}\ln x\right|^2 e^t \\ &= \frac{2}{25}d(gx,fy) + \frac{2}{25}d(gy,fx). \end{aligned}$$

Here $1 \in X$ is the unique common fixed point of f and g.

Example 2.3 Let X be the set of Lebesgue measurable functions on $[0,1]$ such that $\int_0^1 |u(x)|^2\,dx < \infty$, $E = C_{\mathbb{R}}([0,1])$, $P = \{\varphi \in E : \varphi \geq 0\} \subset E$. We define $d : X \times X \to E$ as

$$d\big(u(t), v(t)\big) = e^t \int_0^1 \big|u(s) - v(s)\big|^2\,ds,$$

for all $x, y \in X$. Then (X,d) is a cone b-metric space with the coefficient $s = 2$, but it is not a cone metric space. Considering the functions $fu = \frac{1}{4}u(t)$ and $gu = \frac{1}{2}u(t)$ $(t \in [0,1])$, we have

$$\begin{aligned} d(fu,fv) &= e^t \int_0^1 \left|\frac{1}{4}u(s) - \frac{1}{4}v(s)\right|^2 ds \\ &= \frac{e^t}{4}\int_0^1 \left|\frac{1}{2}u(s) - \frac{1}{2}v(s)\right|^2 ds \\ &= \frac{1}{4}d(gu,gv). \end{aligned}$$

Clearly, $0 \in X$ is the unique common fixed point of f and g.

Remark 2.4 Compared with the common fixed point results on cone metric spaces in [2, 3, 5], the common fixed point theorems in complete b-cone metric spaces in [10] and the fixed point results in cone b-metric spaces in [9], Theorem 2.1 is shown to be a proper generalization by Examples 2.2 and 2.3. Furthermore, Theorem 2.1 generalizes and unifies [9, Theorem 2.1 and 2.3].

Definition 2.5 Let (X,d) be a cone b-metric space with the coefficient $s \geq 1$. A mapping $f : X \to X$ is such that, for some constant $\lambda \in (0, 1/s)$ and for every $x, y \in X$, there exists an element

$$\nu \in C(g,x,y) = \{d(gx,gy), d(gx,fx), d(gy,fy), d(gx,fy), d(gy,fx)\} \tag{2.2}$$

for which $d(fx,fy) \preceq \lambda u$ is called a g-quasi-contraction.

Theorem 2.6 *Let (X,d) be a cone b-metric space with the coefficient $s \geq 1$ and let the mapping $f : X \to X$ be a g-quasi-contraction. If the range of g contains the range of f and $g(X)$ or $f(X)$ is a complete subspace of X, then f and g have a unique point of coincidence in X. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point in X.*

Proof For each $x_0 \in X$, set $gx_1 = fx_0$ and $gx_{n+1} = fx_n$ $(n \in \mathbb{N})$. If $gx_{n_0-1} = gx_{n_0} = fx_{n_0-1}$ for some natural number n_0, then x_{n_0-1} is a coincidence point of f and g in X.

Suppose that $gx_{n-1} \neq gx_n$ for all $n \geq 1$. Now we prove that $\{gx_n\}$ is a Cauchy sequence. First, we show that

$$d(gx_n, gx_1) = d(fx_{n-1}, fx_0) \preceq \frac{s\lambda}{1-s\lambda} d(gx_1, gx_0) \quad \text{for all } n \in \mathbb{N}_+. \tag{2.3}$$

Clearly, we note (2.3) holds when $n = 1$. We assume that (2.3) holds for some $n \leq N-1$ $(N \in \mathbb{N}_+)$, then we prove that (2.3) holds for all $n = N$. Because f is a g-quasi-contractive mapping, there exists a real number $k \leq N$ such that

$$d(gx_N, gx_1) \preceq \lambda d(gx_k, gx_0). \tag{2.4}$$

In order to prove that (2.4) holds, we show that for all $1 \leq i,j \leq N$, there exists $1 \leq k \leq N$ such that

$$d(gx_i, gx_j) \preceq \lambda d(gx_k, gx_0). \tag{2.5}$$

Clearly, (2.5) is true for $N = 1$. Suppose that (2.5) is true for each $N = P \in \mathbb{N}$, that is, for all $1 \leq i,j \leq P$, there exists $1 \leq k \leq P$ such that

$$d(gx_i, gx_j) \preceq \lambda d(gx_k, gx_0). \tag{2.6}$$

Let us prove (2.5) holds for $N = P + 1$.

