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Affine algorithms for the split variational inequality and equilibrium problems

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Abstract
An affine algorithm for the split variational inequality and equilibrium problems is presented. Strong convergence result is given.

Keywords: affine algorithm; split method; variational inequality; equilibrium problem

1 Introduction

In the present manuscript, we focus on the following split variational inequality and equilibrium problem: Finding a point x^* such that

$$x^* \in \mathrm{GVI}(B, \psi, C) \quad \text{and} \quad \psi(x^*) \in \mathrm{EP}(F, A), \tag{1.1}$$

where $\mathrm{GVI}(B, \psi, C)$ is the solution set of the generalized variational inequality of finding $u \in C$, $\psi(u) \in C$ such that

$$\langle Bu, \psi(v) - \psi(u) \rangle \geq 0, \quad \forall \psi(v) \in C, \tag{1.2}$$

and $\mathrm{EP}(F, A)$ is the solution set of the equilibrium problem, which is to find $x^\dagger \in C$ such that

$$F(x^\dagger, y) + \langle Ax^\dagger, y - x^\dagger \rangle \geq 0, \quad \forall y \in C. \tag{1.3}$$

Our main motivations are inspired by the following reasons.

Reason 1 Recently, the split problems have been considered by some authors. Especially, the split feasibility problem which can mathematically be formulated as the problem of finding a point $\tilde{x}$ with the property

$$\tilde{x} \in C \quad \text{and} \quad g(\tilde{x}) \in Q$$

has received much attention due to its applications in signal processing and image reconstruction with particular progress in intensity modulated radiation therapy [1–13]. Note that the involved operator g is a bounded linear operator. However, in the present paper, the involved mapping ψ in (1.1) is a nonlinear mapping.

Reason 2 The variational inequality problem [14–24] and equilibrium problem [23–27], which include the fixed point problems and optimization problems [28–30], have been studied by many authors. It is an interesting topic associated with the analytical and algorithmic approach to the variational inequality and equilibrium problems.

Motivated and inspired by the results in the literature, we present an affine algorithm for solving the split problem (1.1). Strong convergence theorem is given under some mild assumptions.

2 Preliminaries

Let H be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\| \cdot \|$, respectively. Let C be a nonempty closed convex subset of H.

2.1 Monotonicity and convexity

An operator $A : C \to H$ is said to be monotone if $\langle x - y, Ax - Ay \rangle \geq 0$ for all $x, y \in C$. $A : C \to H$ is said to be strongly monotone if there exists a constant $\gamma > 0$ such that $\langle x - y, Ax - Ay \rangle \geq \gamma \|x - y\|^2$ for all $x, y \in C$. $A : C \to H$ is called an inverse-strongly-monotone operator if there exists $\alpha > 0$ such that $\langle x - y, Ax - Ay \rangle \geq \alpha \|Ax - Ay\|^2$ for all $x, y \in C$. Let $g : C \to C$ be a nonlinear operator. $A : C \to H$ is said to be α-inverse strongly g-monotone iff $\langle g(x) - g(y), Ax - Ay \rangle \geq \alpha \|Ax - Ay\|^2$ for all $x, y \in C$ and for some $\alpha > 0$. Let B be a mapping of H into 2^H. The effective domain of B is denoted by $\mathrm{dom}(B)$, that is, $\mathrm{dom}(B) = \{x \in H : Bx \neq \emptyset\}$. A multi-valued mapping B is said to be a monotone operator on H iff $\langle x - y, u - v \rangle \geq 0$ for all $x, y \in \mathrm{dom}(B)$, $u \in Bx$, and $v \in By$. A monotone operator B on H is said to be maximal iff its graph is not strictly contained in the graph of any other monotone operator on H.

A function $F : H \to R$ is said to be convex if for any $x, y \in H$ and for any $\lambda \in [0, 1]$, $F(\lambda x + (1 - \lambda)y) \leq \lambda F(x) + (1 - \lambda)F(y)$.

2.2 Nonexpansivity and continuity

A mapping $T : C \to C$ is said to be nonexpansive [31–38] if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. We use $\mathrm{Fix}(T)$ to denote the set of fixed points of T. $T : C \to C$ is called a firmly nonexpansive mapping if, for all $x, y \in C$, $\|Tx - Ty\|^2 \leq \langle x - y, Tx - Ty \rangle$. It is known that T is firmly nonexpansive if and only if a mapping $2T - I$ is nonexpansive, where I is the identity mapping on H. $T : C \to H$ is said to be L-Lipschitz continuous if there exists a constant $L > 0$ such that $\|Tx - Ty\| \leq L\|x - y\|$ for all $x, y \in C$. In such a case, T is said to be L-Lipschitz continuous. Given a nonempty, closed convex subset C of H, the mapping that assigns every point $x \in H$ to its unique nearest point in C is called a metric projection onto C and denoted by P_C, that is, $P_C x \in C$ and $\|x - P_C x\| = \inf\{\|x - y\| : y \in C\}$. The metric projection P_C is a typical firmly nonexpansive mapping. The characteristic inequality of the projection is $\langle x - P_C x, y - P_C x \rangle \leq 0$ for all $x \in H$, $y \in C$.

2.3 Equilibrium problem

In this paper, we consider the split problem (1.1). In the sequel, we assume that the solution set S of (1.1) is nonempty.

Problem 2.1 Assume that

(A1) $B : C \to H$ is an α-inverse strongly ψ-monotone mapping;

(A2) $\psi : C \to C$ is a weakly continuous and γ-strongly monotone mapping such that $R(\psi) = C$;
(A3) $F : C \times C \to R$ is a bifunction;
(A4) $A : C \to H$ is a β-inverse-strongly monotone mapping.

Our objective is to

$$\text{find } x^* \in \mathrm{GVI}(B, \psi, C) \text{ such that } \psi\left(x^*\right) \in \mathrm{EP}(F, A),$$

where F satisfies the following conditions:
(F1) $F(x,x) = 0$ for all $x \in C$;
(F2) F is monotone, *i.e.*, $F(x,y) + F(y,x) \leq 0$ for all $x, y \in C$;
(F3) for each $x, y, z \in C$, $\lim_{t\downarrow 0} F(tz + (1-t)x, y) \leq F(x,y)$;
(F4) for each $x \in C$, $y \mapsto F(x,y)$ is convex and lower semicontinuous.

In order to solve Problem 2.1, we need the following useful lemmas.

2.4 Useful lemmas

The following three lemmas are important tools for our main results in the next section. Note that these lemmas are used extensively in the literature.

Lemma 2.2 (Combettes and Hirstoaga's lemma [26]) *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $F : C \times C \to R$ be a bifunction which satisfies conditions* (F1)-(F4). *Let $\lambda > 0$ and $x \in C$. Then there exists $z \in C$ such that*

$$F(z,y) + \frac{1}{\lambda}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C.$$

Further, if $T_\lambda(x) = \{z \in C : F(z,y) + \frac{1}{\lambda}\langle y-z, z-x\rangle \geq 0 \text{ for all } y \in C\}$, then the following hold:
(a) *T_λ is single-valued and T_λ is firmly nonexpansive*;
(b) $\mathrm{EP}(F)$ *is closed and convex and* $\mathrm{EP}(F) = \mathrm{Fix}(T_\lambda)$.

