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Strong convergence theorems for modifying Halpern-Mann iterations for a quasi-ϕ-asymptotically nonexpansive multi-valued mapping in Banach spaces

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Abstract

The purpose of this paper is to introduce modifying Halpern-Mann's iterations sequence for a quasi-ϕ-asymptotically nonexpansive multi-valued mapping. Under suitable limit conditions, some strong convergence theorems are proved. The results presented in the paper improve and extend the corresponding results of Chang (Appl. Math. Comput. 218:6489-6497, 2012).

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1 Introduction

Throughout this paper, we denote by N and R the sets of positive integers and real numbers, respectively. Let D be a nonempty closed subset of a real Banach space X. A mapping $T : D \to D$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in D$. Let $N(D)$ and $CB(D)$ denote the families of nonempty subsets and nonempty closed bounded subsets of D, respectively. The Hausdorff metric on $CB(D)$ is defined by

$$H(A_1, A_2) = \max\Big\{\sup_{x \in A_1} d(x, A_2), \sup_{y \in A_2} d(y, A_1)\Big\}$$

for $A_1, A_2 \in CB(D)$, where $d(x, A_1) = \inf\{\|x - y\|, y \in A_1\}$. The multi-valued mapping $T : D \to CB(D)$ is called nonexpansive if $H(Tx, Ty) \leq \|x - y\|$ for all $x, y \in D$. An element $p \in D$ is called a fixed point of $T : D \to N(D)$ if $p \in T(p)$. The set of fixed points of T is represented by $F(T)$.

In the sequel, let $S(X) = \{x \in X : \|x\| = 1\}$. A Banach space X is said to be strictly convex if $\|\frac{x+y}{2}\| \leq 1$ for all $x, y \in S(X)$ and $x \neq y$. A Banach space is said to be uniformly convex if $\lim_{n\to\infty} \|x_n - y_n\| = 0$ for any two sequences $\{x_n\}, \{y_n\} \subset S(X)$ and $\lim_{n\to\infty} \|\frac{x_n+y_n}{2}\| = 0$. The norm of the Banach space X is said to be Gâteaux differentiable if for each $x, y \in S(X)$, the limit

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{1.1}$$

exists. In this case, X is said to be smooth. The norm of the Banach space X is said to be Fréchet differentiable if for each $x \in S(X)$, the limit (1.1) is attained uniformly for $y \in S(x)$, and the norm is uniformly Fréchet differentiable if the limit (1.1) is attained uniformly for $x, y \in S(X)$. In this case, X is said to be uniformly smooth.

Remark 1.1 Let X be a real Banach space with dual X^*. We denote by J the normalized duality mapping from X to 2^{X^*}, which is defined by

$$J(x) = \{x^* \in X^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}, \quad x \in X,$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing.

The following basic properties of the normalized duality mapping J in a Banach space X can be found in Cioranescu [1].

(1) X (X^*, resp.) is uniformly convex if and only if X^* (X, resp.) is uniformly smooth;
(2) If X is smooth, then J is single-valued and norm-to-weak* continuous;
(3) If X is reflexive, then J is onto;
(4) If X is strictly convex, then $Jx \cap Jy \neq \Phi$ for all $x, y \in X$;
(5) If X has a Fréchet differentiable norm, then J is norm-to-norm continuous;
(6) If X is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of X;
(7) Each uniformly convex Banach space X has the Kadec-Klee property, *i.e.*, for any sequence $\{x_n\} \subset X$, if $x_n \rightharpoonup x \in X$ and $\|x_n\| \to \|x\|$, then $x_n \to x \in X$;
(8) If X is a reflexive and strictly convex Banach space with a strictly convex dual X^* and $J^* : X^* \to X$ is the normalized duality mapping in X^*, then $J^{-1} = J^*$, $JJ^* = I_{x^*}$ and $J^*J = I_x$.

Next we assume that X is a smooth, strictly convex, and reflexive Banach space and D is a nonempty closed convex subset of X. In the sequel, we always use $\phi : X \times X \to \overline{R^-}$ to denote the Lyapunov bifunction defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \quad x, y \in X. \tag{1.2}$$

It is obvious from the definition of the function ϕ that

$$(\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2, \tag{1.3}$$

$$\phi(y, x) = \phi(y, z) + \phi(z, x) + 2\langle z - y, Jx - Jz\rangle, \quad x, y, z \in X, \tag{1.4}$$

and

$$\phi\bigl(x, J^{-1}(\alpha Jy + (1 - \alpha)Jz)\bigr) \leq \alpha\phi(x, y) + (1 - \alpha)\phi(x, z) \tag{1.5}$$

for all $\alpha \in [0, 1]$ and $x, y, z \in X$.