By (2.6), we only show that for any $1 \leq i_0 \leq P+1$, there exists $1 \leq k \leq P+1$ such that

$$d(gx_{P+1}, gx_{i_0}) \preceq \lambda d(gx_k, gx_0).$$

Since f is a g-quasi-contractive mapping, there exists

$$\begin{aligned} \nu_{i_0} \in C(g, x_P, x_{i_0-1}) = \{&d(gx_P, gx_{i_0-1}), d(gx_P, gx_{P+1}), \\ &d(gx_{i_0-1}, gx_{i_0}), d(gx_P, gx_{i_0}), d(gx_{i_0-1}, gx_{P+1})\} \end{aligned}$$

such that $d(gx_{P+1}, gx_{i_0}) \preceq \lambda \nu_{i_0}$.

By (2.6), we discuss that there exists an element

$$d(gx_{P+1}, gx_{i_1}) \in \{d(gx_P, gx_{i_0-1}), d(gx_P, gx_{P+1}), d(gx_{i_0-1}, gx_{i_0}), d(gx_P, gx_{i_0}), d(gx_{i_0-1}, gx_{P+1})\}$$

such that $d(gx_{P+1}, gx_{i_0}) \preceq \lambda d(gx_{P+1}, gx_{i_1})$ $(1 \le i_1 \le P+1)$.

If the above inequality does not hold for $1 \le i_1 \le P+1$, then (2.5) is true for $N = P+1$ by (2.6).

We continue in the same way, and after $P+1$ steps, we get $1 \le i_j \le P+1$ $(0 \le j \le P+1)$ such that

$$d(gx_{P+1}, gx_{i_j}) \preceq \lambda d(gx_{P+1}, gx_{i_{j+1}}) \quad (0 \le j \le P).$$

Notice that there exist $0 \le r < s \le P+1$ such that $i_r = i_s$. That is,

$$d(gx_{P+1}, gx_{i_r}) \preceq \lambda^{s-r} d(gx_{P+1}, gx_{i_s}) = \lambda^{s-r} d(gx_{P+1}, gx_{i_r}) \quad (0 \le r < s \le P+1).$$

As $\lambda \in (0,1)$, by Lemma 1.5(5), we get a contradiction. From (2.6), (2.5) is true for $N = P+1$.

Hence, (2.5) is true for all $N \in \mathbb{N}$, which implies that (2.4) holds for $N \in \mathbb{N}$.

Next, let us prove that for all $n \in \mathbb{N}_+$,

$$d(gx_n, gx_0) \preceq \frac{s}{1-s\lambda} d(gx_0, gx_1). \tag{2.7}$$

Using the triangular inequality, from (2.3) we obtain

$$\begin{aligned} d(gx_n, gx_0) &\preceq s[d(gx_n, gx_1) + d(gx_1, gx_0)] \\ &\preceq \frac{s^2\lambda}{1-s\lambda} d(gx_0, gx_1) + sd(gx_1, gx_0) \\ &= \frac{s}{1-s\lambda} d(gx_1, gx_0). \end{aligned}$$

Now, we show that $\{gx_n\}$ is a Cauchy sequence. For all $n > m$, there exists

$$\begin{aligned} \nu_1 \in C(g, x_{m-1}, x_{n-1}) = \{&d(gx_{m-1}, gx_{n-1}), d(gx_{m-1}, gx_m), \\ &d(gx_{n-1}, gx_n), d(gx_{m-1}, gx_n), d(gx_m, gx_{n-1})\} \end{aligned} \tag{2.8}$$

such that $d(gx_m, gx_n) = d(fx_{m-1}, fx_{n-1}) \preceq \lambda \nu_1$.

By the contractive condition, there exist but not all

$$\nu_k \in \{d(gx_i, gx_j) | 0 \le i < j \le n\} \quad (k = 1, 2, 3, \dots, m)$$

such that

$$\nu_k \preceq \lambda \nu_{k+1} \quad (k = 1, 2, 3, \dots, m-1). \tag{2.9}$$

In fact, from (2.8) we have

$$\begin{aligned} v_1 &\in C(g, x_{m-1}, x_{n-1}) \\ &= \{d(gx_{m-1}, gx_{n-1}), d(gx_{m-1}, gx_m), d(gx_{n-1}, gx_n), d(gx_{m-1}, gx_n), d(gx_m, gx_{n-1})\} \\ &\subset A_{m-1,n-1} = \{d(gx_i, gx_j) \,|_{j=m,n,n-1,}^{i=m,m-1,n-1;}\ i < j\}. \end{aligned}$$

