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On the convergence of an iteration method for continuous mappings on an arbitrary interval

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Abstract
In this paper, we consider an iterative method for finding a fixed point of continuous mappings on an arbitrary interval. Then, we give the necessary and sufficient conditions for the convergence of the proposed iterative methods for continuous mappings on an arbitrary interval. We also compare the rate of convergence between iteration methods. Finally, we provide a numerical example which supports our theoretical results.

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1 Introduction

Let E be a closed interval on the real line and let $f : E \to E$ be a continuous mapping. A point $p \in E$ is a fixed point of f if $f(p) = p$. We denote by $F(f)$ the set of fixed points of f. It is known that if E is also bounded, then $F(f)$ is nonempty.

There are many iterative methods for finding a fixed point of f. For example, the Mann iteration (see [1]) is defined by $u_1 \in E$ and

$$u_{n+1} = (1 - \alpha_n)u_n + \alpha_n f(u_n) \tag{1.1}$$

for all $n \geq 1$, where $\{\alpha_n\}_{n=1}^{\infty}$ is a sequence in $[0,1]$. The Ishikawa iteration (see [2]) is defined by $s_1 \in E$ and

$$\begin{cases} t_n = (1 - b_n)s_n + b_n f(s_n), \\ s_{n+1} = (1 - \alpha_n)s_n + \alpha_n f(t_n) \end{cases} \tag{1.2}$$

for all $n \geq 1$, where $\{\alpha_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ are sequences in $[0,1]$. The Noor iteration (see [3]) is defined by $w_1 \in E$ and

$$\begin{cases} r_n = (1 - a_n)w_n + a_n f(w_n), \\ q_n = (1 - b_n)w_n + b_n f(r_n), \\ w_{n+1} = (1 - \alpha_n)w_n + \alpha_n f(q_n) \end{cases} \tag{1.3}$$

for all $n \geq 1$, where $\{\alpha_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ and $\{a_n\}_{n=1}^{\infty}$ are sequences in $[0,1]$. Clearly, Mann and Ishikawa iterations are special cases of the Noor iteration.

In 1974, Roades proved the convergence of the Mann iteration for the class of continuous and nondecreasing functions on a closed unit interval, and then he [4] extended convergence results to Ishikawa iterations. He also proved that the Ishikawa iteration converges faster than the Mann iteration for the class of continuous and nondecreasing mappings. Later, in 1991, Borwein and Borwein [5] proved the convergence of the Mann iteration of continuous mappings on a bounded closed interval. Recently, Qing and Qihou [6] extended their results to an arbitrary interval and to the Ishikawa iteration and a gave necessary and sufficient condition for the convergence of Ishikawa iteration on an arbitrary interval. Recently, Phuengrattana and Suantai [7] proved that the Mann, Ishikawa and Noor iterations are equivalent for the class of continuous and nondecreasing mappings.

In this paper, we are interested in employing the iteration method (1.4) for a continuous mapping on an arbitrary interval. The iteration method was first introduced by Thianwan and Suantai [8] as follows. Let E be a subset of a normed space X and let $f : E \to E$ be a mapping

$$\begin{cases} z_n = (1 - a_n)x_n + a_n f(x_n), \\ y_n = (1 - b_n - c_n)x_n + b_n f(z_n) + c_n f(x_n), \\ x_{n+1} = (1 - \alpha_n - \beta_n)x_n + \alpha_n f(y_n) + \beta_n f(z_n) \end{cases} \tag{1.4}$$

for all $n \geq 1$, where $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$, $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ and $\{c_n\}_{n=1}^{\infty}$ are sequences in $[0,1]$.

Note that (1.4) reduces to (1.1) when $a_n = b_n = c_n = \beta_n = 0$. Similarly, the process (1.4) reduces to (1.2) when $a_n = c_n = \beta_n = 0$. Moreover, for $c_n = \beta_n = 0$, the process (1.4) reduces to (1.3).

The purpose of this article is to give a necessary and sufficient condition for the strong convergence of the iteration method (1.4) of continuous mappings on an arbitrary interval. Our results extend and improve the corresponding results of Rhoades [4], Borwein and Borwein [5], Qing and Qihou [6], Phuengrattana and Suantai [7], and many others.

2 Convergence theorems

We first give a convergence theorem for the iteration method (1.4) for continuous mappings on an arbitrary interval.