Following Alber [2], the generalized projection $\Pi_D : X \to D$ is defined by

$$\Pi_D x = \arg\inf_{y \in D} \phi(y, x), \quad \forall x \in X.$$

Remark 1.2 (see [3]) Let Π_D be the generalized projection from a smooth, reflexive, and strictly convex Banach space X onto a nonempty closed convex subset D of X, then Π_D is closed and quasi-ϕ-nonexpansive from X onto D.

Many problems in nonlinear analysis can be reformulated as a problem of finding a fixed point of a nonexpansive mapping. In 1953, Mann [4] introduced the following iterative sequence $\{x_n\}$:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n,$$

where the initial guess $x_1 \in D$ is arbitrary and $\{\alpha_n\}$ is a real sequence in $[0,1]$. It is known that under appropriate settings the sequence $\{x_n\}$ converges weakly to a fixed point of T. However, even in a Hilbert space, the Mann iteration may fail to converge strongly [5]. Some attempts to construct an iteration method guaranteeing the strong convergence have been made. For example, Halpern [6] proposed the following so-called Halpern iteration:

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)Tx_n,$$

where $u, x_1 \in D$ are arbitrarily given and $\{\alpha_n\}$ is a real sequence in $[0,1]$. Another approach was proposed by Nakajo and Takahashi [7]. They generated a sequence as follows:

$$\begin{cases} x_1 \in X \quad \text{is arbitrary}, \\ y_n = \alpha_n u + (1 - \alpha_n)Tx_n, \\ C_n = \{z \in D : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in D : \langle x_n - z, x_1 - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_1 \quad (n = 1, 2, \ldots), \end{cases} \tag{1.6}$$

where $\{\alpha_n\}$ is a real sequence in $[0,1]$ and P_K denotes the metric projection from a Hilbert space H onto a closed convex subset K of H. It should be noted here that the iteration above works only in a Hilbert space setting. To extend this iteration to a Banach space, the concepts of relatively nonexpansive mappings and quasi-ϕ-nonexpansive mappings have been introduced (see [8–11] and [12]).

Inspired by Matsushita and Takahashi, in this paper, we introduce modifying Halpern-Mann iterations sequence for finding a fixed point of a quasi-ϕ-nonexpansive mappings multi-valued mapping $T : D \to CB(D)$ and prove some strong convergence theorems. The results presented in the paper improve and extend the corresponding results in [13] and other.

2 Preliminaries

In the sequel, we denote the strong convergence and weak convergence of the sequence $\{x_n\}$ by $x_n \to x$ and $x_n \rightharpoonup x$, respectively.

Lemma 2.1 (see [2]) *Let X be a smooth, strictly convex and reflexive Banach space, and let D be a nonempty closed convex subset of X. Then the following conclusions hold*:

(a) $\phi(x, y) = 0$ *if and only if* $x = y$;

(b) $\phi(x, \Pi_D y) + \phi(\Pi_D y, y) \le \phi(x, y)$, $\forall x, y \in D$;
(c) *If* $x \in X$ *and* $z \in D$, *then* $z = \Pi_D x \Leftrightarrow \langle z - y, Jx - Jz \rangle \ge 0$, $\forall y \in D$.

Remark 2.1 If H is a real Hilbert space, then $\phi(x, y) = \|x - y\|^2$ and Π_D is the metric projection P_D of H onto D.

Lemma 2.2 (see [13]) *Let X be a real uniformly smooth and strictly convex Banach space with the Kadec-Klee property, and let D be a nonempty closed convex subset of X. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in D such that $x_n \to p$ and $\phi(x_n, y_n) \to 0$, where ϕ is the function defined by* (1.2), *then $y_n \to p$.*

Definition 2.1 A point $x \in D$ is said to be an asymptotic fixed point of $T : D \to CB(D)$ if there exists a sequence $\{x_n\} \subset D$ such that $x_n \rightharpoonup x \in X$ and $d(x_n, T(x_n)) \to 0$. Denote the set of all asymptotic fixed points of T by $\hat{F}(T)$.