Let $v_1 = d(gx_i, gx_j) = d(fx_{i-1}, fx_{j-1}) \preceq \lambda v_2$, where

$$\begin{aligned} v_2 &\in C(g, x_{i-1}, x_{j-1}) \subset A_{i-1,j-1} = \{d(gx_r, gx_s) \,|_{s=i,j,j-1,}^{r=i,i-1,j-1;}\ r < s\} \\ &= \{d(gx_r, gx_s) \,|_{s=m,m-1,n,n-1,n-2,}^{r=m,m-1,m-2,n-1,n-2;}\ r < s\}. \end{aligned}$$

In general, if there exists

$$v_k \in \{d(gx_i, gx_j) \,|_{j=m,m-1,m-2,\ldots,m-k+1,n,n-1,n-2,\ldots,n-k,}^{i=m,m-1,m-2,\ldots,m-k,n-1,n-2,\ldots,n-k;}\ i < j\} \quad (1 \le k \le m),$$

then we have

$$v_{k+1} \in C(g, x_{i-1}, x_{j-1}) \subset A_{i-1,j-1} = \{d(gx_r, gx_s) \,|_{s=i,j,j-1,}^{r=i,i-1,j-1;}\ r < s\} \quad (1 \le k \le m-1)$$

such that $v_k = d(gx_i, gx_j) = d(fx_{i-1}, fx_{j-1}) \preceq \lambda v_{k+1}$ $(1 \le k \le m-1)$.

As

$$\begin{aligned} &\{d(gx_r, gx_s) \,|_{s=i,j,j-1,}^{r=i,i-1,j-1;}\ r < s\} \\ &\quad \subset \{d(gx_r, gx_s) \,|_{s=m,m-1,m-2,\ldots,m-k+1,m-k,n,n-1,n-2,\ldots,n-k,n-k-1,}^{r=m,m-1,m-2,\ldots,m-k,m-k-1,n-1,n-2,\ldots,n-k,n-k-1;}\ r < s\} \\ &\quad = \{d(gx_i, gx_j) \,|_{j=m,m-1,m-2,\ldots,m-k+1,m-k,n,n-1,n-2,\ldots,n-k,n-k-1,}^{i=m,m-1,m-2,\ldots,m-k,m-k-1,n-1,n-2,\ldots,n-k,n-k-1;}\ i < j\} \\ &\quad \subset \{d(gx_i, gx_j) | 0 \le i < j \le n\} \quad (1 \le k \le m-1), \end{aligned}$$

we can obtain (2.9).

Using the triangular inequality, we get

$$d(gx_i, gx_j) \preceq sd(gx_i, gx_0) + sd(gx_0, gx_j) \quad (0 \le i, j \le n),$$

so we obtain

$$\begin{aligned} d(gx_n, gx_m) &= d(fx_{n-1}, fx_{m-1}) \preceq \lambda v_1 \preceq \lambda^2 v_2 \preceq \cdots \preceq \lambda^m v_m \\ &\preceq \lambda^m sd(gx_i, gx_0) + \lambda^m sd(gx_0, gx_j) \\ &\preceq \frac{2s^2\lambda^m}{1 - s\lambda} d(gx_1, gx_0). \end{aligned}$$

Since $\frac{2s^2\lambda^m}{1-s\lambda} d(gx_1, gx_0) \to \theta$ as $m \to \infty$, by Lemma 1.5, it is easy to see that for any $c \in \operatorname{int} P$, there exists $n_0 \in \mathbb{N}$ such that for all $n > m > n_0$,

$$d(gx_n, gx_m) \preceq \frac{2s^2\lambda^m}{1 - s\lambda} d(gx_1, gx_0) \ll c.$$

So, $\{gx_n\}$ is a Cauchy sequence in $g(X)$. If $g(X) \subset X$ is complete, there exist $q \in g(X)$ and $p \in X$ such that $gx_n \to q$ as $n \to \infty$ and $g(p) = q$.

Now, from (2.2) we get

$$\nu \in C(g, x_n, p) = \{d(gx_n, gp), d(gx_n, fx_n), d(gp, fp), d(gx_n, fp), d(fx_n, gp)\}$$

such that $d(fx_n, fp) \preceq \lambda\nu$.