Lemma 2.3 (Suzuki's lemma [39]) *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$ for all $n \geq 0$ and $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then $\lim_{n\to\infty} \|y_n - x_n\| = 0$.*

Lemma 2.4 (Xu's lemma [40]) *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that $a_{n+1} \leq (1-\gamma_n)a_n + \delta_n\gamma_n$, where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that $\sum_{n=1}^\infty \gamma_n = \infty$ and $\limsup_{n\to\infty} \delta_n \leq 0$ (or $\sum_{n=1}^\infty |\delta_n\gamma_n| < \infty$). Then $\lim_{n\to\infty} a_n = 0$.*

3 Algorithms and convergence analysis

In this section, we first present our algorithm for solving Problem 2.1. Assume that the conditions in Problem 2.1 are all satisfied.

Algorithm 3.1 Let C be a nonempty closed and convex subset of a real Hilbert space H.
Step 0. (Initialization)

$$x_0 \in C.$$

Step 1. (Projection step) For $\{x_n\}$, let the sequence $\{u_n\}$ be generated iteratively by

$$u_n = P_C\big[\alpha_n \delta\varphi(x_n) + (1-\alpha_n)\big(\psi(x_n) - \mu_n Bx_n\big)\big], \quad n \geq 0,$$

where P_C is the metric projection, $\{\alpha_n\} \subset [0.1]$ is a real number sequence, $\varphi : C \to H$ is an L-Lipschitz continuous mapping and $\delta > 0$ is a constant.

Step 2. (Proximal step) Find $\{z_n\}$ such that

$$F(z_n, y) + \langle Au_n, y - z_n\rangle + \frac{1}{\lambda_n}\langle y - z_n, z_n - u_n\rangle \geq 0, \quad \forall y \in C,$$

where $\{\lambda_n\} \subset (0,\infty)$ is a real number sequence.

Step 3. (Affine step) For the above sequences $\{x_n\}$ and $\{z_n\}$, let the $(n+1)$th sequence $\{x_{n+1}\}$ be generated by

$$\psi(x_{n+1}) = \beta_n \psi(x_n) + (1-\beta_n)z_n, \quad n \geq 0,$$

where $\{\beta_n\} \subset [0,1]$ is a real number sequence.

Theorem 3.2 *Suppose $S \neq \emptyset$. Assume that the following restrictions are satisfied*:

(C1) $\lambda_n \in (a,b) \subset (0,2\beta)$, $\mu_n \in (c,d) \subset (0,2\alpha)$ *and* $\gamma \in (L\delta, 2\alpha)$;

(C2) $\lim_{n\to\infty}(\mu_{n+1} - \mu_n) = 0$ *and* $\lim_{n\to\infty}(\lambda_{n+1} - \lambda_n) = 0$;

(C3) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_n \alpha_n = \infty$;

(C4) $\beta_n \in [\xi_1, \xi_2] \subset (0,1)$.

Then the sequence $\{x_n\}$ generated by Algorithm 3.1 *converges strongly to $x^* \in S$, which solves the following variational inequality*:

$$\big\langle \delta\varphi(x^*) - \psi(x^*), \psi(x) - \psi(x^*)\big\rangle \leq 0, \quad \forall x \in \Omega. \tag{3.1}$$

Remark 3.3 The solution of variational inequality (3.1) is unique. As a matter of fact, if $\tilde{x} \in S$ also solves (3.1), we have

$$\big\langle \delta\varphi(x^*) - \psi(x^*), \psi(\tilde{x}) - \psi(x^*)\big\rangle \leq 0 \quad \text{and} \quad \big\langle \delta\varphi(\tilde{x}) - \psi(\tilde{x}), \psi(x^*) - \psi(\tilde{x})\big\rangle \leq 0.$$

Adding up the above two inequalities, we deduce

$$\big\langle \delta\varphi(\tilde{x}) - \psi(\tilde{x}) - \delta\varphi(x^*) + \psi(x^*), \psi(x^*) - \psi(\tilde{x})\big\rangle \leq 0.$$

It follows that

$$\begin{aligned} \|\psi(x^*) - \psi(\tilde{x})\|^2 &\leq \delta\big\langle \varphi(x^*) - \varphi(\tilde{x}), \psi(x^*) - \psi(\tilde{x})\big\rangle \\ &\leq \delta\|\varphi(x^*) - \varphi(\tilde{x})\|\|\psi(x^*) - \psi(\tilde{x})\|, \end{aligned}$$

which implies that

$$\|\psi(x^*) - \psi(\tilde{x})\| \leq \delta\|\varphi(x^*) - \varphi(\tilde{x})\|.$$

Since ψ is γ-strongly monotone, we have

$$\gamma \|x^* - \tilde{x}\|^2 \le \langle \psi(x^*) - \psi(\tilde{x}), x^* - \tilde{x}\rangle \le \|\psi(x^*) - \psi(\tilde{x})\| \|x^* - \tilde{x}\|.$$

Hence,

$$\gamma \|x^* - \tilde{x}\| \le \|\psi(x^*) - \psi(\tilde{x})\| \le \delta \|\varphi(x^*) - \varphi(\tilde{x})\| \le \delta L \|x^* - \tilde{x}\|.$$

This deduces the contraction because of $\delta L < \gamma$ by the assumption. Therefore, $x^* = \tilde{x}$. So, the solution of variational inequality (3.1) is unique.

Remark 3.4 Using the characteristic inequality of the projection, we have

$$\breve{x} \in \mathrm{GVI}(B, \psi, C) \iff \psi(\breve{x}) = P_C(\psi(\breve{x}) - \nu B\breve{x}), \quad \forall \nu > 0.$$

Remark 3.5

$$\|(\psi(x) - \mu Bx) - (\psi(y) - \mu By)\|^2 \le \|\psi(x) - \psi(y)\|^2 + \mu(\mu - 2\alpha)\|Bx - By\|^2.$$

In fact,

$$\begin{aligned}
&\|(\psi(x) - \mu Bx) - (\psi(y) - \mu By)\|^2 \\
&\quad = \|\psi(x) - \psi(y)\|^2 - 2\mu\langle Bx - By, \psi(x) - \psi(y)\rangle + \mu^2\|Bx - By\|^2 \\
&\quad \le \|\psi(x) - \psi(y)\|^2 - 2\mu\alpha\|Bx - By\|^2 + \mu^2\|Bx - By\|^2 \\
&\quad \le \|\psi(x) - \psi(y)\|^2 + \mu(\mu - 2\alpha)\|Bx - By\|^2.
\end{aligned}$$

Next, we prove Theorem 3.2.