Definition 2.2 A multi-valued mapping $T : D \to CB(D)$ is said to be closed if for any sequence $\{x_n\} \subset D$ with $x_n \to x \in X$ and $d(y, T(x_n)) \to 0$, then $d(y, T(x)) = 0$.

Definition 2.3 (1) A multi-valued mapping $T : D \to CB(D)$ is said to be relatively nonexpansive if $F(T) \neq \Phi$, $\hat{F}(T) = F(T)$ and $\phi(p, z) \le \phi(p, x)$, $\forall x \in D, p \in F(T), z \in T(x)$.

(2) A multi-valued mapping $T : D \to CB(D)$ is said to be quasi-ϕ-nonexpansive if $F(T) \neq \Phi$, and $\phi(p, z) \le \phi(p, x)$, $\forall x \in D, p \in F(T), z \in Tx$.

(3) A multi-valued mapping $T : D \to CB(D)$ is said to be quasi-ϕ-asymptotically nonexpansive if $F(T) \neq \Phi$ and there exists a real sequence $k_n \subset [1, +\infty)$, $k_n \to 1$ such that

$$\phi(p, z_n) \le k_n \phi(p, x), \quad \forall x \in D, p \in F(T), z_n \in T^n x. \tag{2.1}$$

Definition 2.4 A mapping $T : D \to CB(D)$ is said to be uniformly L-Lipschitz continuous if there exists a constant $L > 0$ such that $\|x_n - y_n\| \le L\|x - y\|$, where $x, y \in D$, $x_n \in T^n x$, $y_n \in T^n y$.

Next, we present an example of a relatively nonexpansive multi-valued mapping.

Example 2.1 (see [13]) Let $I = [0, 1]$, $X = C(I)$ (the Banach space of continuous functions defined on I with the uniform convergence norm $\|f\|_C = \sup_{t \in I} |f(t)|$), $D = \{f \in X : f(x) \ge 0, x \in I\}$ and let a, b be two constants in $(0, 1)$ with $a < b$. Let $T : D \to N(D)$ be a multi-valued mapping defined by

$$T(f) = \begin{cases} \{g \in D : a \le f(x) - g(x) \le b, \forall x \in I\}, \\ \{0\} \quad \text{otherwise}. \end{cases} \tag{2.2}$$

It is easy to see that $F(T) = \{0\}$, therefore $F(T)$ is nonempty.

From the example in [13], we can see that $T : D \to N(D)$ is a closed quasi-ϕ-asymptotically nonexpansive multi-valued mapping.

Remark 2.2 From the definitions, it is obvious that a relatively nonexpansive multi-valued mapping is a quasi-ϕ-nonexpansive multi-valued mapping, and a quasi-ϕ-nonexpansive multi-valued mapping is a quasi-ϕ-asymptotically nonexpansive multi-valued mapping, and a quasi-ϕ-asymptotically nonexpansive multi-valued mapping is a total quasi-ϕ-asymptotically nonexpansive multi-valued mapping, but the converse is not true.

Lemma 2.3 *Let X and D be as in Lemma* 2.2. *Let $T : D \to CB(D)$ be a closed and quasi-ϕ-asymptotically nonexpansive multi-valued mapping with nonnegative real sequences $\{k_n\}$ with $\{k_n\} \subset [1,\infty)$ and $k_n \to 1$ (as $n \to \infty$), then $F(T)$ is a closed and convex subset of D.*

Proof Let $\{x_n\}$ be a sequence in $F(T)$ such that $x_n \to x^*$. Since T is a quasi-ϕ-asymptotically nonexpansive multi-valued mapping, we have

$$\phi(x_n, z) \leq k_1 \phi\big(x_n, x^*\big)$$

for all $z \in Tx^*$ and for all $n \in N$. Therefore,

$$\phi\big(x^*, z\big) = \lim_{n\to\infty} \phi(x_n, z) \leq \lim_{n\to\infty} k_1 \phi\big(x_n, x^*\big) = k_1 \phi\big(x^*, x^*\big) = 0.$$

By Lemma 2.1(a), we obtain $z = x^*$. Hence, $Tx^* = \{x^*\}$. So, we have $x^* \in F(T)$. This implies $F(T)$ is closed.