Theorem 1 *Let E be a closed interval on the real line and let $f : E \to E$ be a continuous mapping. For $x_1 \in E$, let the iteration $\{x_n\}_{n=1}^{\infty}$ be defined by* (1.4), *and let $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$, $\{c_n\}_{n=1}^{\infty}$, $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ be sequences in* $[0,1]$ *satisfying the following conditions*:

(i) $\lim_{n\to\infty} a_n = 0$, $\lim_{n\to\infty} b_n = 0$, $\lim_{n\to\infty} c_n = 0$ *and* $\sum_{n=1}^{\infty} \beta_n < \infty$,

(ii) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$.

Then $\{x_n\}_{n=1}^{\infty}$ is bounded if and only if $\{x_n\}_{n=1}^{\infty}$ converges to a fixed point of f.

Proof It is obvious that $\{x_n\}_{n=1}^{\infty}$ converges to a fixed point of f. Now, assume that $\{x_n\}_{n=1}^{\infty}$ is bounded. We shall show that $\{x_n\}_{n=1}^{\infty}$ is convergent. To show this, suppose not. Then there exist $a, b \in \mathbb{R}$, $a = \liminf_{n\to\infty} x_n$, $b = \limsup_{n\to\infty} x_n$ and $a < b$. First, we show that if

$a < m < b$, then $f(m) = m$. Suppose that $f(m) \neq m$. Without loss of generality, we suppose that $f(m) - m > 0$. Because $f(x)$ is a continuous mapping, there exists $\delta \in (0, b-a)$ such that

$$f(x) - x > 0 \quad \text{for } |x - m| \leq \delta. \tag{2.1}$$

By the boundedness of $\{x_n\}_{n=1}^{\infty}$, we have $\{x_n\}_{n=1}^{\infty}$ belongs to a bounded closed interval. The continuity of f implies that $\{f(x_n)\}_{n=1}^{\infty}$ belongs to another bounded closed interval, so $\{f(x_n)\}_{n=1}^{\infty}$ is bounded, and since $z_n = (1-a_n)x_n + a_n f(x_n)$, so $\{z_n\}_{n=1}^{\infty}$ is bounded, and thus $\{f(z_n)\}_{n=1}^{\infty}$ is bounded. Similarly, since $y_n = (1-b_n-c_n)x_n + b_n f(z_n) + c_n f(x_n)$, we have $\{y_n\}_{n=1}^{\infty}$ and $\{f(y_n)\}_{n=1}^{\infty}$ are bounded. It follows by (1.4) that $x_{n+1} - x_n = \alpha_n(f(y_n) - x_n) + \beta_n(f(z_n) - x_n)$, $y_n - x_n = b_n(f(z_n) - x_n) + c_n(f(x_n) - x_n)$ and $z_n - x_n = a_n(f(x_n) - x_n)$. By conditions (i) and (ii), we get $|x_{n+1} - x_n| \to 0$, $|y_n - x_n| \to 0$ and $|z_n - x_n| \to 0$. Thus, there exists N such that

$$|x_{n+1} - x_n| < \frac{\delta}{3}, \qquad |y_n - x_n| < \frac{\delta}{3}, \qquad |z_n - x_n| < \frac{\delta}{3} \tag{2.2}$$

for all $n > N$. Since $b = \limsup_{n\to\infty} x_n > m$, there exists $k_1 > N$ such that $x_{n_{k_1}} > m$. Let $k = n_{k_1}$, then $x_k > m$. For x_k, there exist only two cases:

Case 1: $x_k \geq m + \frac{\delta}{3}$, then by (2.2), we have $x_{k+1} - x_k > -\frac{\delta}{3}$, then $x_{k+1} > x_k - \frac{\delta}{3} \geq m$, so $x_{k+1} > m$.

Case 2: $m < x_k < m + \frac{\delta}{3}$, then by (2.2), we have $m - \frac{\delta}{3} < y_k < m + \frac{2\delta}{3}$ and $m - \frac{\delta}{3} < z_k < m + \frac{2\delta}{3}$. So, we have $|x_k - m| < \frac{\delta}{3} < \delta$, $|y_k - m| < \frac{2\delta}{3} < \delta$ and $|z_k - m| < \frac{2\delta}{3} < \delta$. Using (2.1), we get