Proof Let $x^\ddagger \in \Omega$. Hence $x^\ddagger \in \mathrm{GVI}(B, \psi, C)$ and $\psi(x^\ddagger) \in \mathrm{EP}(F, A)$. Since $\mu_n > 0$, from Remark 3.4 we have $\psi(x^\ddagger) = P_C[\psi(x^\ddagger) - \mu_n Bx^\ddagger]$ for all $n \ge 0$. Thus,

$$\begin{aligned}
\|u_n - \psi(x^\ddagger)\| &= \|P_C[\alpha_n \delta\varphi(x_n) + (1 - \alpha_n)(\psi(x_n) - \mu_n Bx_n)] - P_C[\psi(x^\ddagger) - \mu_n Bx^\ddagger]\| \\
&\le \|\alpha_n(\delta\varphi(x_n) - \psi(x^\ddagger) + \mu_n Bx^\ddagger) \\
&\quad + (1 - \alpha_n)((\psi(x_n) - \mu_n Bx_n) - (\psi(x^\ddagger) - \mu_n Bx^\ddagger))\| \\
&\le \alpha_n\|\delta\varphi(x_n) - \delta\varphi(x^\ddagger)\| + \alpha_n\|\delta\varphi(x^\ddagger) - \psi(x^\ddagger) + \mu_n Bx^\ddagger\| \\
&\quad + (1 - \alpha_n)\|(\psi(x_n) - \mu_n Bx_n) - (\psi(x^\ddagger) - \mu_n Bx^\ddagger)\| \\
&\le \alpha_n \delta L\|x_n - x^\ddagger\| + \alpha_n\|\delta\varphi(x^\ddagger) - \psi(x^\ddagger) + \mu_n Bx^\ddagger\| \\
&\quad + (1 - \alpha_n)\|\psi(x_n) - \psi(x^\ddagger)\| \\
&\le \alpha_n \delta L/\gamma\|\psi(x_n) - \psi(x^\ddagger)\| + \alpha_n\|\delta\varphi(x^\ddagger) - \psi(x^\ddagger) + \mu_n Bx^\ddagger\| \\
&\quad + (1 - \alpha_n)\|\psi(x_n) - \psi(x^\ddagger)\| \\
&= [1 - (1 - \delta L/\gamma)\alpha_n]\|\psi(x_n) - \psi(x^\ddagger)\| + \alpha_n\|\delta\varphi(x^\ddagger) - \psi(x^\ddagger) + \mu_n Bx^\ddagger\|
\end{aligned}$$

$$\begin{aligned}&\le \bigl[1-(1-\delta L/\gamma)\alpha_n\bigr]\bigl\|\psi(x_n)-\psi\bigl(x^\ddagger\bigr)\bigr\| \\ &\quad + \alpha_n\bigl(\bigl\|\delta\varphi\bigl(x^\ddagger\bigr)-\psi\bigl(x^\ddagger\bigr)\bigr\| + 2\alpha\bigl\|Bx^\ddagger\bigr\|\bigr). \end{aligned} \tag{3.2}$$

By Algorithm 3.1, we have $z_n = T_{\lambda_n}(I-\lambda_n A)u_n$ for all $n \ge 0$. Noting that $\psi(x^\ddagger) \in \mathrm{EP}(F,A)$, we deduce $\psi(x^\ddagger) = T_{\lambda_n}(I-\lambda_n A)\psi(x^\ddagger)$ for all $n \ge 0$. It follows that

$$\begin{aligned}\bigl\|\psi(x_{n+1})-\psi\bigl(x^\ddagger\bigr)\bigr\| &\le \beta_n\bigl\|\psi(x_n)-\psi\bigl(x^\ddagger\bigr)\bigr\| \\ &\quad + (1-\beta_n)\bigl\|T_{\lambda_n}(I-\lambda_n A)u_n - T_{\lambda_n}(I-\lambda_n A)\psi\bigl(x^\ddagger\bigr)\bigr\| \\ &\le \beta_n\bigl\|\psi(x_n)-\psi\bigl(x^\ddagger\bigr)\bigr\| + (1-\beta_n)\bigl\|u_n-\psi\bigl(x^\ddagger\bigr)\bigr\| \\ &\le \beta_n\bigl\|\psi(x_n)-\psi\bigl(x^\ddagger\bigr)\bigr\| \\ &\quad + (1-\beta_n)\bigl[1-(1-\delta L/\gamma)\alpha_n\bigr]\bigl\|\psi(x_n)-\psi\bigl(x^\ddagger\bigr)\bigr\| \\ &\quad + (1-\beta_n)\alpha_n\bigl(\bigl\|\delta\varphi\bigl(x^\ddagger\bigr)-\psi\bigl(x^\ddagger\bigr)\bigr\| + 2\alpha\bigl\|Bx^\ddagger\bigr\|\bigr) \\ &= \bigl[1-(1-\delta L/\gamma)(1-\beta_n)\alpha_n\bigr]\bigl\|\psi(x_n)-\psi\bigl(x^\ddagger\bigr)\bigr\| \\ &\quad + (1-\delta L/\gamma)(1-\beta_n)\alpha_n\frac{\|\delta\varphi(x^\ddagger)-\psi(x^\ddagger)\| + 2\alpha\|Bx^\ddagger\|}{1-\delta L/\gamma}. \end{aligned}$$

By induction

$$\bigl\|\psi(x_n)-\psi\bigl(x^\ddagger\bigr)\bigr\| \le \max\left\{\bigl\|\psi(x_0)-\psi\bigl(x^\ddagger\bigr)\bigr\|, \frac{\|\delta\varphi(x^\ddagger)-\psi(x^\ddagger)\| + 2\alpha\|Bx^\ddagger\|}{1-\delta L/\gamma}\right\}.$$

Hence, $\{\psi(x_n)\}$ is bounded. Since ψ is γ-strongly monotone, we can get (by a similar technique as that in Remark 3.3) $\gamma\|x_n - x^\ddagger\| \le \|\psi(x_n)-\psi(x^\ddagger)\|$. So, $\|x_n - x^\ddagger\| \le \frac{1}{\gamma}\|\psi(x_n)-\psi(x^\ddagger)\| \le \frac{1}{\gamma}\max\{\|\psi(x_0)-\psi(x^\ddagger)\|, \frac{\|\delta\varphi(x^\ddagger)-\psi(x^\ddagger)\|+2\alpha\|Bx^\ddagger\|}{1-\delta L/\gamma}\}$. This implies that $\{x_n\}$ is bounded. Next, we show $\|x_{n+1}-x_n\| \to 0$. From Step 2 in Algorithm 3.1, we have

$$F(z_n, y) + \frac{1}{\lambda_n}\bigl\langle y - z_n, z_n - (u_n - \lambda_n A u_n)\bigr\rangle \ge 0, \quad \forall y \in C.$$

Taking $y = z_{n+1}$, we get

$$F(z_n, z_{n+1}) + \frac{1}{\lambda_n}\bigl\langle z_{n+1} - z_n, z_n - (u_n - \lambda_n A u_n)\bigr\rangle \ge 0.$$