Let $p, q \in F(T)$ and $t \in (0,1)$, and put $w = tp + (1-t)q$. Next we prove that $w \in F(T)$. Indeed, in view of the definition of ϕ, let $z_n \in T^n w$, we have

$$\begin{aligned} \phi(w, z_n) &= \|w\|^2 - 2\langle w, Jz_n\rangle + \|z_n\|^2 \\ &= \|w\|^2 - 2\big\langle tp + (1-t)q, Jz_n\big\rangle + \|z_n\|^2 \\ &= \|w\|^2 + t\phi(p, z_n) + (1-t)\phi(q, z_n) - t\|p\|^2 - (1-t)\|q\|^2. \end{aligned} \tag{2.3}$$

Since

$$\begin{aligned} &t\phi(p, z_n) + (1-t)\phi(q, z_n) \\ &\quad \leq tk_n\phi(p, w) + (1-t)k_n\phi(q, w) \\ &\quad = t\big\{\|p\|^2 - 2\langle p, Jw\rangle + \|w\|^2 + (k_n - 1)\phi(p, w)\big\} \\ &\qquad + (1-t)\big\{\|q\|^2 - 2\langle q, Jw\rangle + \|w\|^2 + (k_n - 1)\phi(q, w)\big\} \\ &\quad = t\|p\|^2 + (1-t)\|q\|^2 - \|w\|^2 + t(k_n - 1)\phi(p, w) + (1-t)(k_n - 1)\phi(q, w). \end{aligned} \tag{2.4}$$

Substituting (2.3) into (2.4) and simplifying it, we have

$$\phi(w, z_n) \leq t(k_n - 1)\phi(p, w) + (1-t)(k_n - 1)\phi(q, w) \to 0 \quad (\text{as } n \to \infty).$$

Hence, by Lemma 2.2, we have $z_n \to w$. This implies that $z_{n+1} (\in TT^n w) \to w$. Since T is closed, we have $w \in Tw$, *i.e.*, $w \in F(T)$. This completes the proof of Lemma 2.3. □

Lemma 2.4 ([13]) *Let X be a uniformly convex Banach space, $r > 0$ be a positive number and $B_r(0)$ be a closed ball of X. Then, for any given sequence $\{x_n\}_{n=1}^\infty \subset B_r(0)$ and for any given sequence $\{x_n\}_{n=1}^\infty$ of positive numbers with $\sum_{n=1}^\infty \lambda_n = 1$, there exists a continuous,* *strictly increasing, and convex function* $g:[0,2r)\to[0,\infty)$ *with* $g(0)=0$ *such that for any positive integers* i, j *with* $i<j$,

$$\left\|\sum_{n=1}^{\infty}\lambda_n x_n\right\|^2\leq\sum_{n=1}^{\infty}\lambda_n\|x_n\|^2-\lambda_i\lambda_j g\|x_i-x_j\|^2. \tag{2.5}$$

Lemma 2.5 ([14]) *Let* X *be a uniformly convex and smooth Banach space and let* $\{x_n\}$ *and* $\{y_n\}$ *be two sequences of* X *such that* $\{x_n\}$ *or* $\{y_n\}$ *is bounded. If* $\lim_{n\to\infty}\phi(x_n,y_n)=0$, *then* $\lim_{n\to\infty}\|x_n-y_n\|=0$.

Let X be a reflexive, strictly convex, and smooth Banach space. The duality mapping J^* from X^* onto $X^{**}=X$ coincides with the inverse of the duality mapping J from E onto E^*. We make use the following mapping $v:X\times X^*\to R$ studied in Alber [2]:

$$v(x,x^*)=\|x\|^2-2\langle x,x^*\rangle+\|x^*\|^2 \tag{2.6}$$

for all $x\in X$, $x^*\in X^*$. Obviously, $v(x,x^*)=\phi(x,J^{-1}x^*)$.

Lemma 2.6 ([15]) *Let* X *be a reflexive, strictly convex, and smooth Banach space, and let* v *as in* (2.6). *Then*

$$v(x,x^*)+2\langle J^{-1}x^*-x,y^*\rangle\leq v(x,x^*+y^*) \tag{2.7}$$

for all $x\in X$, $x^*,y^*\in X^*$.

Lemma 2.7 ([16]) *Assume that* $\{\alpha_n\}$ *is a sequence of nonnegative real numbers such that*

$$\alpha_{n+1}\leq(1-\gamma_n)\alpha_n+\gamma_n\delta_n,$$

where $\{\gamma_n\}$ *is a sequence in* $(0,1)$ *and* $\{\delta_n\}$ *is a sequence such that*
(a) $\lim_{n\to\infty}\gamma_n=0$, $\sum_{n=1}^{\infty}\gamma_n=\infty$;
(b) $\limsup_{n\to\infty}\delta_n\leq 0$.
Then $\lim_{n\to\infty}\alpha_n=0$.