$$f(x_k) - x_k > 0, \qquad f(y_k) - y_k > 0, \qquad f(z_k) - z_k > 0.$$

By (1.4), we have

$$\begin{aligned}
x_{k+1} &= (1 - \alpha_k - \beta_k)x_k + \alpha_k f(y_k) + \beta_k f(z_k) \\
&= x_k + \alpha_k\bigl(f(y_k) - x_k\bigr) + \beta_k\bigl(f(z_k) - x_k\bigr) \\
&= x_k + \alpha_k\bigl(f(y_k) - y_k\bigr) + \alpha_k(y_k - x_k) + \beta_k\bigl(f(z_k) - z_k\bigr) + \beta_k(z_k - x_k) \\
&= x_k + \alpha_k\bigl[b_k\bigl(f(z_k) - x_k\bigr) + c_k\bigl(f(x_k) - x_k\bigr)\bigr] + \beta_k a_k\bigl(f(x_k) - x_k\bigr) \\
&\quad + \alpha_k\bigl(f(y_k) - y_k\bigr) + \beta_k\bigl(f(z_k) - z_k\bigr) \\
&= x_k + a_k b_k\bigl(f(z_k) - z_k\bigr) + a_k b_k(z_k - x_k) + \alpha_k c_k\bigl(f(x_k) - x_k\bigr) \\
&\quad + \beta_k a_k\bigl(f(x_k) - x_k\bigr) + \alpha_k\bigl(f(y_k) - y_k\bigr) + \beta_k\bigl(f(z_k) - z_k\bigr) \\
&= x_k + a_k^2 b_k\bigl(f(x_k) - x_k\bigr) + a_k b_k\bigl(f(z_k) - z_k\bigr) + \alpha_k c_k\bigl(f(x_k) - x_k\bigr) \\
&\quad + \beta_k a_k\bigl(f(x_k) - x_k\bigr) + \alpha_k\bigl(f(y_k) - y_k\bigr) + \beta_k\bigl(f(z_k) - z_k\bigr).
\end{aligned}$$

By Case 1 and Case 2, we can conclude that $x_{k+1} > m$. By using the above argument, we obtain $x_{k+2} > m, x_{k+3} > m, x_{k+4} > m, \ldots$. Thus we get $x_n > m$ for all $n > k = n_{k_1}$. So, $a = \liminf_{n\to\infty} x_n \geq m$, which is a contradiction with $a < m$. Thus $f(m) = m$.

For the sequence $\{x_n\}_{n=1}^{\infty}$, we consider the following two cases.

Case 1′: There exists x_m such that $a < x_m < b$.

Then $f(x_m) = x_m$. Thus

$$\begin{aligned} z_m &= (1 - a_m)x_m + a_m f(x_m) = x_m, \\ y_m &= (1 - b_m - c_m)x_m + b_m f(z_m) + c_m f(x_m) \\ &= (1 - b_m - c_m)x_m + b_m f(x_m) + c_m f(x_m) = x_m, \\ x_{m+1} &= (1 - \alpha_m - \beta_m)x_m + \alpha_m f(y_m) + \beta_m f(z_m) \\ &= (1 - \alpha_m - \beta_m)x_m + \alpha_m f(x_m) + \beta_m f(x_m) = x_m. \end{aligned}$$

By induction, we obtain $x_m = x_{m+1} = x_{m+2} = x_{m+3} = \cdots$, so, $x_n \to x_m$. This implies that $x_m = a$ and $x_n \to a$, which contradicts our assumption.

Case 2′: For all n, $x_n \le a$ or $x_n \ge b$.

Because $b - a > 0$ and $|x_{n+1} - x_n| \to 0$, so there exists N_0 such that $|x_{n+1} - x_n| < \frac{b-a}{3}$ for all $n > N_0$. It implies that either $x_n \le a$ for all $n > N_0$ or $x_n \ge b$ for all $n > N_0$. If $x_n \le a$ for $n > N_0$, then $b = \limsup_{n\to\infty} x_n \le a$, which is a contradiction with $a < b$. If $x_n \ge b$ for $n > N_0$, so we have $a = \liminf_{n\to\infty} x_n \ge b$, which is a contradiction with $a < b$.

Hence, we have $\{x_n\}_{n=1}^{\infty}$ is convergent.