Similarly, we also have

$$F(z_{n+1}, z_n) + \frac{1}{\lambda_{n+1}}\bigl\langle z_n - z_{n+1}, z_{n+1} - (u_{n+1} - \lambda_{n+1} A u_{n+1})\bigr\rangle \ge 0.$$

Adding up the above two inequalities, we get

$$F(z_n, z_{n+1}) + F(z_{n+1}, z_n) + \langle Au_n - Au_{n+1}, z_{n+1} - z_n\rangle + \left\langle z_{n+1} - z_n, \frac{z_n - u_n}{\lambda_n} - \frac{z_{n+1} - u_{n+1}}{\lambda_{n+1}}\right\rangle \ge 0.$$

By the monotonicity of F, we have

$$F(z_n, z_{n+1}) + F(z_{n+1}, z_n) \le 0.$$

So,

$$\langle Au_n - Au_{n+1}, z_{n+1} - z_n\rangle + \left\langle z_{n+1} - z_n, \frac{z_n - u_n}{\lambda_n} - \frac{z_{n+1} - u_{n+1}}{\lambda_{n+1}}\right\rangle \geq 0.$$

Thus,

$$\lambda_n\langle Au_n - Au_{n+1}, z_{n+1} - z_n\rangle + \left\langle z_{n+1} - z_n, z_n - z_{n+1} + z_{n+1} - u_n - \frac{\lambda_n}{\lambda_{n+1}}(z_{n+1} - u_{n+1})\right\rangle \geq 0.$$

It follows that

$$\begin{aligned}
\|z_{n+1} - z_n\|^2 &\leq \lambda_n\langle Au_n - Au_{n+1}, z_{n+1} - z_n\rangle \\
&\quad + \left\langle z_{n+1} - z_n, u_{n+1} - u_n + \left(1 - \frac{\lambda_n}{\lambda_{n+1}}\right)(z_{n+1} - u_{n+1})\right\rangle \\
&= \big\langle (I - \lambda_n A)u_{n+1} - (I - \lambda_n A)u_n, z_{n+1} - z_n\big\rangle \\
&\quad + \left\langle z_{n+1} - z_n, \left(1 - \frac{\lambda_n}{\lambda_{n+1}}\right)(z_{n+1} - u_{n+1})\right\rangle \\
&\leq \big\|(I - \lambda_n A)u_{n+1} - (I - \lambda_n A)u_n\big\|\|z_{n+1} - z_n\| \\
&\quad + \left|1 - \frac{\lambda_n}{\lambda_{n+1}}\right|\|z_{n+1} - z_n\|\|z_{n+1} - u_{n+1}\| \\
&\leq \|z_{n+1} - z_n\|\left(\|u_{n+1} - u_n\| + \left|1 - \frac{\lambda_n}{\lambda_{n+1}}\right|\|z_{n+1} - u_{n+1}\|\right)
\end{aligned}$$

and hence

$$\begin{aligned}
\|z_{n+1} - z_n\| &\leq \|u_{n+1} - u_n\| + \left|1 - \frac{\lambda_n}{\lambda_{n+1}}\right|\|z_{n+1} - u_{n+1}\| \\
&\leq \|u_{n+1} - u_n\| + \frac{1}{a}|\lambda_{n+1} - \lambda_n|\|z_{n+1} - u_{n+1}\|.
\end{aligned}$$

By Algorithm 3.1, we have

$$\begin{aligned}
\|u_{n+1} - u_n\| &= \big\|P_C\big[\alpha_{n+1}\delta\varphi(x_{n+1}) + (1 - \alpha_{n+1})\big(\psi(x_{n+1}) - \mu_{n+1}Bx_{n+1}\big)\big] \\
&\quad - P_C\big[\alpha_n\delta\varphi(x_n) + (1 - \alpha_n)\big(\psi(x_n) - \mu_n Bx_n\big)\big]\big\| \\
&\leq \big\|\big[\alpha_{n+1}\delta\varphi(x_{n+1}) + (1 - \alpha_{n+1})\big(\psi(x_{n+1}) - \mu_{n+1}Bx_{n+1}\big)\big] \\
&\quad - \big[\alpha_n\delta\varphi(x_n) + (1 - \alpha_n)\big(\psi(x_n) - \mu_n Bx_n\big)\big]\big\| \\
&\leq \alpha_{n+1}\delta\big\|\varphi(x_{n+1}) - \varphi(x_n)\big\| + \delta|\alpha_{n+1} - \alpha_n|\big\|\varphi(x_n)\big\| \\
&\quad + (1 - \alpha_{n+1})\big\|\psi(x_{n+1}) - \mu_{n+1}Bx_{n+1} - \big(\psi(x_n) - \mu_{n+1}Bx_n\big)\big\| \\
&\quad + |\alpha_{n+1} - \alpha_n|\big\|\psi(x_n)\big\| + |\mu_{n+1} - \mu_n|\big\|B(x_n)\big\| + |\alpha_{n+1}\mu_{n+1} - \alpha_n\mu_n|\big\|B(x_n)\big\| \\
&\leq \alpha_{n+1}\delta L\|x_{n+1} - x_n\| + (1 - \alpha_{n+1})\big\|\psi(x_{n+1}) - \psi(x_n)\big\| \\
&\quad + |\alpha_{n+1} - \alpha_n|\big(\delta\big\|\varphi(x_n)\big\| + \big\|\psi(x_n)\big\|\big) + |\mu_{n+1} - \mu_n|\big\|B(x_n)\big\| \\
&\quad + |\alpha_{n+1}\mu_{n+1} - \alpha_n\mu_n|\big\|B(x_n)\big\|
\end{aligned}$$

$$\begin{aligned}
&\le \alpha_{n+1}(\delta L/\gamma)\big\|\psi(x_{n+1})-\psi(x_n)\big\| + (1-\alpha_{n+1})\big\|\psi(x_{n+1})-\psi(x_n)\big\| \\
&\quad + |\alpha_{n+1}-\alpha_n|\big(\delta\big\|\varphi(x_n)\big\| + \big\|\psi(x_n)\big\|\big) + |\mu_{n+1}-\mu_n|\big\|B(x_n)\big\| \\
&\quad + |\alpha_{n+1}\mu_{n+1}-\alpha_n\mu_n|\big\|B(x_n)\big\| \\
&= \big[1-(1-\delta L/\gamma)\alpha_{n+1}\big]\big\|\psi(x_{n+1})-\psi(x_n)\big\| \\
&\quad + |\alpha_{n+1}-\alpha_n|\big(\delta\big\|\varphi(x_n)\big\| + \big\|\psi(x_n)\big\|\big) \\
&\quad + |\mu_{n+1}-\mu_n|\big\|B(x_n)\big\| + |\alpha_{n+1}\mu_{n+1}-\alpha_n\mu_n|\big\|B(x_n)\big\|.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|z_{n+1}-z_n\| &\le \big[1-(1-\delta L/\gamma)\alpha_{n+1}\big]\big\|\psi(x_{n+1})-\psi(x_n)\big\| \\
&\quad + |\alpha_{n+1}-\alpha_n|\big(\delta\big\|\varphi(x_n)\big\| + \big\|\psi(x_n)\big\|\big) \\
&\quad + |\mu_{n+1}-\mu_n|\big\|B(x_n)\big\| + |\alpha_{n+1}\mu_{n+1}-\alpha_n\mu_n|\big\|B(x_n)\big\| \\
&\quad + \frac{1}{a}|\lambda_{n+1}-\lambda_n|\|z_{n+1}-u_{n+1}\|.
\end{aligned}$$