Lemma 2.8 ([17]) *Let* $\{\alpha_n\}$ *be a sequence of real numbers such that there exists a subsequence* $\{n_i\}$ *of* $\{n\}$ *such that* $\alpha_{n_i}\leq\alpha_{n_i+1}$ *for all* $i\in N$. *Then there exists a nondecreasing* $\{m_k\}\subset N$ *such that* $m_k\to\infty$ *and the following properties are satisfied for all* (*sufficiently large*) *numbers sequence* $k\subset N$:

$$\alpha_{m_k}\leq\alpha_{m_k+1}\quad\textit{and}\quad\alpha_k\leq\alpha_{m_k+1}.$$

In fact, $m_k=\max\{j\leq k:\alpha_j\leq\alpha_{j+1}\}$.

3 Main results

Theorem 3.1 *Let* X *be a real uniformly smooth and strictly convex Banach space with the Kadec-Klee property, let* D *be a nonempty closed convex subset of* X, *and let* $T:D\to CB(D)$ *be a closed and uniformly* L*-Lipschitz continuous quasi-*ϕ*-asymptotically nonexpansive* *multi-valued mapping with nonnegative real sequences* $\{k_n\} \subset [0,+\infty)$, $k_n \to 1$ *(as* $n \to \infty$*) such that condition* (2.1) *and* $\prod_{n=1}^{\infty} k_n < +\infty$. *Let* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *be two sequences in* $(0,1)$ *satisfying*

(R1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\lim_{n\to\infty} \frac{k_n-1}{\alpha_n} = 0$;

(R2) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(R3) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$.

If $\{x_n\}$ *is the sequence generated by*

$$x_{n+1} = \Pi_D J^{-1}\big[\alpha_n Jx_1 + (1-\alpha_n)\big(\beta_n Jx_n + (1-\beta_n)Jz_n\big)\big], \quad z_n \in T^n x_n, \tag{3.1}$$

where $x_1 \in X$ *is arbitrary,* $F(T)$ *is the fixed point set of* T, *and* Π_D *is the generalized projection of* X *onto* D. *If* $I - T$ *is demi-closed at zero and* $F(T) \neq \Phi$, *then* $\lim_{n\to\infty} x_n = \Pi_{F(T)} x_1$.

Remark 3.1 We can present an example of $\{k_n\}$ satisfying the conditions $\{k_n\} \subset [0,+\infty)$, $k_n \to 1$ (as $n \to \infty$) and $\prod_{n=1}^{\infty} k_n < +\infty$. For instance, if $k_n = 1 + \frac{1}{2^{2^{n-1}}}$, then $\prod_{n=1}^{\infty} k_n = 2 < +\infty$.

Proof First, we prove that $\{x_n\}$ is a bounded sequence in D.

Let $p \in F(T)$ and $y_n = J^{-1}[\beta_n Jx_n + (1-\beta_n)Jz_n]$ for any $n \in N$. Then

$$x_{n+1} = \Pi_D J^{-1}\big[\alpha_n Jx_1 + (1-\alpha_n)Jy_n\big]$$

for any $n \in N$. Using (2.1) and (1.5), we have

$$\begin{aligned} \phi(p,y_n) &= \phi\big(p, J^{-1}\big[\beta_n Jx_n + (1-\beta_n)Jz_n\big]\big) \\ &\le \beta_n \phi(p,x_n) + (1-\beta_n)\phi(p,z_n) \\ &\le \beta_n \phi(p,x_n) + (1-\beta_n)k_n\phi(p,x_n) \\ &\le k_n \phi(p,x_n), \end{aligned}$$

and

$$\begin{aligned} \phi(p,x_{n+1}) &= \phi\big(p, \Pi_D J^{-1}\big[\alpha_n Jx_1 + (1-\alpha_n)Jy_n\big]\big) \\ &\le \phi\big(p, J^{-1}\big[\alpha_n Jx_1 + (1-\alpha_n)Jy_n\big]\big) \\ &\le \alpha_n \phi(p,x_1) + (1-\alpha_n)\phi(p,y_n) \\ &\le \alpha_n \phi(p,x_1) + (1-\alpha_n)k_n\phi(p,x_n) \\ &\le \max\big\{\phi(p,x_1), k_n\phi(p,x_n)\big\} \\ &\le \max\big\{\phi(p,x_1), k_n k_{n-1}\phi(p,x_{n-1})\big\}. \end{aligned}$$

By induction, we have

$$\phi(p,x_{n+1}) \le k_n k_{n-1} \cdots k_1 \phi(p,x_1).$$

Since $\lim_{n\to\infty} k_n = 1$ and $\prod_{n=1}^{\infty} k_n < +\infty$, then $k_n k_{n-1} \cdots k_1$ is bounded, and we get $\phi(p,x_{n+1})$ is bounded. This implies that $\{x_n\}$ is bounded, so is $\{z_n\}$.