We have the following five cases:

(1) $d(fx_n, fp) \preceq \lambda d(gx_n, gp) \preceq s\lambda d(gx_{n+1}, gp) + s\lambda d(gx_{n+1}, gx_n)$;

(2) $d(fx_n, fp) \preceq \lambda d(gx_n, fx_n) = \lambda d(gx_n, gx_{n+1})$;

(3) $d(fx_n, fp) \preceq \lambda d(gp, fp) \preceq s\lambda d(gx_{n+1}, gp) + s\lambda d(gx_{n+1}, fp)$, that is, $d(fx_n, fp) \preceq \frac{s\lambda}{1-s\lambda} d(gx_{n+1}, gp)$;

(4) $d(fx_n, fp) \preceq \lambda d(gx_n, fp) \preceq s\lambda d(gx_{n+1}, fp) + s\lambda d(gx_{n+1}, gx_n)$, that is, $d(fx_n, fp) \preceq \frac{s\lambda}{1-s\lambda} d(gx_{n+1}, gx_n)$;

(5) $d(fx_n, fp) \preceq \lambda d(fx_n, gp) = \lambda d(gx_{n+1}, gp)$.

As $\frac{s\lambda}{1-s\lambda} > s\lambda$, then we obtain that

$$d(gx_{n+1}, fp) \preceq \frac{s\lambda}{1 - s\lambda}\left[d(gx_{n+1}, gx_n) + d(gx_{n+1}, q)\right].$$

Since $gx_n \to q$ as $n \to \infty$, for any $c \in \operatorname{int} P$, there exists $n_1 \in \mathbb{N}$ such that for all $n > n_1$,

$$d(gx_{n+1}, gx_n) \ll \frac{(1 - s\lambda)c}{2s\lambda} \quad \text{and} \quad d(gx_{n+1}, q) \ll \frac{(1 - s\lambda)c}{2s\lambda}.$$

By Lemmas 1.5 and 1.6, we have $gx_n \to fp$ as $n \to \infty$ and $q = fp$.

Now, if w is another point such that $gu = fu = w$, then

$$d(w, q) = d(fu, fp) \preceq \lambda\nu,$$

where $\lambda \in (0, \frac{1}{s})$ and

$$\nu \in C(f; u, p) = \{d(gu, gp), d(gu, fu), d(gp, fp), d(gu, fp), d(fu, gp)\}.$$

It is obvious that $d(w, q) = \theta$, *i.e.*, $w = q$. Therefore, q is the unique point of coincidence of f, g in X. Moreover, the mappings f and g are weakly compatible, by Lemma 1.9 we know that q is the unique common fixed point of f and g.

Similarly, if $f(X)$ is complete, the above conclusion is also established. □

Example 2.7 Let $X = \mathbb{R}$, $E = C^1_{\mathbb{R}}[0,1]$ and $P = \{f \in E : f \geq 0\}$. Define $d : X \times X \to E$ by $d(x, y) = |x - y|^{\frac{3}{2}}\varphi$ where $\varphi : [0,1] \to \mathbb{R}$ such that $\varphi(t) = e^t$. It is easy to see that (X, d) is a cone b-metric space with the coefficient $s = 2^{\frac{1}{2}}$, but it is not a cone metric space. The mappings $f, g : X \to X$ are defined by $fx = \alpha x$ and $gx = \sqrt{\alpha} x$ ($\alpha \in [\frac{1}{\sqrt[3]{8}}, \frac{1}{\sqrt[3]{4}})$). The mapping f is a g-quasi-contraction with the constant $\lambda = \alpha^{\frac{3}{4}} \in [\frac{1}{2}, \frac{\sqrt{2}}{2})$. Moreover, $0 \in X$ is the unique common fixed point of f and g.

Remark 2.8 Kadelburg and Radenovi [11] obtained a fixed point result without the normality of the underlying cone, but only in the case of a quasi-contractive constant $\lambda \in$

$(0, 1/2)$ (see [11, Theorem 2.2]). However, Ljiljana [7] proved the result is true for $\lambda \in (0, 1)$ on a cone metric space by a new way. Referring to this way, Theorem 2.6 presents a similar common fixed point result in the case of the contractive constant $\lambda \in (0, 1/s)$ in cone b-metric spaces without the assumption of normality. Moreover, it is obvious that Example 2.7 given above shows that Theorem 2.6 not only improves and generalizes [11, Theorem 2.2], but also generalizes and unifies [7, Theorem 3].

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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