It follows that

$$\begin{aligned}
\|z_{n+1}-z_n\| - \big\|\psi(x_{n+1})-\psi(x_n)\big\| &\le |\alpha_{n+1}-\alpha_n|\big(\delta\big\|\varphi(x_n)\big\| + \big\|\psi(x_n)\big\|\big) \\
&\quad + |\mu_{n+1}-\mu_n|\big\|B(x_n)\big\| + |\alpha_{n+1}\mu_{n+1}-\alpha_n\mu_n|\big\|B(x_n)\big\| \\
&\quad + \frac{1}{a}|\lambda_{n+1}-\lambda_n|\|z_{n+1}-u_{n+1}\|.
\end{aligned}$$

Since $\lim_{n\to\infty}\alpha_n = 0$, $\lim_{n\to\infty}(\mu_{n+1}-\mu_n) = 0$, $\lim_{n\to\infty}(\lambda_{n+1}-\lambda_n) = 0$ and the sequences $\{\varphi(x_n)\}$, $\{\psi(x_n)\}$, $\{z_n\}$, $\{u_n\}$ and $\{Bx_n\}$ are bounded, we have

$$\limsup_{n\to\infty}\big(\|z_{n+1}-z_n\| - \big\|\psi(x_{n+1})-\psi(x_n)\big\|\big) \le 0.$$

By Lemma 2.3, we obtain

$$\lim_{n\to\infty}\big\|z_n-\psi(x_n)\big\| = 0.$$

Hence,

$$\lim_{n\to\infty}\big\|\psi(x_{n+1})-\psi(x_n)\big\| = \lim_{n\to\infty}(1-\beta_n)\big\|z_n-\psi(x_n)\big\| = 0.$$

This together with the γ-strong monotonicity of ψ implies that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0.$$

By the convexity of the norm, we have

$$\begin{aligned}
&\big\|\psi(x_{n+1})-\psi\big(x^{\ddagger}\big)\big\|^2 \\
&\quad = \big\|\beta_n\big(\psi(x_n)-\psi\big(x^{\ddagger}\big)\big) + (1-\beta_n)\big(z_n-\psi\big(x^{\ddagger}\big)\big)\big\|^2
\end{aligned}$$

$$\begin{aligned}
&\le \beta_n \|\psi(x_n) - \psi(x^\ddagger)\|^2 + (1-\beta_n)\|z_n - \psi(x^\ddagger)\|^2 \\
&\le \beta_n \|\psi(x_n) - \psi(x^\ddagger)\|^2 + (1-\beta_n)\big[\|\alpha_n(\delta\varphi(x_n) - \psi(x^\ddagger) + \mu_n Bx^\ddagger) \\
&\quad + (1-\alpha_n)\big((\psi(x_n) - \mu_n Bx_n) - (\psi(x^\ddagger) - \mu_n Bx^\ddagger)\big)\|\big]^2 \\
&\le \beta_n \|\psi(x_n) - \psi(x^\ddagger)\|^2 + (1-\beta_n)\big[\alpha_n\|\delta\varphi(x_n) - \psi(x^\ddagger) + \mu_n Bx^\ddagger\|^2 \\
&\quad + (1-\alpha_n)\|(\psi(x_n) - \mu_n Bx_n) - (\psi(x^\ddagger) - \mu_n Bx^\ddagger)\|^2 \\
&\quad + 2\alpha_n(1-\alpha_n)\|\delta\varphi(x_n) - \psi(x^\ddagger) + \mu_n Bx^\ddagger\|\|(\psi(x_n) - \mu_n Bx_n) - (\psi(x^\ddagger) - \mu_n Bx^\ddagger)\|\big] \\
&\le \beta_n \|\psi(x_n) - \psi(x^\ddagger)\|^2 \\
&\quad + (1-\beta_n)(1-\alpha_n)\|(\psi(x_n) - \mu_n Bx_n) - (\psi(x^\ddagger) - \mu_n Bx^\ddagger)\|^2 + \alpha_n M,
\end{aligned} \tag{3.3}$$

where $M > 0$ is some constant. From Remark 3.5, we derive

$$\|(\psi(x_n) - \mu_n Bx_n) - (\psi(x^\ddagger) - \mu_n Bx^\ddagger)\|^2 \le \|\psi(x_n) - \psi(x^\ddagger)\|^2 + \mu_n(\mu_n - 2\alpha)\|Bx_n - Bx^\ddagger\|^2.$$

Thus,

$$\begin{aligned}
\|\psi(x_{n+1}) - \psi(x^\ddagger)\|^2 &\le \beta_n\|\psi(x_n) - \psi(x^\ddagger)\|^2 + (1-\beta_n)(1-\alpha_n)\big(\|\psi(x_n) - \psi(x^\ddagger)\|^2 \\
&\quad + \mu_n(\mu_n - 2\alpha)\|Bx_n - Bx^\ddagger\|^2\big) + \alpha_n M \\
&\le \|\psi(x_n) - \psi(x^\ddagger)\|^2 \\
&\quad + (1-\beta_n)(1-\alpha_n)\mu_n(\mu_n - 2\alpha)\|Bx_n - Bx^\ddagger\|^2 + \alpha_n M.
\end{aligned}$$

So,

$$\begin{aligned}
&(1-\beta_n)(1-\alpha_n)\mu_n(2\alpha - \mu_n)\|Bx_n - Bx^\ddagger\|^2 \\
&\quad \le \|\psi(x_n) - \psi(x^\ddagger)\|^2 - \|\psi(x_{n+1}) - \psi(x^\ddagger)\|^2 + \alpha_n M \\
&\quad \le \big(\|\psi(x_n) - \psi(x^\ddagger)\| + \|\psi(x_{n+1}) - \psi(x^\ddagger)\|\big)\|\psi(x_{n+1}) - \psi(x_n)\| + \alpha_n M.
\end{aligned}$$