Next, let $g : [0, 2r] \to [0, \infty)$ be a function satisfying the properties of Lemma 2.5, where $r = \sup\{\|x_1\|, \|x_n\|, \|z_n\|\}$. Put $p = \Pi_{F(T)}x_1$ and $y_n = J^{-1}(\beta_n Jx_n + (1-\beta_n)Jz_n)$. Then

$$\begin{aligned}
\phi(p, y_n) &= \phi\bigl(p, J^{-1}\bigl(\beta_n Jx_n + (1-\beta_n)Jz_n\bigr)\bigr) \\
&\le \beta_n\phi(p, x_n) + (1-\beta_n)k_n\phi(p, x_n) - \beta_n(1-\beta_n)g\bigl(\|Jx_n - Jz_n\|\bigr) \\
&\le k_n\phi(p, x_n) - \beta_n(1-\beta_n)g\bigl(\|Jx_n - Jz_n\|\bigr),
\end{aligned}$$

and

$$\begin{aligned}
\phi(p, x_{n+1}) &= \phi\bigl(p, \Pi_D J^{-1}\bigl[\alpha_n Jx_1 + (1-\alpha_n)Jy_n\bigr]\bigr) \\
&\le \phi\bigl(p, J^{-1}\bigl[\alpha_n Jx_1 + (1-\alpha_n)Jy_n\bigr]\bigr) \\
&\le \alpha_n\phi(p, x_1) + (1-\alpha_n)\phi(p, y_n) \\
&\le \alpha_n\phi(p, x_1) + (1-\alpha_n)\bigl[k_n\phi(p, x_n) - \beta_n(1-\beta_n)g\bigl(\|Jx_n - Jz_n\|\bigr)\bigr] \\
&\le \alpha_n k_n\bigl(\phi(p, x_1) - \phi(p, x_n)\bigr) + (k_n - 1)\phi(p, x_n) + \phi(p, x_n) \\
&\quad - (1-\alpha_n)\beta_n(1-\beta_n)g\bigl(\|Jx_n - Jz_n\|\bigr). \qquad (3.2)
\end{aligned}$$

Letting

$$M = \sup_{n\in N}\left\{\left|k_n\bigl(\phi(p, x_1) - \phi(p, x_n)\bigr) + \frac{k_n - 1}{\alpha_n}\phi(p, x_n)\right| + \beta_n(1-\beta_n)g\bigl(\|Jx_n - Jz_n\|\bigr)\right\},$$

by (3.2), we have

$$\beta_n(1-\beta_n)g\bigl(\|Jx_n - Jz_n\|\bigr) \le \phi(p, x_n) - \phi(p, x_{n+1}) + \alpha_n M.$$

Let $u_n = J^{-1}[\alpha_n Jx_1 + (1-\alpha_n)Jy_n]$. Then $x_{n+1} = \Pi_D u_n$ for all $n \in N$. It follows from (2.6) and (2.7) that