Finally, we show that $\{x_n\}_{n=1}^{\infty}$ converges to a fixed point of f. Let $x_n \to p$ and suppose that $f(p) \neq p$. By the continuity of f, we have $\{f(x_n)\}_{n=1}^{\infty}$ is bounded. From $z_n = (1 - a_n)x_n + a_n f(x_n)$ and $a_n \to 0$, we obtain $z_n \to p$. Similarly, by $y_n = (1 - b_n - c_n)x_n + b_n f(z_n) + c_n f(x_n)$ and $b_n \to 0$ and $c_n \to 0$, it follows that $y_n \to p$. Let $r_k = f(y_k) - x_k$ and $s_k = f(z_k) - x_k$. By the continuity of f, we have $\lim_{k\to\infty} r_k = \lim_{k\to\infty}(f(y_k) - x_k) = f(p) - p \neq 0$ and $\lim_{k\to\infty} s_k = \lim_{k\to\infty}(f(z_k) - x_k) = f(p) - p \neq 0$. Put $w = f(p) - p$. Then $w \neq 0$. By (1.4), we get $x_{n+1} = x_n + \alpha_n(f(y_n) - x_n) + \beta_n(f(z_n) - x_n)$. It follows that

$$x_n = x_1 + \sum_{k=1}^{n-1}(\alpha_k r_k + \beta_k s_k). \tag{2.3}$$

By $r_k \to w \neq 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$, we have that $\sum_{k=1}^{\infty} \alpha_k r_k$ is divergent, $\sum_{k=1}^{\infty} \beta_k s_k$ is convergent. It follows by (2.3) that $\{x_n\}_{n=1}^{\infty}$ is divergent, which is a contradiction with $x_n \to p$. Thus $f(p) = p$, that is, $\{x_n\}_{n=1}^{\infty}$ converges to a fixed point of f. □

The following corollaries are obtained directly by Theorem 1.

Corollary 1 [7] *Let E be a closed interval on the real line and let $f : E \to E$ be a continuous mapping. For $x_1 \in E$, let the Noor iteration $\{x_n\}_{n=1}^{\infty}$ be defined by* (1.3), *where $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$, $\{\gamma_n\}_{n=1}^{\infty}$ are sequences in* $[0,1]$ *satisfying the following conditions*:

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty$ *and* $\lim_{n\to\infty} \alpha_n = 0$;

(ii) $\lim_{n\to\infty} \beta_n = 0$ *and* $\lim_{n\to\infty} \gamma_n = 0$.

Then $\{x_n\}_{n=1}^{\infty}$ is bounded if and only if $\{x_n\}_{n=1}^{\infty}$ converges to a fixed point of f.

Proof By putting $c_n = \beta_n = 0$ for all $n \ge 1$ in Theorem 1, we obtain the required result directly from Theorem 1. □

Corollary 2 [6] *Let E be a closed interval on the real line and let $f : E \to E$ be a continuous mapping. For $x_1 \in E$, let the Ishikawa iteration $\{x_n\}_{n=1}^{\infty}$ be defined by* (1.2), *where $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ are sequences in* $[0,1]$ *satisfying the following conditions*:

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty$ *and* $\lim_{n\to\infty} \alpha_n = 0$;
(ii) $\lim_{n\to\infty} \beta_n = 0$.
Then $\{x_n\}_{n=1}^{\infty}$ *is bounded if and only if* $\{x_n\}_{n=1}^{\infty}$ *converges to a fixed point of* f.

Proof By putting $a_n = c_n = 0$ and $\beta_n = 0$ for all $n \geq 1$ in Theorem 1, we obtain the desired result. □

Corollary 3 [5] *Let E be a closed interval on the real line and let $f : E \to E$ be a continuous mapping. For $x_1 \in E$, let the Mann iteration $\{x_n\}_{n=1}^{\infty}$ be defined by* (1.1), *where $\{\alpha_n\}_{n=1}^{\infty}$ is a sequence in* $[0,1]$ *satisfying the following conditions*:
(i) $\lim_{n\to\infty} \alpha_n = 0$ *and*
(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$.
Then $\{x_n\}_{n=1}^{\infty}$ *is bounded if and only if* $\{x_n\}_{n=1}^{\infty}$ *converges to a fixed point of* f.

Proof It is the special case $a_n = b_n = c_n = \beta_n = 0$ in Theorem 1. □

3 Rate of convergence

In this section, we compare the rate of convergence of the iteration (1.4) with the Mann, Ishikawa and Noor iterations. We show that the iteration (1.4) converges faster than the others.

In order to compare the rate of convergence of continuous self-mappings defined on a closed interval, we use the following definition introduced by Rhoades [4].