Since $\alpha_n \to 0$, $\|\psi(x_{n+1}) - \psi(x_n)\| \to 0$ and $\liminf_{n\to\infty}(1-\beta_n)(1-\alpha_n)\mu_n(2\alpha - \mu_n) > 0$, we obtain

$$\lim_{n\to\infty}\|Bx_n - Bx^\ddagger\| = 0.$$

Set $y_n = \psi(x_n) - \mu_n Bx_n - (\psi(x^\ddagger) - \mu_n Bx^\ddagger)$ for all n. By using the property of projection, we get

$$\begin{aligned}
\|u_n - \psi(x^\ddagger)\|^2 &= \|P_C[\alpha_n\delta\varphi(x_n) + (1-\alpha_n)(\psi(x_n) - \mu_n Bx_n)] - P_C[\psi(x^\ddagger) - \mu_n Bx^\ddagger]\|^2 \\
&\le \langle \alpha_n(\delta\varphi(x_n) - \psi(x^\ddagger) + \mu_n Bx^\ddagger) + (1-\alpha_n)y_n, u_n - \psi(x^\ddagger)\rangle \\
&= \frac{1}{2}\big\{\|\alpha_n(\delta\varphi(x_n) - \psi(x^\ddagger) + \mu_n Bx^\ddagger) + (1-\alpha_n)y_n\|^2 + \|u_n - \psi(x^\ddagger)\|^2 \\
&\quad - \|\alpha_n(\delta\varphi(x_n) - \psi(x^\ddagger) + \mu_n Bx^\ddagger) + (1-\alpha_n)y_n - u_n + \psi(x^\ddagger)\|^2\big\} \\
&\le \frac{1}{2}\big\{\alpha_n\|\delta\varphi(x_n) - \psi(x^\ddagger) + \mu_n Bx^\ddagger\|^2
\end{aligned}$$

$$\begin{aligned}
&+ (1-\alpha_n)\|\psi(x_n) - \psi(x^\ddagger)\|^2 + \|u_n - \psi(x^\ddagger)\|^2 \\
&- \|\alpha_n(\delta\varphi(x_n) - \psi(x^\ddagger) + \mu_n Bx^\ddagger - y_n) \\
&+ \psi(x_n) - u_n - \mu_n(Bx_n - Bx^\ddagger)\|^2\} \\
= {}& \frac{1}{2}\{\alpha_n\|\delta\varphi(x_n) - \psi(x^\ddagger) + \mu_n Bx^\ddagger\|^2 \\
&+ (1-\alpha_n)\|\psi(x_n) - \psi(x^\ddagger)\|^2 + \|u_n - \psi(x^\ddagger)\|^2 \\
&- \|\psi(x_n) - u_n\|^2 - \mu_n^2\|Bx_n - Bx^\ddagger\| \\
&- \alpha_n^2\|\delta\varphi(x_n) - \psi(x^\ddagger) + \mu_n Bx^\ddagger - y_n\|^2 \\
&+ 2\mu_n\alpha_n\langle Bx_n - Bx^\ddagger, \delta\varphi(x_n) - \psi(x^\ddagger) + \mu_n Bx^\ddagger - y_n\rangle \\
&+ 2\mu_n\langle\psi(x_n) - u_n, Bx_n - Bx^\ddagger\rangle \\
&- 2\alpha_n\langle\psi(x_n) - u_n, \delta\varphi(x_n) - \psi(x^\ddagger) + \mu_n Bx^\ddagger - y_n\rangle\}.
\end{aligned} \tag{3.4}$$

It follows that

$$\begin{aligned}
\|u_n - \psi(x^\ddagger)\|^2 \le {}& \alpha_n\|\delta\varphi(x_n) - \psi(x^\ddagger) + \mu_n Bx^\ddagger\|^2 \\
&+ (1-\alpha_n)\|\psi(x_n) - \psi(x^\ddagger)\|^2 - \|\psi(x_n) - u_n\|^2 \\
&+ 2\mu_n\alpha_n\|Bx_n - Bx^\ddagger\|\|\delta\varphi(x_n) - \psi(x^\ddagger) + \mu_n Bx^\ddagger - y_n\| \\
&+ 2\mu_n\|\psi(x_n) - u_n\|\|Bx_n - Bx^\ddagger\| \\
&+ 2\alpha_n\|\psi(x_n) - u_n\|\|\delta\varphi(x_n) - \psi(x^\ddagger) + \mu_n Bx^\ddagger - y_n\|.
\end{aligned} \tag{3.5}$$

From (3.3) and (3.5), we have

$$\begin{aligned}
&\|\psi(x_{n+1}) - \psi(x^\ddagger)\|^2 \\
&\quad\le \beta_n\|\psi(x_n) - \psi(x^\ddagger)\|^2 + (1-\beta_n)\|u_n - \psi(x^\ddagger)\|^2 \\
&\quad\le \beta_n\|\psi(x_n) - \psi(x^\ddagger)\|^2 + (1-\beta_n)\alpha_n\|\delta\varphi(x_n) - \psi(x^\ddagger) + \mu_n Bx^\ddagger\|^2 \\
&\qquad+ (1-\alpha_n)(1-\beta_n)\|\psi(x_n) - \psi(x^\ddagger)\|^2 - (1-\beta_n)\|\psi(x_n) - u_n\|^2 \\
&\qquad+ 2\mu_n(1-\beta_n)\alpha_n\|Bx_n - Bx^\ddagger\|\|\delta\varphi(x_n) - \psi(x^\ddagger) + \mu_n Bx^\ddagger - y_n\| \\
&\qquad+ 2\mu_n(1-\beta_n)\|\psi(x_n) - u_n\|\|Bx_n - Bx^\ddagger\| \\
&\qquad+ 2(1-\beta_n)\alpha_n\|\psi(x_n) - u_n\|\|\delta\varphi(x_n) - \psi(x^\ddagger) + \mu_n Bx^\ddagger - y_n\| \\
&\quad\le \|\psi(x_n) - \psi(x^\ddagger)\|^2 + \alpha_n\|\delta\varphi(x_n) - \psi(x^\ddagger) + \mu_n Bx^\ddagger\|^2 \\
&\qquad- (1-\beta_n)\|\psi(x_n) - u_n\|^2 \\
&\qquad+ 2\mu_n\alpha_n\|Bx_n - Bx^\ddagger\|\|\delta\varphi(x_n) - \psi(x^\ddagger) + \mu_n Bx^\ddagger - y_n\| \\
&\qquad+ 2\mu_n\|\psi(x_n) - u_n\|\|Bx_n - Bx^\ddagger\| \\
&\qquad+ 2\alpha_n\|\psi(x_n) - u_n\|\|\delta\varphi(x_n) - \psi(x^\ddagger) + \mu_n Bx^\ddagger - y_n\|.
\end{aligned}$$

Then we obtain

$$\begin{aligned}(1-\beta_n)\|\psi(x_n)-u_n\|^2 \leq{} & \left(\|\psi(x_n)-\psi(x^\ddagger)\| + \|\psi(x_{n+1})-\psi(x^\ddagger)\|\right)\|\psi(x_{n+1})-\psi(x_n)\| \\ & + \alpha_n\|\delta\varphi(x_n)-\psi(x^\ddagger)+\mu_n Bx^\ddagger\|^2 \\ & + 2\mu_n\alpha_n\|Bx_n-Bx^\ddagger\|\|\delta\varphi(x_n)-\psi(x^\ddagger)+\mu_n Bx^\ddagger-y_n\| \\ & + 2\mu_n\|\psi(x_n)-u_n\|\|Bx_n-Bx^\ddagger\| \\ & + 2\alpha_n\|\psi(x_n)-u_n\|\|\delta\varphi(x_n)-\psi(x^\ddagger)+\mu_n Bx^\ddagger-y_n\|.\end{aligned}$$