$$\begin{aligned}
\phi(p, x_{n+1}) &\le \phi\bigl(p, J^{-1}\bigl(\alpha_n Jx_1 + (1-\alpha_n)Jy_n\bigr)\bigr) = \nu\bigl(p, \alpha_n Jx_1 + (1-\alpha_n)Jy_n\bigr) \\
&\le \nu\bigl(p, J^{-1}\bigl(\alpha_n Jx_1 + (1-\alpha_n)Jy_n\bigr) - \alpha_n(Jx_1 - Jp)\bigr) \\
&\quad - 2\bigl\langle J^{-1}\bigl(\alpha_n Jx_1 + (1-\alpha_n)Jy_n\bigr) - p, -\alpha_n(Jx_1 - Jp)\bigr\rangle \\
&= \nu\bigl(p, \alpha_n Jp + (1-\alpha_n)Jy_n\bigr) + 2\alpha_n\langle u_n - p, Jx_1 - Jp\rangle \\
&= \phi\bigl(p, J^{-1}\bigl[\alpha_n Jp + (1-\alpha_n)Jy_n\bigr]\bigr) + 2\alpha_n\langle u_n - p, Jx_1 - Jp\rangle \\
&\le \|p\|^2 - 2\bigl\langle p, \alpha_n Jp + (1-\alpha_n)Jy_n\bigr\rangle + \bigl\|\alpha_n Jp + (1-\alpha_n)Jy_n\bigr\|^2 \\
&\quad + 2\alpha_n\langle u_n - p, Jx_1 - Jp\rangle \\
&\le \|p\|^2 - 2\alpha_n\langle p, Jp\rangle - 2(1-\alpha_n)\langle p, y_n\rangle + \alpha_n\|Jp\|^2 + (1-\alpha_n)\|Jy_n\|^2 \\
&\quad + 2\alpha_n\langle u_n - p, Jx_1 - Jp\rangle \\
&= \alpha_n\phi(p, p) + (1-\alpha_n)\phi(p, y_n) + 2\alpha_n\langle u_n - p, Jx_1 - Jp\rangle \\
&\le (1-\alpha_n)k_n\phi(p, x_n) + 2\alpha_n\langle u_n - p, Jx_1 - Jp\rangle. \qquad (3.3)
\end{aligned}$$

The rest of the proof will be divided into two parts.

Case (1). Suppose that there exists n_0 such that $\{\phi(p,x_n)\}_{n_0}^{\infty}$ is nonincreasing. In this situation, $\{\phi(p,x_n)\}_{n_0}^{\infty}$ is convergent. Together with (R1), (R3), and (3.2), we obtain

$$\lim_{n\to\infty} g\big(\|Jx_n - Jz_n\|\big) = 0. \tag{3.4}$$

Therefore, $\lim_{n\to\infty}\|Jx_n - Jz_n\| = 0$ and $\lim_{n\to\infty}\|x_n - z_n\| = 0$. Since $d(x_n, Tx_n) \le \|x_n - z_n\|$, we obtain

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0. \tag{3.5}$$

Then

$$\begin{aligned}\phi(z_n,y_n) &= \phi\big(z_n,J^{-1}\big[\beta_n Jx_n + (1-\beta_n)Jz_n\big]\big)\\ &\le \beta_n\phi(z_n,x_n) + (1-\beta_n)\phi(z_n,z_n)\\ &= \beta_n\phi(z_n,x_n) \to 0\end{aligned} \tag{3.6}$$

and

$$\begin{aligned}\phi(y_n,u_n) &= \phi\big(y_n,J^{-1}\big[\alpha_n Jx_1 + (1-\alpha_n)Jy_n\big]\big)\\ &\le \alpha_n\phi(y_n,x_1) + (1-\alpha_n)\phi(y_n,y_n)\\ &= \alpha_n\phi(y_n,x_1) \to 0.\end{aligned} \tag{3.7}$$

From (3.6), (3.7) and Lemma 2.5, we have

$$\lim_{n\to\infty}\|y_n - z_n\| = 0 \quad\text{and}\quad \lim_{n\to\infty}\|y_n - u_n\| = 0.$$

From (3.4), we have

$$\lim_{n\to\infty}\|x_n - u_n\| = 0. \tag{3.8}$$

Since $I - T$ is demi-closed at zero, we choose a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $x_{n_i} \rightharpoonup p^* \in F(T)$. By Lemma 2.2(c), we have

$$\begin{aligned}&\limsup_{n\to\infty}\langle u_n - p, Jx_1 - Jp\rangle\\ &\quad= \limsup_{n\to\infty}\langle x_n - p, Jx_1 - Jp\rangle\\ &\quad= \lim_{i\to\infty}\langle x_{n_i} - p, Jx_1 - Jp\rangle = \big\langle p^* - p, Jx_1 - Jp\big\rangle \le 0.\end{aligned} \tag{3.9}$$

Hence the conclusion follows.