Definition 1 Let E be a closed interval on the real line and let $f : E \to E$ be a continuous mapping. Suppose that $\{x_n\}_{n=1}^{\infty}$ and $\{u_n\}_{n=1}^{\infty}$ are two iterations which converge to the fixed point p of f. We say that $\{x_n\}_{n=1}^{\infty}$is better than $\{u_n\}_{n=1}^{\infty}$ if

$$|x_n - p| \leq |u_n - p| \quad \text{for all } n \geq 1.$$

Lemma 1 [7] *Let E be a closed interval on the real line and let $f : E \to E$ be a continuous and nondecreasing mapping. Let $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$, $\{a_n\}_{n=1}^{\infty}$ be sequences in* $[0,1)$. *Let $\{u_n\}_{n=1}^{\infty}$, $\{s_n\}_{n=1}^{\infty}$ and $\{w_n\}_{n=1}^{\infty}$ be defined by* (1.1)-(1.3), *respectively. Then the following hold*:
(i) *If $f(u_1) < u_1$, then $f(u_n) < u_n$ for all $n \geq 1$ and $\{u_n\}_{n=1}^{\infty}$ is nonincreasing.*
(ii) *If $f(u_1) > u_1$, then $f(u_n) > u_n$ for all $n \geq 1$ and $\{u_n\}_{n=1}^{\infty}$ is nondecreasing.*
(iii) *If $f(s_1) < s_1$, then $f(s_n) < s_n$ for all $n \geq 1$ and $\{s_n\}_{n=1}^{\infty}$ is nonincreasing.*
(iv) *If $f(s_1) > s_1$, then $f(s_n) > s_n$ for all $n \geq 1$ and $\{s_n\}_{n=1}^{\infty}$ is nondecreasing.*
(v) *If $f(w_1) < w_1$, then $f(w_n) < w_n$ for all $n \geq 1$ and $\{w_n\}_{n=1}^{\infty}$ is nonincreasing.*
(vi) *If $f(w_1) > w_1$, then $f(w_n) > w_n$ for all $n \geq 1$ and $\{w_n\}_{n=1}^{\infty}$ is nondecreasing.*

Lemma 2 *Let E be a closed interval on the real line and let $f : E \to E$ be a continuous and nondecreasing mapping. Let $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$, $\{c_n\}_{n=1}^{\infty}$, $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ be sequences in* $[0,1)$. *Let $\{x_n\}_{n=1}^{\infty}$ be defined by* (1.4). *Then the following hold*:
(i) *If $f(x_1) < x_1$, then $f(x_n) < x_n$ for all $n \geq 1$ and $\{x_n\}_{n=1}^{\infty}$ is nonincreasing.*
(ii) *If $f(x_1) > x_1$, then $f(x_n) > x_n$ for all $n \geq 1$ and $\{x_n\}_{n=1}^{\infty}$ is nondecreasing.*

Proof (i) Let $f(x_1) < x_1$. Then $f(x_1) < z_1 \leq x_1$. Since f is nondecreasing, we have $f(z_1) \leq f(x_1) < z_1 \leq x_1$. This implies $f(z_1) < y_1 \leq z_1$. Thus $f(y_1) \leq f(x_1) < z_1 \leq x_1$. For y_1, we consider the following cases.

Case 1: $f(z_1) < y_1 \leq z_1$. Then $f(y_1) \leq f(z_1) < z_1 < x_1$. It follows that if $f(y_1) < x_2 \leq y_1$, then $f(x_2) \leq f(y_1) < x_2$, if $y_1 < x_2 \leq z_1$, then $f(x_2) \leq f(z_1) < y_1 < x_2$ and if $z_1 < x_2 \leq x_1$, then $f(x_2) \leq f(x_1) < z_1 < x_2$. Thus, we have $f(x_2) < x_2$.

Case 2: $z_1 < y_1 \leq x_1$. Then $f(y_1) \leq f(x_1) < z_1 \leq x_1$. This implies $f(y_1) < x_2 \leq x_1$. Thus $f(x_2) \leq f(x_1) < z_1 < y_1 \leq x_1$. It follows that if $f(y_1) < x_2 \leq y_1$, then $f(x_2) \leq f(y_1) < x_2$ and if $y_1 < x_2 \leq x_1$, then $f(x_2) \leq f(x_1) < y_1 < x_2$. Hence, we have $f(x_2) < x_2$.

From Cases 1 and 2, we have $f(x_2) < x_2$. By induction, we conclude that $f(x_n) < x_n$ for all $n \geq 1$. This implies $z_n \leq x_n$ for all $n \geq 1$. Since f is nondecreasing, we have $f(z_n) \leq f(x_n) < x_n$ for all $n \geq 1$. Thus $y_n \leq x_n$ for all $n \geq 1$, then $f(y_n) \leq f(x_n) < x_n$ for all $n \geq 1$. Hence, we have $x_{n+1} \leq x_n$ for all $n \geq 1$, that is, $\{x_n\}_{n=1}^{\infty}$ is nonincreasing.