Since $\lim_{n\to\infty}\alpha_n = 0$, $\lim_{n\to\infty}\|\psi(x_{n+1})-\psi(x_n)\| = 0$ and $\lim_{n\to\infty}\|Bx_n - Bx^\ddagger\| = 0$, we have

$$\lim_{n\to\infty}\|\psi(x_n)-u_n\| = 0. \tag{3.6}$$

Next, we prove $\limsup_{n\to\infty}\langle\delta\varphi(x^*)-\psi(x^*), u_n-\psi(x^*)\rangle \leq 0$, where x^* is the unique solution of (3.1). We take a subsequence $\{u_{n_i}\}$ of $\{u_n\}$ such that

$$\begin{aligned}&\limsup_{n\to\infty}\langle\delta\varphi(x^*)-\psi(x^*), u_n-\psi(x^*)\rangle \\ &\quad= \lim_{i\to\infty}\langle\delta\varphi(x^*)-\psi(x^*), u_{n_i}-\psi(x^*)\rangle \\ &\quad= \lim_{i\to\infty}\langle\delta\varphi(x^*)-\psi(x^*), \psi(x_{n_i})-\psi(x^*)\rangle.\end{aligned} \tag{3.7}$$

Since $\{x_{n_i}\}$ is bounded, there exists a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ which converges weakly to some point $z \in C$. Without loss of generality, we may assume that $x_{n_i} \rightharpoonup z$. This implies that $\psi(x_{n_i}) \rightharpoonup \psi(z)$ due to the weak continuity of ψ. Now, we show $z \in S$. We firstly show $z \in \mathrm{EP}(F, A)$. Since $z_n = T_{\lambda_n}(u_n - \lambda_n Au_n)$, for any $y \in C$, we have

$$F(z_n, y) + \frac{1}{\lambda_n}\langle y - z_n, z_n - (u_n - \lambda_n Au_n)\rangle \geq 0.$$

From the monotonicity of F, we have

$$\frac{1}{\lambda_n}\langle y - z_n, z_n - (u_n - \lambda_n Au_n)\rangle \geq F(y, z_n), \quad \forall y \in C.$$

Hence,

$$\left\langle y - z_{n_i}, \frac{z_{n_i}-u_{n_i}}{\lambda_{n_i}} + Au_{n_i}\right\rangle \geq F(y, z_{n_i}), \quad \forall y \in C. \tag{3.8}$$

Put $v_t = ty + (1-t)z$ for all $t \in (0,1]$ and $y \in C$. Then we have $v_t \in C$. So, from (3.8) we have

$$\begin{aligned}\langle v_t - z_{n_i}, Av_t\rangle &\geq \langle v_t - z_{n_i}, Av_t\rangle - \left\langle v_t - z_{n_i}, \frac{z_{n_i}-u_{n_i}}{\lambda_{n_i}} + Au_{n_i}\right\rangle + F(v_t, z_{n_i}) \\ &= \langle v_t - z_{n_i}, Av_t - Az_{n_i}\rangle + \langle v_t - z_{n_i}, Az_{n_i} - Au_{n_i}\rangle \\ &\quad - \left\langle v_t - z_{n_i}, \frac{z_{n_i}-u_{n_i}}{\lambda_{n_i}}\right\rangle + F(v_t, z_{n_i}).\end{aligned} \tag{3.9}$$

Note that $\|Az_{n_i} - Au_{n_i}\| \le \frac{1}{\beta}\|z_{n_i} - u_{n_i}\| \to 0$. Further, from the monotonicity of A, we have $\langle v_t - z_{n_i}, Av_t - Az_{n_i}\rangle \ge 0$. Letting $i \to \infty$ in (3.9), we have $\langle v_t - z, Av_t\rangle \ge F(v_t, z)$. This together with (F1), (F4) implies that

$$\begin{aligned} 0 &= F(v_t, v_t) \le tF(v_t, y) + (1-t)F(v_t, z) \\ &\le tF(v_t, y) + (1-t)\langle v_t - z, Av_t\rangle \\ &= tF(v_t, y) + (1-t)t\langle y - z, Av_t\rangle, \end{aligned}$$

and hence $0 \le F(v_t, y) + (1-t)\langle Av_t, y - z\rangle$. Letting $t \to 0$, we have $0 \le F(z, y) + \langle y - z, Az\rangle$. This implies that $z \in \mathrm{EP}(F, A)$. Next, we only need to prove $z \in \mathrm{GVI}(B, \psi, C)$. Set

$$Rv = \begin{cases} Bv + N_C(v), & v \in C, \\ \emptyset, & v \notin C. \end{cases}$$

By [41], we know that R is maximal ψ-monotone. Let $(v, w) \in G(R)$. Since $w - Bv \in N_C(v)$ and $x_n \in C$, we have $\langle \psi(v) - \psi(x_n), w - Bv\rangle \ge 0$. Noting that $u_n = P_C[\alpha_n \delta\varphi(x_n) + (1 - \alpha_n)(\psi(x_n) - \mu_n Bx_n)]$, we get

$$\bigl\langle \psi(v) - u_n, u_n - \bigl[\alpha_n\delta\varphi(x_n) + (1-\alpha_n)\bigl(\psi(x_n) - \mu_n Bx_n\bigr)\bigr]\bigr\rangle \ge 0.$$

It follows that

$$\left\langle \psi(v) - u_n, \frac{u_n - \psi(x_n)}{\mu_n} + Bx_n - \frac{\alpha_n}{\mu_n}\bigl(\delta\varphi(x_n) - \psi(x_n) + \mu_n Bx_n\bigr)\right\rangle \ge 0.$$