Case (2). Suppose that there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $\phi(p,x_{n_i}) \le \phi(p,x_{n_i+1})$. Then, by Lemma 2.8, there exists a nondecreasing sequence $\{m_k\} \subset N$, $m_k \to \infty$ such that

$$\phi(p,x_{m_k}) \le \phi(p,x_{m_k+1}) \quad\text{and}\quad \phi(p,x_k) \le \phi(p,x_{m_k+1}).$$

This together with (3.2) gives

$$\beta_{m_k}(1-\beta_{m_k})g(\|Jx_{m_k}-Jz_{m_k}\|) \le \phi(p,x_{m_k}) - \phi(p,x_{m_k+1}) + \alpha_{m_k}M$$

for all $k \in N$. Then, by conditions (R1) and (R3),

$$\lim_{k\to\infty} g(\|Jx_{m_k}-Jz_{m_k}\|) = 0.$$

By the same argument as Case (1), we get

$$\lim_{k\to\infty}\sup\langle u_{m_k}-p, Jx_1-Jp\rangle \le 0. \tag{3.10}$$

From (3.3), we get

$$\phi(p,x_{m_k+1}) \le (1-\alpha_{m_k})k_{m_k}\phi(p,x_{m_k}) + 2\alpha_{m_k}\langle u_{m_k}-p, Jx_1-Jp\rangle$$

and

$$\alpha_{m_k}k_{m_k}\phi(p,x_{m_k}) \le k_{m_k}\phi(p,x_{m_k}) - \phi(p,x_{m_k+1}) + 2\alpha_{m_k}\langle u_{m_k}-p, Jx_1-Jp\rangle.$$

Since $\phi(p,x_{m_k}) - \phi(p,x_{m_k+1}) \le 0$, we have

$$\alpha_{m_k}k_{m_k}\phi(p,x_{m_k}) \le (k_{m_k}-1)\phi(p,x_{m_k}) + 2\alpha_{m_k}\langle u_{m_k}-p, Jx_1-Jp\rangle.$$

This implies that

$$\phi(p,x_{m_k}) \le \frac{k_{m_k}-1}{\alpha_{m_k}}\phi(p,x_{m_k}) + 2\langle u_{m_k}-p, Jx_1-Jp\rangle.$$

From (3.10) and (R1), we get $\lim_{k\to\infty}\phi(p,x_{m_k}) \le 0$ and $x_{m_k} \to p$. This implies that $\lim_{n\to\infty} x_n = p$, which yields that $p = w = \Pi_{F(T)}x_1$. Therefore, $x_n \to \Pi_{F(T)}x_1$. The proof of Theorem 3.1 is completed. □

By Remark 2.2, the following corollaries are obtained.

Corollary 3.1 *Let X and D be as in Theorem* 3.1, *and let* $T : D \to CB(D)$ *be a closed and uniformly L-Lipschitz continuous relatively nonexpansive multi-valued mapping. Let* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *be two sequences in* $(0,1)$ *satisfying*
(R1) $\lim_{n\to\infty}\alpha_n = 0$;
(R2) $\sum_{n=1}^{\infty}\alpha_n = \infty$;
(R3) $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$.
Let $\{x_n\}$ *be the sequence generated by* (3.1), *where* $F(T)$ *is the set of fixed points of* T, *and* Π_D *is the generalized projection of X onto D, then* $\{x_n\}$ *converges strongly to* $\Pi_{F(T)}x_1$.

If we take $\beta_n = \beta$, the following result is obtained.

Corollary 3.2 *Let X be a real uniformly smooth and strictly convex Banach space with the Kadec-Klee property, let D be a nonempty closed convex subset of X and let $T : D \to CB(D)$ be a closed and uniformly L-Lipschitz continuous quasi-ϕ-asymptotically nonexpansive multi-valued mapping with nonnegative real sequences $\{k_n\} \subset [0, +\infty)$, $k_n \to 1$ (as $n \to \infty$) satisfying condition* (2.1). *Let $\{\alpha_n\}$ be a sequence in* $(0, 1)$ *satisfying*

(Q1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\lim_{n\to\infty} \frac{k_n-1}{\alpha_n} = 0$;

(Q2) $\sum_{n=1}^{\infty} \alpha_n = \infty$.

If $\beta \in (0, 1)$ and $\{x_n\}$ is the sequence generated by

$$x_{n+1} = \Pi_D J^{-1}\big[\alpha_n Jx_1 + (1-\alpha_n)J\big(\beta Jx_n + (1-\beta)Jz_n\big)\big], \quad z_n \in T^n x_n, \tag{3.11}$$

where $x_1 \in X$ is arbitrary, $F(T)$ is the fixed point set of T, and Π_D is the generalized projection of X onto D; if $I - T$ is demi-closed at zero and $F(T)$ is nonempty, then $\lim_{n\to\infty} x_n = \Pi_{F(T)} x_1$.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

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