(ii) Following the line of (i), we can show the desired result. □

Lemma 3 *Let E be a closed interval on the real line and let $f : E \to E$ be a continuous and nondecreasing mapping. Let $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$, $\{c_n\}_{n=1}^{\infty}$, $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ be sequences in* $[0,1)$. *For $w_1 = x_1 \in E$, let $\{w_n\}_{n=1}^{\infty}$ and $\{x_n\}_{n=1}^{\infty}$ be the sequences defined by* (1.3) *and* (1.4), *respectively. Then the following are satisfied*:

(i) *If $f(w_1) < w_1$, then $x_n \leq w_n$ for all $n \geq 1$.*

(ii) *If $f(w_1) > w_1$, then $x_n \geq w_n$ for all $n \geq 1$.*

Proof (i) Let $f(w_1) < w_1$. Since $w_1 = x_1$, we get $f(x_1) < x_1$. From (1.4), we have $f(x_1) < z_1 \leq x_1$. Since f is nondecreasing, we obtain $f(z_1) \leq f(x_1) < z_1 \leq x_1$. This implies $f(z_1) < y_1 \leq z_1$. Using (1.3) and (1.4), we have

$$z_1 - r_1 = (1 - c_1)(x_1 - w_1) + c_1\big[f(x_1) - f(w_1)\big] = 0,$$

that is, $z_1 = r_1$, and we get

$$y_1 - q_1 = (1 - b_1)(x_1 - w_1) + b_1\big(f(z_1) - f(r_1)\big) + c_1\big(f(x_1) - x_1\big) \leq 0.$$

Since f is nondecreasing, we have $f(y_1) \leq f(q_1)$. This implies

$$\begin{aligned} x_2 - w_2 &= (1 - \alpha_1 - \beta_1)x_1 + \alpha_1 f(y_1) + \beta_1 f(z_1) - (1 - \alpha_1)w_1 - \alpha_1 f(q_1) \\ &= (1 - \alpha_1)(x_1 - w_1) + \alpha_1\big(f(y_1) - f(q_1)\big) + \beta_1\big(f(z_1) - x_1\big) \\ &\leq 0, \end{aligned}$$

that is, $x_2 \leq w_2$. Assume that $x_k \leq w_k$. Thus $f(x_k) \leq f(w_k)$.

By Lemma 1(v) and Lemma 1(i), we have $f(w_k) < f(w_k)$ and $f(x_k) < f(w_k)$. This implies $f(x_k) < z_k \leq x_k$ and $f(z_k) \leq f(x_k) < z_k$. Thus

$$z_k - r_k = (1 - \gamma_k)(x_k - w_k) + \gamma_k\big(f(x_k) - f(w_k)\big) \leq 0.$$

That is, $z_k \leq r_k$. Since $f(z_k) \leq f(r_k)$, we have

$$y_k - q_k = (1 - b_k)(x_k - w_k) + b_k\big(f(z_k) - f(r_k)\big) + c_{k1}\big(f(x_k) - x_k\big) \leq 0,$$

so $y_k \leq q_k$, which implies $f(y_k) - f(q_k)$. It follows that

$$\begin{aligned} x_{k+1} - w_{k+1} &= (1-\alpha_k)(x_k - w_k) + \alpha_k\bigl(f(y_k) - f(q_k)\bigr) + \beta_k\bigl(f(z_k) - x_k\bigr) \\ &\leq 0, \end{aligned}$$

that is, $x_{k+1} \leq w_{k+1}$. By mathematical induction, we obtain $x_n \leq w_n$ for all $n \geq 1$.

(ii) By using Lemma 1(vi) and Lemma 1(ii) and the same argument as in (i), we can show that $x_n \geq w_n$ for all $n \geq 1$. □

Theorem 2 *Let E be a closed interval on the real line and let $f : E \to E$ be a continuous and nondecreasing mapping such that $F(f)$ is nonempty and bounded. Let $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$, $\{c_n\}_{n=1}^{\infty}$, $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ be sequences in* $[0,1)$. *For $w_1 = x_1 \in E$, let $\{w_n\}_{n=1}^{\infty}$ and $\{x_n\}_{n=1}^{\infty}$ be the sequences defined by* (1.3) *and* (1.4), *respectively. Let $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$, $\{c_n\}_{n=1}^{\infty}$, $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ be sequences in* $[0,1)$. *Then the following is satisfied*:

The iteration (1.4) *$\{x_n\}_{n=1}^{\infty}$ converges to $p \in F(f)$ if and only if the Noor iteration $\{w_n\}_{n=1}^{\infty}$ converges to p. Moreover, the iteration* (1.4) *$\{x_n\}_{n=1}^{\infty}$ is better than the Noor iteration.*

Proof Put $L = \inf\{p \in E : p = f(p)\}$ and $U = \sup\{p \in E : p = f(p)\}$.