Then

$$\begin{aligned} \bigl\langle \psi(v) - \psi(x_{n_i}), w\bigr\rangle &\ge \bigl\langle \psi(v) - \psi(x_{n_i}), Bv\bigr\rangle \\ &\ge \bigl\langle \psi(v) - \psi(x_{n_i}), Bv\bigr\rangle - \left\langle \psi(v) - u_{n_i}, \frac{u_{n_i} - \psi(x_{n_i})}{\mu_{n_i}}\right\rangle \\ &\quad - \bigl\langle \psi(v) - u_{n_i}, Bx_{n_i}\bigr\rangle + \frac{\alpha_{n_i}}{\mu_{n_i}}\bigl\langle \psi(v) - u_{n_i}, \delta\varphi(x_{n_i}) - \psi(x_{n_i}) + \mu_{n_i}Bx_{n_i}\bigr\rangle \\ &= \bigl\langle \psi(v) - \psi(x_{n_i}), Bv - Bx_{n_i}\bigr\rangle + \bigl\langle \psi(v) - \psi(x_{n_i}), Bx_{n_i}\bigr\rangle \\ &\quad - \left\langle \psi(v) - u_{n_i}, \frac{u_{n_i} - \psi(x_{n_i})}{\mu_{n_i}}\right\rangle - \bigl\langle \psi(v) - u_{n_i}, Bx_{n_i}\bigr\rangle \\ &\quad + \frac{\alpha_{n_i}}{\mu_{n_i}}\bigl\langle \psi(v) - u_{n_i}, \delta\varphi(x_{n_i}) - \psi(x_{n_i}) + \mu_{n_i}Bx_{n_i}\bigr\rangle \\ &\ge -\left\langle \psi(v) - u_{n_i}, \frac{u_{n_i} - \psi(x_{n_i})}{\mu_{n_i}}\right\rangle - \bigl\langle \psi(x_{n_i}) - u_{n_i}, Bx_{n_i}\bigr\rangle \\ &\quad + \frac{\alpha_{n_i}}{\mu_{n_i}}\bigl\langle \psi(v) - u_{n_i}, \delta\varphi(x_{n_i}) - \psi(x_{n_i}) + \mu_{n_i}Bx_{n_i}\bigr\rangle. \end{aligned} \tag{3.10}$$

Since $\|\psi(x_{n_i}) - u_{n_i}\| \to 0$ and $\psi(x_{n_i}) \rightharpoonup \psi(z)$, we deduce that $\langle \psi(v) - \psi(z), w\rangle \ge 0$ by taking $i \to \infty$ in (3.10). Thus, $z \in R^{-1}0$ by the maximal ψ-monotonicity of R. Hence,

$z \in \mathrm{GVI}(B, \psi, C)$. Therefore, $z \in S$. From (3.7), we obtain

$$\begin{aligned}
&\limsup_{n\to\infty}\langle \delta\varphi(x^*) - \psi(x^*), u_n - \psi(x^*)\rangle \\
&\quad = \lim_{i\to\infty}\langle \delta\varphi(x^*) - \psi(x^*), \psi(x_{n_i}) - \psi(x^*)\rangle \\
&\quad = \langle \delta\varphi(x^*) - \psi(x^*), \psi(z) - \psi(x^*)\rangle \le 0.
\end{aligned}$$

Note that

$$\begin{aligned}
\|u_n - \psi(x^*)\|^2 &\le \langle \alpha_n(\delta\varphi(x_n) - \psi(x^*)) + (1-\alpha_n)y_n, u_n - \psi(x^*)\rangle \\
&\le \alpha_n\delta\langle \varphi(x_n) - \varphi(x^*), u_n - \psi(x^*)\rangle + \alpha_n\langle \delta\varphi(x^*) - \psi(x^*), u_n - \psi(x^*)\rangle \\
&\quad + (1-\alpha_n)\|\psi(x_n) - \mu_n Bx_n - (\psi(x^*) - \mu_n Bx^*)\|\|u_n - \psi(x^*)\| \\
&\le \alpha_n L\delta\|x_n - x^*\|\|u_n - \psi(x^*)\| + \alpha_n\langle \delta\varphi(x^*) - \psi(x^*), u_n - \psi(x^*)\rangle \\
&\quad + (1-\alpha_n)\|\psi(x_n) - \psi(x^*)\|\|u_n - \psi(x^*)\| \\
&\le \alpha_n(\delta L/\gamma)\|\psi(x_n) - \psi(x^*)\|\|u_n - \psi(x^*)\| \\
&\quad + \alpha_n\langle \delta\varphi(x^*) - \psi(x^*), u_n - \psi(x^*)\rangle \\
&\quad + (1-\alpha_n)\|\psi(x_n) - \psi(x^*)\|\|u_n - \psi(x^*)\| \\
&= \bigl[1 - (1 - L\delta/\gamma)\alpha_n\bigr]\|\psi(x_n) - \psi(x^*)\|\|u_n - \psi(x^*)\| \\
&\quad + \alpha_n\langle \delta\varphi(x^*) - \psi(x^*), u_n - \psi(x^*)\rangle \\
&= \frac{1 - (1 - L\delta/\gamma)\alpha_n}{2}\|\psi(x_n) - \psi(x^*)\|^2 + \frac{1}{2}\|u_n - \psi(x^*)\|^2 \\
&\quad + \alpha_n\langle \delta\varphi(x^*) - \psi(x^*), u_n - \psi(x^{\ddagger})\rangle.
\end{aligned}$$

It follows that

$$\begin{aligned}
\|u_n - \psi(x^*)\|^2 &\le \bigl[1 - (1 - L\delta/\gamma)\alpha_n\bigr]\|\psi(x_n) - \psi(x^*)\|^2 \\
&\quad + 2\alpha_n\langle \delta\varphi(x^*) - \psi(x^*), u_n - \psi(x^*)\rangle.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|\psi(x_{n+1}) - \psi(x^*)\|^2 &\le \beta_n\|\psi(x_n) - \psi(x^*)\|^2 + (1-\beta_n)\|u_n - \psi(x^*)\|^2 \\
&\le \beta_n\|\psi(x_n) - \psi(x^*)\|^2 \\
&\quad + (1-\beta_n)\bigl[1 - (1 - \delta L/\gamma)\alpha_n\bigr]\|\psi(x_n) - \psi(x^*)\|^2 \\
&\quad + 2(1-\beta_n)\alpha_n\langle \delta\varphi(x^*) - \psi(x^*), u_n - \psi(x^*)\rangle \\
&= \bigl[1 - (1 - \delta L/\gamma)(1-\beta_n)\alpha_n\bigr]\|\psi(x_n) - \psi(x^*)\|^2 \\
&\quad + 2(1-\beta_n)\alpha_n\langle \delta\varphi(x^*) - \psi(x^*), u_n - \psi(x^*)\rangle \\
&= \bigl[1 - (1 - \delta L/\gamma)(1-\beta_n)\alpha_n\bigr]\|\psi(x_n) - \psi(x^*)\|^2 \\
&\quad + (1 - \delta L/\gamma)(1-\beta_n)
\end{aligned}$$

$$\begin{aligned} &\times \alpha_n\left(\frac{2}{1-\delta L/\gamma}\left\langle \delta\varphi\left(x^*\right)-\psi\left(x^*\right), u_n-\psi\left(x^*\right)\right\rangle\right) \\ &= (1-\gamma_n)\left\|\psi(x_n)-\psi\left(x^*\right)\right\|^2+\delta_n\gamma_n, \end{aligned}$$

where $\gamma_n = (1-\delta L/\gamma)(1-\beta_n)\alpha_n$ and $\delta_n = \frac{2}{1-\delta L/\gamma}\langle \delta\varphi(x^*)-\psi(x^*), u_n-\psi(x^*)\rangle$. It is easily seen that $\sum_n \gamma_n = \infty$ and $\limsup_{n\to\infty}\delta_n \le 0$. We can therefore apply Lemma 2.4 to conclude that $\psi(x_n) \to \psi(x^*)$ and $x_n \to x^*$. This completes the proof. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

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