($\Rightarrow$) If the iteration (1.4) $\{x_n\}_{n=1}^{\infty}$ converges to $p \in F(f)$, then setting $c_n = \beta_n = 0$ for all $n \geq 1$ in (1.4), we can get the convergence of the Noor iteration.

($\Leftarrow$) Suppose that the Noor iteration $\{w_n\}_{n=1}^{\infty}$ converges to $p \in F(f)$. We divide our proof into the following three cases:

Case 1: $w_1 = x_1 > U$, Case 2: $w_1 = x_1 < L$, Case 3: $L \leq w_1 = x_1 \leq U$.

Case 1: $w_1 = x_1 > U$. By Proposition 3.5 in [7], we get $f(w_1) < w_1$ and $f(x_1) < x_1$. Using Lemma 3(i), we get that $x_n \leq w_n$ for all $n \geq 1$. Following the line of the proof of Theorem 3.7 in [7], we have $U \leq x_n$ for all $n \geq 1$. Then we have $0 \leq x_n - p \leq w_n - p$, so

$$|x_n - p| \leq |w_n - p| \quad \text{for all } n \geq 1.$$

We can see that the iteration (1.4) $\{x_n\}_{n=1}^{\infty}$ is better than the Noor iteration.

Case 2: $w_1 = x_1 < L$. By Proposition 3.5 in [7], we get $f(w_1) > w_1$ and $f(x_1) > x_1$. Using Lemma 3(ii), we get $x_n \geq w_n$ for all $n \geq 1$. Following the line of the proof of Theorem 3.7 in [7], we get $x_n \leq L$ for all $n \geq 1$. So,

$$|x_n - p| \leq |w_n - p| \quad \text{for all } n \geq 1.$$

We can see that the iteration (1.4) $\{x_n\}_{n=1}^{\infty}$ is better than the Noor iteration.

Case 3: $L \leq w_1 = x_1 \leq U$. Suppose that $f(w_1) \neq w_1$. If $f(w_1) < w_1$, we have by Lemma 1(v) that $\{w_n\}_{n=1}^{\infty}$ is nondecreasing with limit p. By Lemma 3(i), we have $p \leq x_n \leq w_n$ for all $n \geq 1$. It follows that $|x_n - p| \leq |w_n - p|$ for all $n \geq 1$. Hence we have that the iteration (1.4) $\{x_n\}_{n=1}^{\infty}$ is better than the Noor iteration $\{w_n\}_{n=1}^{\infty}$. If $f(w_1) > w_1$, we have by Lemma 1(vi) that $\{w_n\}_{n=1}^{\infty}$ in nondecreasing with limit p. By Lemma 3(ii), we have $p \geq x_n \geq w_n$ for all $n \geq 1$. It follows that $|x_n - p| \leq |w_n - p|$ for all $n \geq 1$. Hence, we have that the iteration (1.4) $\{x_n\}_{n=1}^{\infty}$ is better than the Noor iteration $\{w_n\}_{n=1}^{\infty}$. □

Next, we present a numerical example for comparing the rate of convergence between the Mann (1.1), Ishikawa (1.2), Noor (1.3) iterations and the iteration (1.4).

Table 1 Comparison of rate of convergence of the Mann, Ishikawa, Noor and iteration (1.4) for the given function in Example 1

n	Mann	Ishikawa	Noor	Iteration (1.4)
1	3.750000	3.588542	3.489803	2.934030
5	2.861209	2.565875	2.445855	1.684223
10	2.047043	1.815025	1.732658	1.275403
15	1.569324	1.424585	1.376349	1.127905
20	1.312865	1.228173	1.200831	1.065029
25	1.176804	1.127450	1.111775	1.035290
30	1.103122	1.073872	1.064664	1.020127
35	1.061984	1.044248	1.038692	1.011940
40	1.038280	1.027271	1.023833	1.007314
45	1.024214	1.017229	1.015052	1.004602
50	1.015642	1.011122	1.009714	1.002962

Example 1 Let $f : [0,4] \to [0,4]$ be defined by $f(x) = \frac{x^2+5}{6}$. Then f is a continuous and nondecreasing mapping with a fixed point $p = 1$. Use the initial point $x_1 = 4$ and control condition $a_n = b_n = c_n = \beta_n = \frac{1}{n^2+1}$ and $\alpha_n = \frac{1}{n^{0.5}+1}$.

Remark 1 From the example above, we see that the iteration (1.4) is better than the Mann, Ishikawa and Noor iterations under the same control conditions (see Table 1).

4 Convergence theorems for modified iteration methods

Now, we give a convergence theorem for continuous mappings on an arbitrary interval by using the following modified iteration method defined by Suantai [9].

$$\begin{cases} z_n = (1-a_n)x_n + a_n f^n(x_n), \\ y_n = (1-b_n-c_n)x_n + b_n f^n(z_n) + c_n f^n(x_n), \\ x_{n+1} = (1-\alpha_n-\beta_n)x_n + \alpha_n f^n(y_n) + \beta_n f^n(z_n) \end{cases} \tag{4.1}$$

for all $n \geq 1$, where $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$, $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ and $\{c_n\}_{n=1}^{\infty}$ are sequences in $[0,1]$ and $f^n = f \circ f \circ \cdots \circ f$. Also, it is an open problem whether it is possible to prove the convergence theorem of a finite family of continuous mappings on an arbitrary interval by using the iteration methods in this paper.

Theorem 3 *Let E be a closed interval on the real line and let $f : E \to E$ be an injective and continuous mapping. If the sequence $\{x_n\}_{n=1}^{\infty}$ satisfying the conditions of Theorem 1 is bounded, then it converges to a fixed point of f.*

Proof Suppose that $\{x_n\}_{n=1}^{\infty}$ is not convergent. Let $a = \liminf_{n\to\infty} x_n$ and $b = \limsup_{n\to\infty} x_n$. Then $a < b$. Next, we show that

$$\text{if } m \in (a,b), \quad \text{then } f(m) = m.$$

To show this, suppose that $f(m) \neq m$ for some $m \in (a,b)$. Because f is an injective mapping,

$$f^n(m) \neq m \quad \text{for } f(m) \neq m.$$

Without loss of generality, we suppose that $f^n(m) - m > 0$. From the continuity of f, we know that f^n is a continuous function. So, there exists δ, $0 < \delta < b - a$, such that

$$f^n(x) - x > 0 \quad \text{for } |x - m| \leq \delta.$$

If we use a similar method as the proof of Theorem 1, the proof of this theorem follows. That is, $\{x_n\}_{n=1}^{\infty}$ converges to a fixed point of f. □

The following three corollaries follow from Theorem 3.

Corollary 4 *Let E be a closed interval on the real line and let $f : E \to E$ be an injective and continuous mapping. For $x_1 \in E$, let $\{x_n\}_{n=1}^{\infty}$ be the sequence defined by*

$$\begin{cases} z_n = (1 - a_n)x_n + a_n f^n(x_n), \\ y_n = (1 - b_n)x_n + b_n f^n(z_n), \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n f^n(y_n). \end{cases}$$

If the sequence $\{x_n\}_{n=1}^{\infty}$ satisfying the conditions of Corollary 1 is bounded, then it converges to a fixed point of f.

Proof If we take $c_n = \beta_n = 0$ for all $n \geq 1$ in Theorem 3, we obtain the desired result. □

Corollary 5 *Let E be a closed interval on the real line and let $f : E \to E$ be an injective and continuous mapping. For $x_1 \in E$, let $\{x_n\}_{n=1}^{\infty}$ be a sequence defined by*

$$\begin{cases} y_n = (1 - b_n)x_n + b_n f^n(x_n), \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n f^n(y_n). \end{cases}$$

If the sequence $\{x_n\}_{n=1}^{\infty}$ satisfying the conditions of Corollary 2 is bounded, then it converges to a fixed point of f.

Proof It follows directly from Theorem 3 by setting $a_n = c_n = \beta_n = 0$ for all $n \geq 1$. □

Corollary 6 *Let E be a closed interval on the real line and let $f : E \to E$ be an injective and continuous mapping. For $x_1 \in E$, let $\{x_n\}_{n=1}^{\infty}$ be a sequence defined by*

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n f^n(x_n).$$

If the sequence $\{x_n\}_{n=1}^{\infty}$ satisfying the conditions of Corollary 3 is bounded, then it converges to a fixed point of f.

Proof By putting $a_n = b_n = c_n = \beta_n = 0$ for all $n \geq 1$ in Theorem 3, we obtain the desired result. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

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